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N-Homoclinic Bifurcations of Piecewise Linear Vector Fields

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Abstract

N-homoclinic bifurcations (N>2) are found and studied in a piecewise-linear vector field on $\mathbb{R}^3$.

1. Introduction

Consider a two parameter family of vector fields on $\mathbb{R}^n$;

$$ \dot{x} = F(x; \mu) $$

Assume:

(i) $F(0, \mu) = 0, \mu \in \mathbb{R}^2, \mu = (\mu_1, \mu_2) \in \mathbb{R}^2$

(ii) $DF(x, \mu)$ has real eigenvalues $\lambda_1, \lambda_2, \lambda_3$ satisfying

$$ \text{Re}(\lambda_i) < \lambda_2 < \lambda_1 < 0 < \lambda_3 < \text{Re}(\lambda_j) $$

where $\text{Re}(\lambda_i)$ indicates the real part of other eigenvalues.

(iii) The dynamics has a homoclinic orbit through the origin $x = 0$ at some $\mu$

Homoclinic doubling bifurcation is the phenomenon schematically drawn in Fig1. Namely, a homoclinic orbit of the simplest type (1-homoclinic) orbit bifurcates into a "double-loop" (2-homoclinic orbit) orbit.

![Fig1-1 Simplest homoclinic orbit](image1.jpg)

![Fig1-2 "Double-loop" homoclinic orbit](image2.jpg)

![Fig1 Schematic picture of homoclinic doubling bifurcation](image3.jpg)

This phenomenon was first found and analyzed by Yanagida [1] during
Fig 2  Critically twisted homoclinic orbit

This strip represents a family of solutions of variational equation along the 1-homoclinic orbit.

Fig 3  Non-principal homoclinic orbit

This strip represents a family of solutions of variational equation along the 1-homoclinic orbit.
the course of his studies on generalized nerve axon equation. Analyzing with the original partial differential equation, Yanagida derived an ordinary differential equation and proved the existence of a double-pulse traveling wave solution, which corresponds to the homoclinic doubling bifurcation. Yanagida observed that there are three cases in which homoclinic doubling bifurcation can occur:

The original 1-homoclinic orbit is
(1) a homoclinic orbit with resonant eigenvalues, or
(2) a critically twisted homoclinic orbit, or
(3) a non-principal homoclinic orbit.

Case(1) refers to $\lambda_1 + \lambda_3 = 0$ while cases (2) and (3) are schematically shown in Fig2 and Fig3, respectively. M. Kisaka [3] proved that an N(>2)-homoclinic orbit dose not bifurcate from a 1-homoclinic orbit in case (1) and (2). Nothing is known about N(>2)-homoclinic orbits for case(3), however. Details are found in [1],[2],[3],[4]. The purpose of this paper is to give an example which suggests that N(>2)-homoclinic orbit bifurcate from 1-homoclinic orbit for case(3).

2. Normal Forms of 2-Region Continuous Piecewise-Linear Vector Field.

Consider the 2-region continuous piecewise-linear vector field in $\mathbb{R}^3$:

$$\dot{x}' = f(x') = \begin{cases} A'x' & (<\alpha', x'> - 1 \leq 0) \\ B'x' - p' & (<\alpha', x'> - 1 \geq 0) \end{cases}$$

(2.1)

where $A'$ and $B'$ are $3 \times 3$ matrices and $p' \in \mathbb{R}^3$. The plane $<\alpha', x'> = 1$ is the boundary of the vector field. Assume that $A'$ has 3 real eigenvalues $\lambda_1$, $\lambda_2$, $\lambda_3$ ($\lambda_3 > 0 > \lambda_1 > \lambda_2$) and $B'$ has a pair of complex conjugate eigenvalues $\sigma_1 + i\omega_1$ and a real eigenvalue $\gamma_1$. ($\sigma_1 < 0, \omega_1 > 0, \gamma_1 > 0$). According to the normal form theorem [5],[6], $f$ is uniquely determined up to linearly conjugacy as follows (provided that $f$ has no eigenspace parallel to the boundary):

$$\dot{x}'' = S_Ax'' + \frac{1}{2}p'' \left[ |<\alpha'', x''| - 1| + (<\alpha'', x''| - 1) \right]$$

$$= \begin{cases} S_Ax'' & (x'' \in \mathbb{R}_-) \\ S_B(x'' - P'') & (x'' \in \mathbb{R}_+) \end{cases}$$

(2.2)
where
\[ R_{\pm} = \{ x'' \in \mathbb{R}^3 : \pm(\langle \alpha'', x'' \rangle - 1) > 0 \} \quad \alpha'' = (1, 0, 0) \]
\[ p''^T = (c_1, c_2, c_3) \]
\[ p''^T = (1 - \frac{a_3}{b_3}, \frac{c_1a_3}{b_3}, \frac{c_2a_3}{b_3}) \]
\[
\begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
_3 & _2 & _1 \\
\end{bmatrix}
\]
\[
\begin{bmatrix}
0 & 1 & 0 \\
_2 & 0 & 1 \\
_3 + _3 & _2 & _1 \\
\end{bmatrix}
\]
\[ a_1 = \lambda_1 + \lambda_2 + \lambda_3, \quad a_2 = -((\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1), \quad a_3 = \lambda_1\lambda_2\lambda_3 \]
\[ b_1 = 2\sigma_1 + \gamma_1, \quad b_2 = -(\sigma_1^2 + \omega_1^2 + 2\gamma_1\sigma_1), \quad b_3 = (\sigma_1^2 + \omega_1^2)\gamma_1 \]
\[ c_1 = b_1 - a_1, \quad c_2 = b_2 - a_2 + c_1a_1, \quad c_3 = b_3 - a_3 + c_1a_2 + c_2a_1 \]

Fig. 4 shows the geometric structure of (2.2). The vector field defined by (2.2) is transformed via
\[ x'' = H_A x \]
(2.3)
where
\[
\begin{bmatrix}
1 & 1 & 1 \\
\lambda_1 & \lambda_2 & \lambda_3 \\
\lambda_1^2 & \lambda_2^2 & \lambda_3^2 \\
\end{bmatrix}
\]
to the vector field
\[
\dot{x} = Ax + \frac{1}{2} p(\|\langle \alpha, x \rangle - 1\| + (\langle \alpha, x \rangle - 1))
\]
\[
= \begin{cases} 
Ax & (x \in \mathbb{R}_-) \\
B(x - p) & (x \in \mathbb{R}_+)
\end{cases}
\]
(2.4)
where
\[
A = \begin{bmatrix}
\lambda_1 & 0 & 0 \\
0 & \lambda_2 & 0 \\
0 & 0 & \lambda_3
\end{bmatrix}
\]
\[ \alpha = (1,1,1) \quad \beta = H_A \beta'' \quad P = B^{-1}p \]

\[ B = A + p^T \alpha \]

\[ R_\pm = \{ x \in \mathbb{R}^3 : \pm <\alpha, x > -1 > 0 \} \]

\[ V = \{ x \in \mathbb{R}^3 : <\alpha, x >= 1 \} \]

\[ V_- = \{ x \in V : x^T A x < 0 \} \]

\[ V_+ = \{ x \in V : x^T A x > 0 \} \]

This is called the normal form of 2-region continuous piecewise-linear vector field.

Fig 5. shows the geometric structure of (2.4).

### 3. Bifurcation equations.

#### 3.1 Return time coordinate.

Consider a point \( \bar{x} \) lying on the boundary \( V \). Let \( \bar{y} \) and \( \bar{z} \) be the points at which the trajectory starting from \( \bar{x} \) hits \( V \) again at positive time \( s \) and negative time \( -t \), respectively. Since the system is linear in each region, one has

\[ \bar{y} = e^{B_s}(\bar{x} - P) + P \tag{3.1.1} \]

\[ \bar{z} = e^{-A_t} \bar{x} \tag{3.1.2} \]

Since the vector field is continuous,

\[ A \bar{x} = B(\bar{x} - P) \quad \bar{x} \in V \tag{3.1.3} \]

\[ A \bar{y} = B(\bar{y} - P) \quad \bar{y} \in V \tag{3.1.4} \]

Using (3.1.4) and (3.1.1), one has

\[ A \bar{y} = B e^{B_s}(\bar{x} - P) \]

Since \( A \) is non-singular,

\[ \bar{y} = A^{-1} e^{B_s} B(\bar{x} - P) \tag{3.1.5} \]

Moreover, by (3.1.3), one has

\[ \bar{y} = A^{-1} e^{B_s} A \bar{x} = e^{C_s} \bar{x} \]

where

\[ C = A^{-1} BA \]

Since \( \bar{x}, \bar{y}, \bar{z} \) are on the boundary \( V \),

\[ ^T c \alpha e^{-At} \bar{x} = 1 \quad ^T c \alpha \bar{x} = 1 \quad ^T c \alpha e^{Cs} \bar{x} = 1 \]

so that

\[ \left[ e_1 ^T c \alpha e^{-At} + e_2 ^T c \alpha + e_3 ^T c \alpha e^{Cs} \right] \bar{x} = h \tag{3.1.6} \]

where

\[ e_1 = ^T (1,0,0), \quad e_2 = ^T (0,1,0), \quad e_3 = ^T (0,0,1), \quad h = ^T (1,1,1) \]

If

\[ K(s,t) = \left[ e_1 ^T c \alpha e^{-At} + e_2 ^T c \alpha + e_3 ^T c \alpha e^{Cs} \right]^{-1} \tag{3.1.7} \]
is non singular, then
\[ \tilde{x} = K(s, t)h \] (3.1.8)
The pair \((s, t)\) is called the return time coordinate of \(\tilde{x}\) on \(V\).
(See Fig6. and Fig7).

### 3.2 Homoclinic bifurcation equations.

If a trajectory starting from \((0,1,1)\) hits \(E^C(0)\) on the boundary \(V\), then it is a 1-homoclinic orbit through the origin (Fig4.) which is characterized by

\[
\begin{align*}
\tau \alpha e^{Cs} e_3 - 1 &= 0 \\
\tau e_3 e^{Cs} e_3 &= 0 \\
\end{align*}
\] (3.2.1)

Fig8. shows a 1-homoclinic orbit. Similarly, an \(N\)-homoclinic orbit through the origin is characterized by

\[
\begin{align*}
\tau \alpha e^{Cs} e_3 - 1 &= 0 \\
N(e^{Cs} e_3 - e^{-At} K(s_2, t_2) h) &= 0 \\
N(e^{Cs} K(s_1, t_1) h - e^{-At} K(s_{i+1}, t_{i+1}) h) &= 0 \\
(2 \leq i \leq m - 1) \\
\tau e_3 e^{Cs} K(s_m, t_m) h &= 0 \\
\end{align*}
\] (3.2.2)

where

\[
N = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
\end{bmatrix}
\]

Fig9 shows a typical 3-homoclinic orbit.

### 3.3 Tangent map

Assume that there exists an \(s_0\) such that
\[
\begin{align*}
z_0 &= e^{Bs_0} (y_0 - P) + P \quad (y_0 \in V_+, z_0 \in V_-), \\
\tau \alpha (e^{Bs} (y_0 - P) + P) - 1 &\neq 0, \quad \forall s \in (0, s_0) \\
\end{align*}
\]

Let
\[
H(y, s) = \tau \alpha (e^{Bs} (y - P)) + P - 1
\]

Since
\[
H(y_0, s_0) = \tau \alpha z_0 - 1 = 0
\]

and since
\[
\frac{\partial H}{\partial t}(y_0, s_0) = \tau \alpha Be^{Bs} (y_0 - P) = \tau \alpha B(z_0 - P) = \tau \alpha Az_0 \neq 0
\]

there exist a neighborhood \(V_+(y_0)\) of \(y_0\) on \(V_+\) and function (called a return time function)
\[
s: V_+(y_0) \to R
\]
such that
\[ H(y, s(y)) = 0, \quad s(y_0) = s_0, \]

Then,
\[
D_s(y_0) = -\left[ \frac{\partial H}{\partial t}(y_0, s_0) \right]^{-1} \frac{\partial H}{\partial y}(y_0, s_0)
= -\left[ T^\top \alpha A z_0 \right]^{-1} T^\top \alpha e^{Bs_s}
\]

Let
\[
g(y) = e^{Bs_s(y)}(y - p) + p,
\]

Then one can show that the tangent map is given by,
\[
D_g(y_0) = B e^{Bs_s} (y_0 - P) D_s(y_0) + e^{Bs_s}
= B(z_0 - P) D_s(y_0) + e^{Bs_s}
= \left\{ I - \frac{A z_0^\top \alpha}{\alpha A z_0} \right\} e^{Bs_s}
(3.3.1)
\]

3.4 Conditions for homoclinic doubling bifurcation.

Define (See Fig.10)
\[
\begin{align*}
&h_1(\lambda_1, \lambda_2, \lambda_3, \sigma_1, \omega_1, \gamma_1) = e_1^T e^{Bs_s} e_3 \\
&h_2(\lambda_1, \lambda_2, \lambda_3, \sigma_1, \omega_1, \gamma_1) = e_3^T \left\{ I - \frac{A z_0^\top \alpha}{\alpha A z_0} \right\} e^{Bs_s} (e_1 - e_3)
\end{align*}
(3.4.1)
\]

Then, a homoclinic doubling bifurcation is characterized by

1. homoclinic orbit with resonant eigenvalues;
\[
h_1(\lambda_1, \lambda_2, \lambda_3, \sigma_1, \omega_1, \gamma_1) \times h_2(\lambda_1, \lambda_2, \lambda_3, \sigma_1, \omega_1, \gamma_1) < 0
\]
and
\[
|\lambda_1| = |\lambda_3|
(3.4.2)
\]

2. critically twisted homoclinic orbit;
\[
h_2(\lambda_1, \lambda_2, \lambda_3, \sigma_1, \omega_1, \gamma_1) = 0
\]
and
\[
|\lambda_1| < |\lambda_3|
(3.4.3)
\]

3. non-principal homoclinic orbit;
\[
h_1(\lambda_1, \lambda_2, \lambda_3, \sigma_1, \omega_1, \gamma_1) = 0
\]
and
\[
|\lambda_1| < |\lambda_3|
(3.4.4)
\]

4. Bifurcation sets of N-homoclinic orbits.

4.1 Two parameter diagram.
Fig 11 shows N-homoclinic bifurcation sets, for $N=1$ to $7$, in the $(\lambda_1, \sigma_1)$-space obtained by solving (3.2.1) and (3.2.2). The vertical axis is $\sigma_1$ while the horizontal axis is $\lambda_1$. The other eigenvalues are fixed as

$$\omega_1 = 1.0, \gamma_1 = -0.01, \lambda_2 = -0.32, \lambda_3 = 0.3$$  \hspace{1cm} (4.1.1)  

Fig 12 shows details of Fig 11 where bifurcation sets for $N=8$ and $9$ are discernible. Fig 13 shows the same bifurcation sets in the range $-0.8 < \lambda_1 < -0.4$, whereas Fig 14 shows details of Fig 13. NH in these figures indicates N-homoclinic bifurcation sets. For $N=3$ and $5$ to $9$, homoclinic bifurcation sets form a loop while 4-homoclinic bifurcation sets consist of two loops. Moreover, it appears that all the $N(3 \sim 9)$-homoclinic bifurcation sets bifurcate from a point on the 1-homoclinic bifurcation set. Fig 15 shows the orbits corresponding to the bifurcation sets. For 1H in Fig 15, the numbers 1, 2 and 3 correspond to those in Fig 3.

4.2 Non-principal homoclinic orbit.

Solving the set of Eqs. (3.2.1) and (3.4.4) by Newton method, we obtained the following set of values:

$$\sigma_1 = 0.0137, \lambda_1 = -0.01$$

These are the values on which non-principal homoclinic orbit exists. Now let us look at this point in Fig 12. It appears that all the $N(>2)$-homoclinic bifurcation sets accumulate towards this point. This phenomenon suggests that there is a close relationship between $N(>2)$-homoclinic orbits and non-principal homoclinic orbit.

4.3 Three dimensional bifurcation diagram.

Fig 16 shows a three dimensional bifurcation diagram of 3-homoclinic bifurcation set. Here $\gamma_1$ is fixed as $\gamma_1 = -0.04$ while others are the same as in (4.1.1). This figure shows that 3-homoclinic bifurcation sets vanish if $\lambda_2$ is sufficiently larger than -0.3. Kisaka [3] proved under several conditions of eigenvalues including the case $|\lambda_2| > |\lambda_3|$ that $N(>2)$-homoclinic orbit dose not bifurcate from 1-homoclinic orbit for the critically twisted case. This, however, does not contradict our numerical results because for the latter, Kisaka's conditions are not satisfied.
Fig. 7  Poincare full return map.

Fig. 6  Poincare half return map.

Fig. 5  Geometric structure of (2.3).

boundary

boundary

boundary
Fig. 11 Two parameter bifurcation diagram.
$N(1^-7)$-homoclinic bifurcation set.
Fig. 12. Details of Fig. 11.
Fig 13 Two parameter bifurcation diagram.

Fig 14 Details of Fig 13.
Fig 16. Three-dimensional bifurcation diagram.
3-homoclinic orbit.
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References.