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Nonlinear Associative Dynamics and Pattern Representations in Chaotic Neural Networks

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Abstract

We breifly review our model of chaotic neural networks and apply its dynamics to associative memory. In particular, we examine the influence of different encoding schemes of pattern vectors; namely orthogonal, nonorthogonal and sparse coding. The results obtained in this paper show that the output patterns of chaotic neural networks can be not only periodic but non-periodic (chaotic) and recall the stored patterns intermittently and succesively. With our model of chaotic neural networks, dynamical associative memory can be realized.

1 Introduction

Deterministic chaos are widely discovered in various kinds of fields of sciences and engineering, such as fluid systems, chemical reaction systems, control systems and biological systems[1]. Especially, neural systems are essentially nonlinear and nonequillibrium systems and therefore it is very reasonable to expect that there exist deterministic chaos. It has been clarified that both experimentally with squid giant axons and numerically with the Hodgkin - Huxley equations[2] that deterministic chaos is easily and reproducivery observed in nerve membranes[3]-[7]. From these experimental researches, it is confirmed that single neurons, the fundamental processing element of neural systems can have chaotic dynamics. Moreover, it is also suggestted that electorencephalographic potentials, which represent the macroscopic behavior of our brain, can have chaotic dynamics by a lot of dimensional analyses[8], and from the view point of the function of brains, it is suggested that deterministic chaos has important role of the information processing in our brain[9]–[11].

We have proposed a model of a chaotic neural networks composed of neuron models with chaotic dynamics, on the basis of experiments on chaotic response in nerve membranes of squid axons qualitatively[12],[13]. In ref. [14], we have applied this model to associative memory,the stored patterns of which are mutually orthogonal, and examined its chaotic memory dynamics.

In this paper, we apply the chaotic neural network to associative memory, the stored patterns of which are not only mutually orthogonal but nonorthogonal and sparse. In order to clarify the chaotic dynamics, we calculate the Lyapunov spectrum, the temporal changes of the distance between the output pattern of chaotic neural network and stored pattern, and the total number of successful retrievals and the transition probabilities in the retrievals.

2 The model of a chaotic neuron

The model of a chaotic neuron was proposed to qualitatively describe chaotic response observed with squid giant axons and the Hodgkin-Huxley equations[12],[13]. The dynamics of a chaotic neuron with graded output and exponentially decaying refractoriness is described by the following equation[12],[13]:

$$x(t+1) = f[A(t) - \alpha \sum_{d=0}^{t} k^d g\{x(t-d)\} - \theta] \tag{1}$$

where $x(t+1)$ is the output which takes an analog value between 0 and 1 at discrete time $t+1$; $A(t)$ is an externally applied input at t; f is the continuous output function; g is the refractory function; α, k, and θ are the scaling parameter of refractoriness, the decay parameter of refractoriness, and the threshold, respectively.

Defining an internal state $y(t+1)$ of the neuron as

$$y(t+1) = A(t) - \alpha \sum_{d=0}^{t} k^d g\{x(t-d)\} - \theta, \tag{2}$$

we can simplify the dynamics of eq.(1) and get the reduced one dimensional map as follow[12]:

$$y(t+1) = ky(t) - \alpha g[f\{y(t)\}] + a(t), \tag{3}$$

$$x(t+1) = f\{y(t+1)\} \tag{4}$$

where

$$a(t) = A(t) - kA(t-1) - \theta(1-k). \tag{5}$$

When we assume the externally applied input is temporally constant, $a(t)$ is in eq.(5) becomes constant a. By changing the parameters a, k and α, we can obtain not only periodic solutions but chaotic solutions. For example, Fig.1 shows the two typical solutions of chaotic neuron model when we change the parameter a. Fig.2 is the bifurcation diagram of a chaotic neuron model in case of changing the parameter a. In Fig.2(a) the internal states $y(t), t = 250, \cdots, 450$ are plotted. Fig.2(b) and (c) are the Lyapunov exponent defined by

$$\lambda = \lim_{N\to\infty} \frac{1}{N} \sum_{t=0}^{N-1} \left| \frac{dy(t+1)}{dy(t)} \right| \tag{6}$$

and the average firing rates,

$$\rho = \lim_{N\to\infty} \frac{1}{N} \sum_{t=0}^{N-1} h\{x(t)\}, \tag{7}$$

respectively, where $h(t)$ is defined by

$$h(x) = \begin{cases} 1 & x \geq 0.5 \\ 0 & x < 0 \end{cases} \tag{8}$$

3 The model of a chaotic neural network

We can extend the model of a chaotic neuron, eqs.(3) and (4), to an articficial neural network model, which we call a "chaotic neural network", by considering two kinds of input; namely, feedback inputs from component neurons and externally applied inputs[12],[13]. The dynamics of the i th chaotic neuron in the neural network composed of n chaotic neurons can be modeled as eq.(9) [13],

$$\begin{aligned} x_i(t+1) &= f_i \left[\sum_{j=1}^{m} V_{ij} \sum_{d=0}^{t} k_s^d A_j(t-d) \right. \\ &+ \sum_{j=1}^{n} W_{ij} \sum_{d=0}^{t} k_m^d h_j\{x_j(t-d)\} \\ &- \left. \alpha \sum_{d=0}^{t} k_r^d g_i\{x_i(t-d)\} - \theta_i \right] \end{aligned} \tag{9}$$

where $x_i(t+1)$ is the output of the i th chaotic neuron at the discrete time $t+1$; $A_j(t-d)$ is the strength of the j th input externally applied at the discrete time $t-d$ and m is the number of the externally applied inputs; V_{ij} is the connection weight from the j th externally applied input to the i th chaotic neuron; W_{ij} is the connection weight from the j th chaotic neuron to the i th chaotic neuron; f_i is the continuous output function of the i th chaotic neuron; h_j is the transfer function of the axon of the j th chaotic neuron for the propagating action potentials; g_i is the refractory function of the i th chaotic neuron; α is the scaling parameter; θ_i is the threshold; k_s, k_m and k_r are the decay parameters for the external inputs, the feedback inputs and the refractoriness, respectively.

We define three internal states by ξ_i,η_i and ζ_i as follows:

$$\xi_i(t+1) = \sum_{j=1}^{m} V_{ij} \sum_{d=0}^{t} k_s^d A_j(t-d), \tag{10}$$

$$\eta_i(t+1) = \sum_{j=1}^{n} W_{ij} \sum_{d=0}^{t} k_m^d h_j\{x_j(t-d)\}, \tag{11}$$

$$\zeta_i(t+1) = -\alpha \sum_{d=0}^{t} k_r^d g_i\{x_i(t-d)\} - \theta_i. \tag{12}$$

Similarly to the previous section, we can reduce eqs.(10)–(12) to the following equations under the assumption of exponential decay [12],[13]:

$$\xi_i(t+1) = k_s \xi_i(t) + \sum_{j=1}^{m} V_{ij} A_j(t), \tag{13}$$

$$\eta_i(t+1) = k_m \eta_i(t) + \sum_{j=1}^{n} W_{ij} h_j\{x_j(t)\}, \tag{14}$$

$$\zeta_i(t+1) = k_r \zeta_i(t) - \alpha g_i\{x_i(t)\} - \theta_i(1-k_r), \tag{15}$$

$$x_i(t+1) = f[\xi_i(t+1) + \eta_i(t+1) + \zeta_i(t+1)]. \tag{16}$$

Eqs.(10)–(16) include some of the usual neural network models such as the McCulloch-Pitts model[15] and the feedforward perceptrons[16]; namely our model of a chaotic neural network is a natural extention of the former models for producing chaotic dynamics and is easy to adjust to these neural networks models by changing the parameter values. In this paper, we assume that there do not exist any externally applied inputs of eq.(9) and that the term of threshold in eq.(11) is spatially constant as follows:

$$a_i = -\theta_i(1-k_r) \equiv a(const.). \tag{17}$$

Moreover, for the sake of simplicity, we also assume that $h_j(z) = z$, $g_i(z) = z$ and $f_i(y) = 1/\{1 + exp(-y/\epsilon)\}$ where ϵ is the steepness parameter. Then we can get a simpler model of eqs.(18)–(20) which we use in the following simulation:

$$\eta_i(t+1) = k_m \eta_i(t) + \sum_{j=1}^{n} W_{ij} x_j(t), \tag{18}$$

$$\zeta_i(t+1) = k_r \zeta_i(t) - \alpha x_i(t) + a, \tag{19}$$

$$x_i(t+1) = f[\eta_i(t+1) + \zeta_i(t+1)]. \tag{20}$$

4 Associative Memory Composed by a Chaotic Neural Network

We apply the chaotic neural network composed of sixteen chaotic neurons to associative memory[17]. By the following superimposed auto-correlation matrix, we make three networks with different kinds of encoding schemes on the synaptic weights W_{ij}, namely mutually orthogonal, nonorthogonal and sprase coding,

$$W_{ij} = \frac{1}{M}\sum_{l=1}^{M}(2p_i^l - 1)(2p_j^l - 1), \tag{21}$$

where p_i^l is the i th component of the l th stored pattern and M is the number of the stored patterns. The examples of the stored pattern in each case are shown in Figs. 3–5. In these figures, each square indicates the component of the stored patterns and the value 1 is represented by black and 0 white.

In the following simulation, we examine associative dynamics by changing a as the bifurcation parameter. The other parameter values are fixed to $k_m = 0.3, k_r = 0.95, \alpha = 1.6, \epsilon = 0.015$, respectively. In each simulation, we consider the first 10,000 steps as transient and the next 10,000 steps are used for calculations of severals quantities.

5 Nonlinear Dynamics of Chaotic Associative Memory

In order to confirm the existense of the chaotic dynamics, we calculate the Lyapunov spectrum. In this paper, we used the model of the chaotic neural network with two kinds of internal states, $\eta_i(t)$ and $\zeta_i(t)$. Defining the $2n$-dimensional state $\boldsymbol{y}(t)$ as,

$$\boldsymbol{y}(t) = (\eta_1(t), \ldots, \eta_n(t), \zeta_1(t), \ldots, \zeta_n(t)), \tag{22}$$

we can interpret the chaotic neural network as a $2n$-dimensional discrete-time dynamical system,

$$\boldsymbol{y}(t+1) = \boldsymbol{F}(\boldsymbol{y}(t)), \boldsymbol{y} \in \boldsymbol{R}^{2n} \tag{23}$$

where $\boldsymbol{F} : \boldsymbol{R}^{2n} \rightarrow \boldsymbol{R}^{2n}$ is a nonlinear map in the $2n$-dimensional state space. By linerization of eq.(23) with the infinitesimal deviation $\delta \boldsymbol{y}(t)$ from $\boldsymbol{y}(t)$, the following linear map is obtained:

$$\delta \boldsymbol{y}(t+1) = \boldsymbol{DF}\{\boldsymbol{y}(t)\}\delta \boldsymbol{y}(t). \tag{24}$$

where $\boldsymbol{DF}$ is a Jacobian matrix of $\boldsymbol{F}$ at $\boldsymbol{y}(t)$.

To calculate the Lyapunov spectrum, the orthonormal bases $\boldsymbol{u}_i(t), (i = 1, 2, \ldots, 2n)$, in the $2n$-dimensional space, are given as $\delta \boldsymbol{y}(t)$ of eq.(24). With the linear map $\boldsymbol{DF}$ of eq.(24), $\boldsymbol{u}_i(t), (i = 1, 2, \ldots, 2n)$ are transformed as follows:

$$\boldsymbol{e}_i(t+1) = \boldsymbol{DF}\boldsymbol{u}_i(t) \quad for \quad i = 1, 2, \ldots, 2n. \tag{25}$$

Since these $\boldsymbol{e}_i(t+1), (i = 1, 2, \ldots, 2n)$ are not mutually orthogonal,the Gram-Schmidt procedure is applied to form new orthonormal bases[18],

$$\begin{aligned} \boldsymbol{e}'_i(t+1) &= \boldsymbol{e}_i(t+1) \\ &\quad - \sum_{j=1}^{i-1} \langle \boldsymbol{e}_i(t+1), \boldsymbol{u}_j(t+1) \rangle \boldsymbol{u}_j(t+1), \end{aligned} \tag{26}$$

$$\boldsymbol{u}_i(t+1) = \frac{\boldsymbol{e}'_i(t+1)}{|\boldsymbol{e}'_i(t+1)|} \tag{27}$$

where $\langle , \rangle$ denotes the inner product.

Repeating this procedure, the Lyapunov spectrum is calculated as follow:

$$\lambda_i = \lim_{N \to \infty} \frac{1}{N} \sum_{t=1}^{N} \log |\boldsymbol{e}'_i(t)| \quad for \quad i = 1, 2, \ldots, 2n. \tag{28}$$

If the largest Lyapunov exponent, λ_1, is at least positive, the chaotic neural network has the orbital instability which is one of fundumental characteristics of deterministic chaos.

The Lyapunov dimension defined by Kaplan and York[20] is also calculated from the Lyapunov spectrum as,

$$D_L = j + \frac{\sum_{i=1}^{j} \lambda_i}{|\lambda_{j+1}|} \tag{29}$$

where j is the largest integer such that the sum of the Lyapunov exponents in descending order is not negative.

Next, in order to analyse how frequently the chaotic neural network recalls the stored patterns, we compute the distances between the output pattern of the chaotic neural network and four patterns of Figs.3–5 stored with the synaptic weights. This distance between the l th stored pattern and the output pattern at the discrete time t is defined as:

$$d_l(t) = \frac{1}{n}\sum_{i=1}^{n}|x_i(t) - p_i^l|. \qquad (30)$$

When $d_l(t) = 0$ (or 1), the chaotic neural network recalls the l th stored patterns (or its reversed pattern) exactly. We also calculate the total number of the retrievals to examine how the chaotic neural networks retrieve the stored patterns. We define this number by the following way. When the distance $d_l(t)$ of eq.(30) is less than 0.1, the chaotic neural network recalls the l th stored pattern, therefore the total number of retrievals is a natural number. We also compute the transition probabilities in retrieving, which are defined by the two succsesive retrievals; after retrieving the i th stored pattern, if the chaotic neural network recalls the j th stored pattern, the transition probabilities,r_{ij} is counted up.

Figs.6–8 show the typical examples of the spatio-temporal dynamics and the temporal changes of the distances the distances $d_l(t)$,$(l = 1, \cdots, 4)$ with the parameter values described above. In these figures, the outputs pattern, $x_i(t), (i = 1, 2, \cdots, 16)$, is displayed in each column, and the horizontal direction indicates the discrete time t. The strength of each output is proportional to the size of each square; the full black is nearly equal to 1 and white 0. In case of the nonorthogonal coding, the temporal changes is non-periodic with positive Lyapunov exponent, but the chaotic neural network recalls only the first and second patterns. However, in case of the orthogonal coding, the chaotic neural network only recalls the first pattern and its reversed one, which is applied as the initial condition. In this case, the largest

Lyapunov exponent is calculated as negative which shows this response is really periodic. Fig.8 is for the sparse coding, which exhibits very different response among three coding representations. For example, the chaotic neural network retrieves all stored patterns with dynamical recalling. The largest Lyapunov exponent is 0.034. These results are also confirmed by transition probabilities in Tables.1–3. Although, the total number of retrievals in case of the sparse coding is relatively less than in other two cases, the chaotic neural network can retrieve all stored patterns.

Figs.9–11 are the diagrams of the largest Lyapunov exponent and the total number of retrievals in each case by changing a as the bifurcation parameter. We can see that there are two different regions in the range of a. When a is greater than 1.6, the chaotic neural network cannot recall any stored patterns. However, in case that a is less than 1.6, the chaotic neural network can exhibit dynamical recallings. In this region, while in case of the orthogonal coding it is relatively hard to observe chaotic response, in case of the sparse coding, there are more positive Lyapunov exponents and the total number of retrievals are decreased.

6 Conclusions

We have briefly reviewed our model of chaotic neural networks and have applied the model of the chaotic neural network to associative memory and analysed its dynamical behavior. We examine the influences of the different kinds of encoding schemes of pattern vectors. As a result, we can observe not only periodic ut chaotic responses and the spatio-temporal pattern of the chaotic neural network exibits the dynamical recalling with positive Lyapunov exponents. It is shown that this model enables us to design dynamical associative memory with chaotic dynamics.

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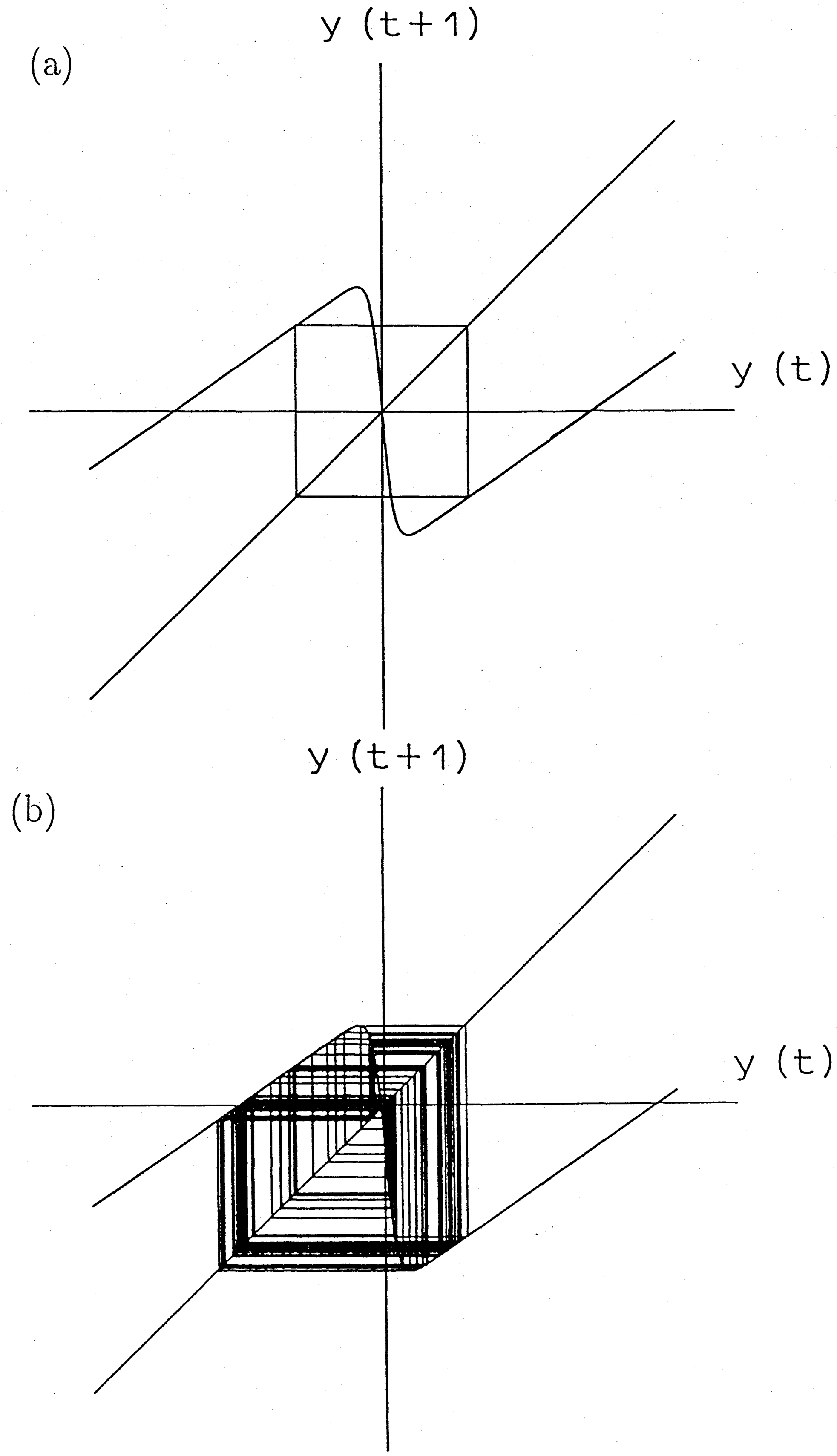

Figure 1: Examples of responses of a chaotic neuron. (a) Periodic and (b) chaotic solutions where $k = 0.7, \alpha = 1.0, \epsilon = 0.02$ and $a = 0.5$ in (a) and $k = 0.7, \alpha = 1.0, \epsilon = 0.02$ and $a = 0.35$.

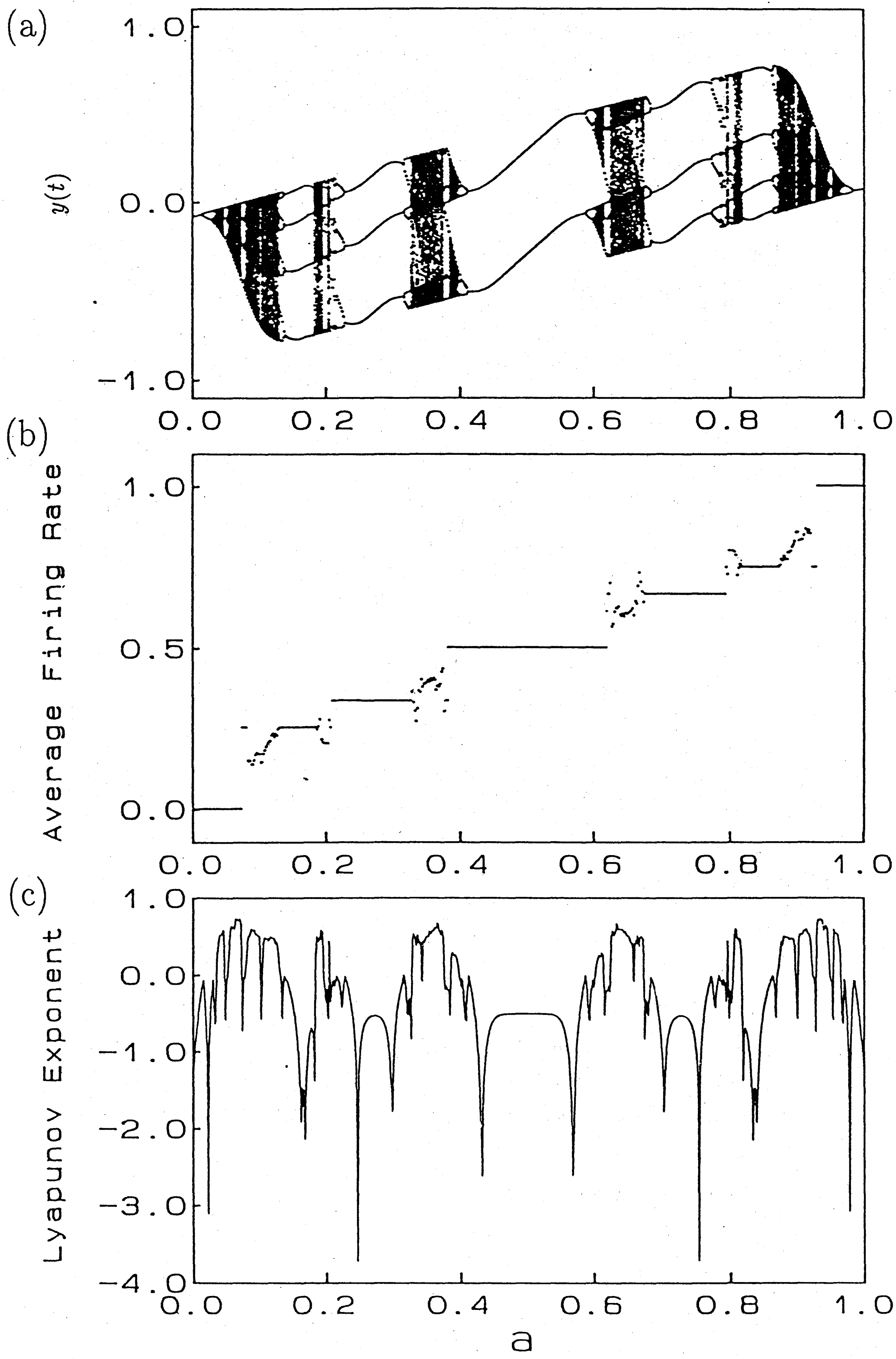

Figure 2: The bifurcation diagram of a chaotic neuron when the parameter a is changed. (a) The internal states, (b) the Lyapunov exponent and (c) the average firing rate.

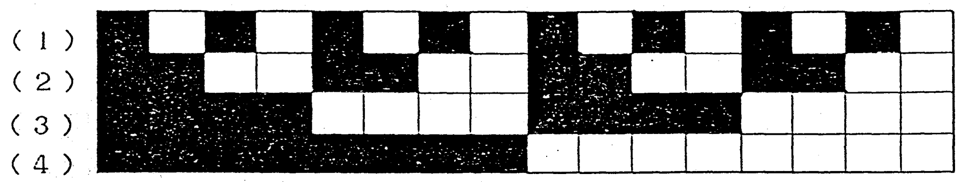

Figure 3: The stored patterns of orthogonal coding.

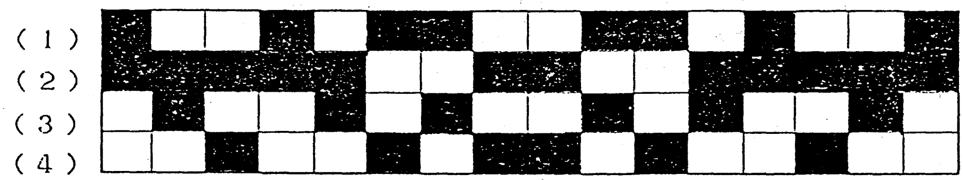

Figure 4: The stored patterns of nonorthogonal coding.

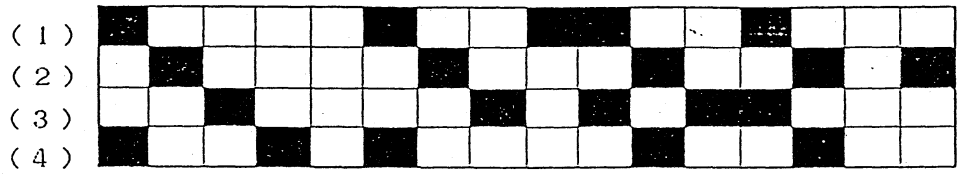

Figure 5: The stored patterns of sparse coding.

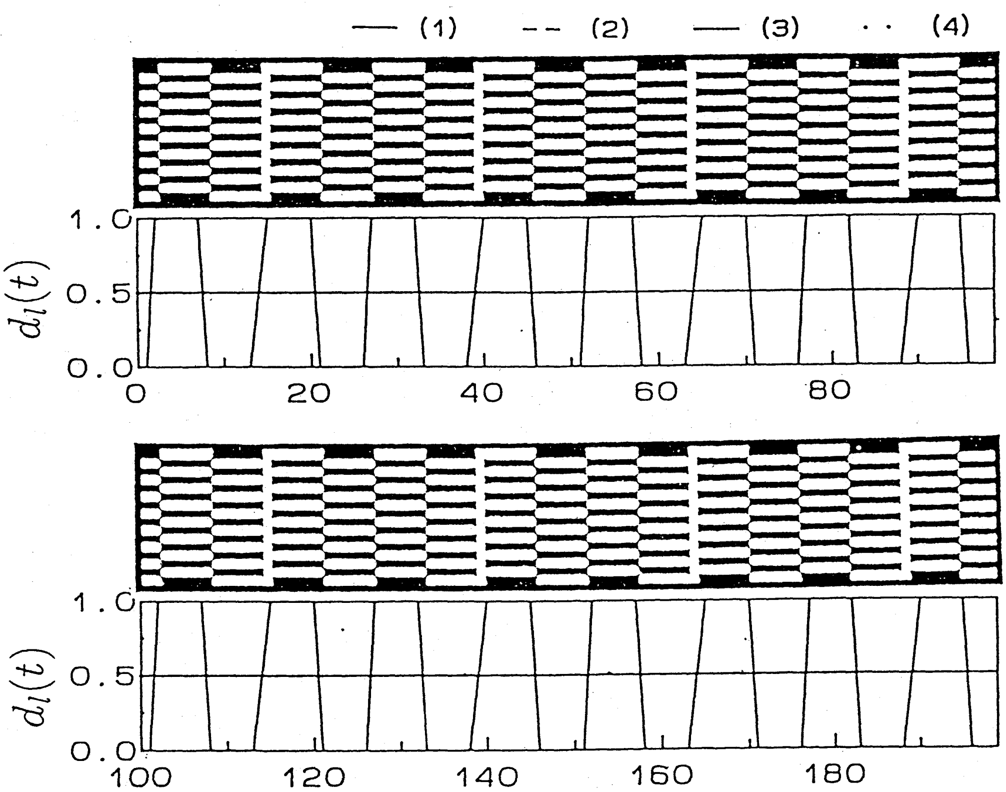

Figure 6: The associative dynamics of the chaotic neural network in case of the orthogonal coding in Fig.3, a=0.8, $\lambda_1 = -0.076$

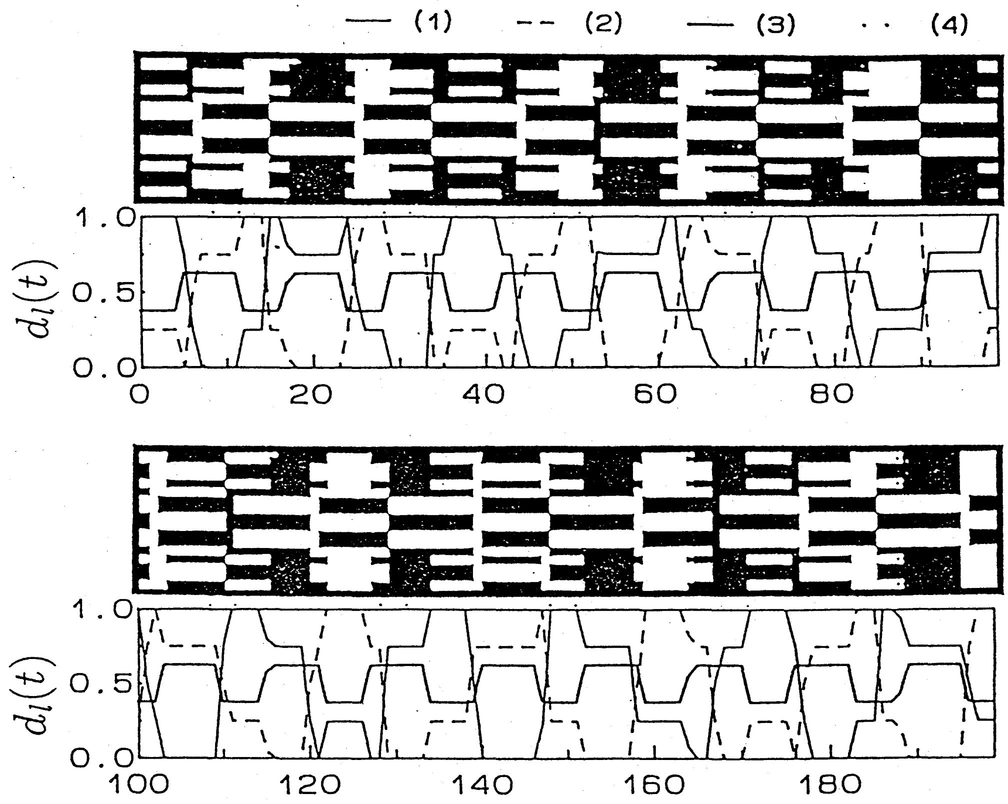

Figure 7: The associative dynamics of the chaotic neural network in case of the nonorthogonal coding in Fig.4, a=0.8, $\lambda_1 = 0.037$

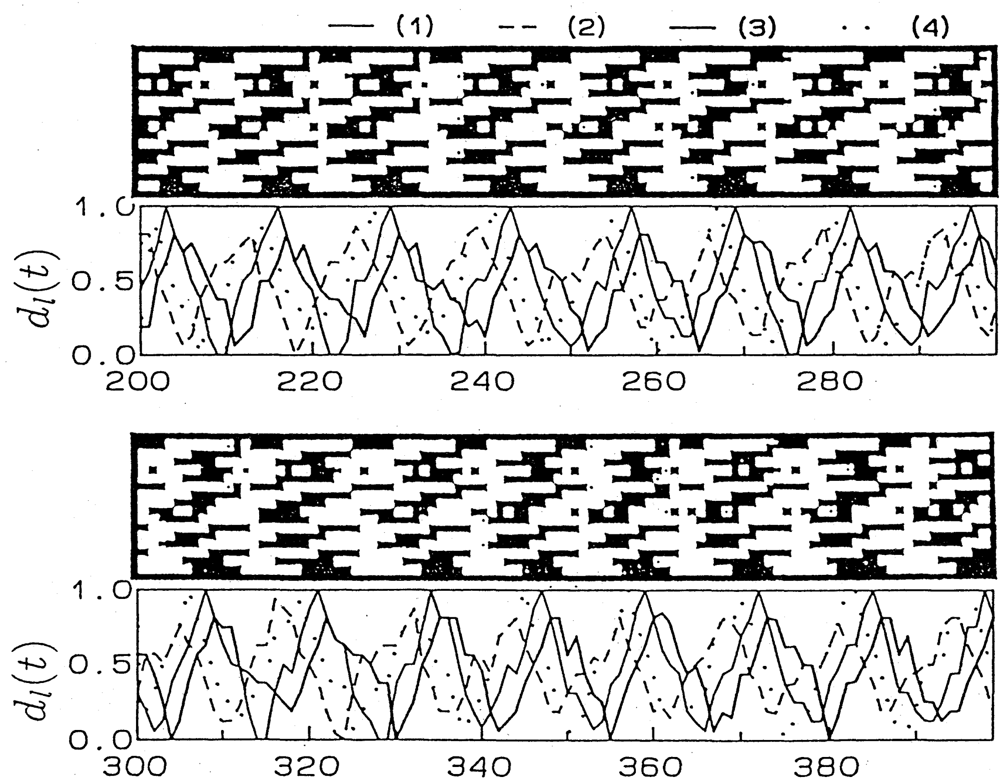

Figure 8: The associative dynamics of the chaotic neural network in case of the sparse coding in Fig.5, a=0.6, $\lambda_1 = 0.034$

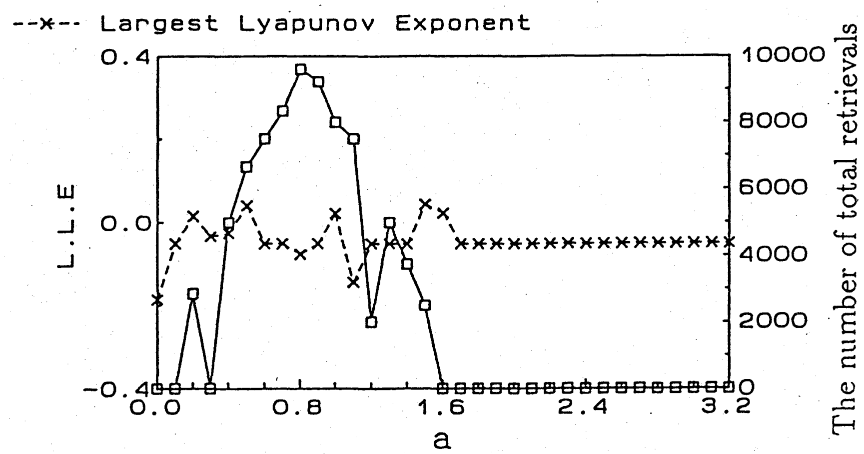

Figure 9: The diagram of the largest Lyapunov exponent and the total numbers of retrievals in case of the orthogonal coding.

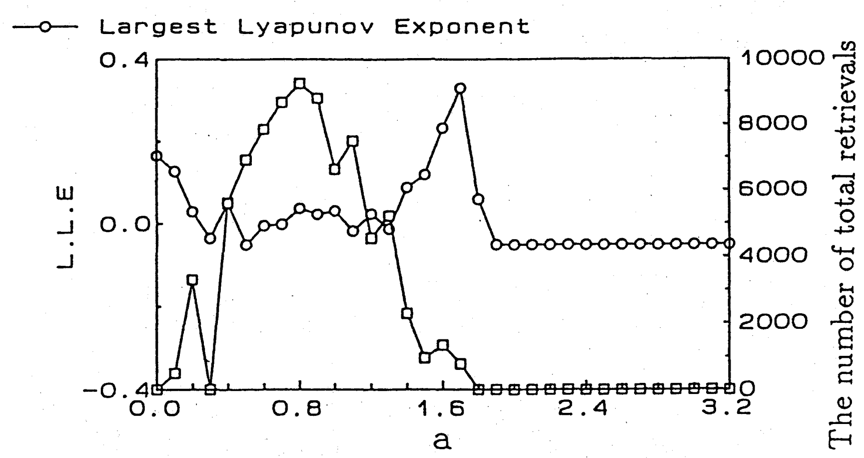

Figure 10: The diagram of the largest Lyapunov exponent and the total numbers of retrievals in case of the nonorthogonal coding.

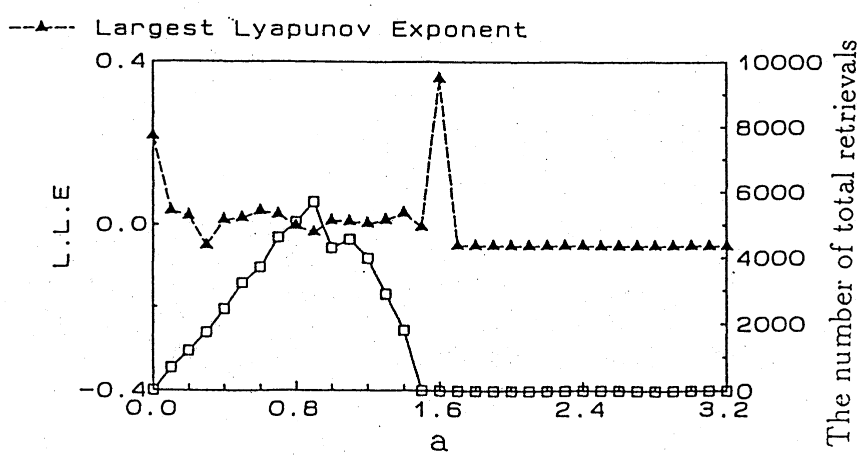

Figure 11: The diagram of the largest Lyapunov exponent and the total numbers of retrievals in case of the sparse coding.

Table 1: The transition probabilities (%) in case of the orthogonal coding, $a = 0.8, \lambda_1 = -0.076$. The total number of retrievals is 9600.

μ_j μ_i	(1)	(2)	(3)	(4)	($\bar{1}$)	($\bar{2}$)	($\bar{3}$)	($\bar{4}$)
(1)	—	0.0	0.0	0.0	8.3	0.0	0.0	0.0
(2)	0.0	—	0.0	0.0	0.0	0.0	0.0	0.0
(3)	0.0	0.0	—	0.0	0.0	0.0	0.0	0.0
(4)	0.0	0.0	0.0	—	0.0	0.0	0.0	0.0
($\bar{1}$)	8.3	0.0	0.0	0.0	—	0.0	0.0	0.0
($\bar{2}$)	0.0	0.0	0.0	0.0	0.0	—	0.0	0.0
($\bar{3}$)	0.0	0.0	0.0	0.0	0.0	0.0	—	0.0
($\bar{4}$)	0.0	0.0	0.0	0.0	0.0	0.0	0.0	—

Table 2: The transition probabilities (%) in case of the nonorthogonal coding, $a = 0.8, \lambda_1 = 0.037$. The total number of retrieval is 9264.

μ_j μ_i	(1)	(2)	(3)	(4)	($\bar{1}$)	($\bar{2}$)	($\bar{3}$)	($\bar{4}$)
(1)	—	2.4	0.0	0.0	0.9	3.2	0.0	0.0
(2)	2.6	—	0.0	0.0	4.0	0.6	0.0	0.0
(3)	0.0	0.0	—	0.0	0.0	0.0	0.0	0.0
(4)	0.0	0.0	0.0	—	0.0	0.0	0.0	0.0
($\bar{1}$)	0.7	4.0	0.0	0.0	—	1.8	0.0	0.0
($\bar{2}$)	3.2	0.8	0.0	0.0	1.6	—	0.0	0.0
($\bar{3}$)	0.0	0.0	0.0	0.0	0.2	0.0	—	0.0
($\bar{4}$)	0.0	0.0	0.0	0.0	0.0	0.0	0.0	—

Table 3: The transition probabilities (%) in case of the sparse coding, $a = 0.6, \lambda = 0.034$. The total number of retrievals is 3706.

μ_j μ_i	(1)	(2)	(3)	(4)	($\bar{1}$)	($\bar{2}$)	($\bar{3}$)	($\bar{4}$)
(1)	—	10.6	0.0	7.7	0.3	0.0	0.0	0.0
(2)	0.2	—	0.2	0.1	16.8	0.1	0.0	0.1
(3)	5.2	0.0	—	1.5	0.0	0.0	0.0	0.0
(4)	0.0	6.7	0.0	—	2.6	0.0	0.2	0.2
($\bar{1}$)	5.3	0.0	2.6	0.0	—	0.4	0.0	11.5
($\bar{2}$)	1.4	0.0	0.0	0.0	0.0	—	0.0	0.0
($\bar{3}$)	0.0	0.0	0.0	0.0	0.2	0.0	—	0.0
($\bar{4}$)	6.5	0.0	4.0	0.4	0.0	0.9	0.0	—