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Characters of cuspidal unramified series for central simple algebras of prime degree

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INTRODUCTION

Let $A$ be a central simple algebra of dimension $n^2$ over a non-archimedean local field $F$ and $L$ be a maximal unramified extension of $F$ in $A$. Gerardin [G] constructed an irreducible supercuspidal representation $\pi_\theta$ of $A^\times$ associated with a regular quasi-character $\theta$ of $L^\times$. ($\theta$ is regular $\iff \theta^\sigma \neq \theta \forall \sigma \in \text{Gal}(L/F)$).

The aim of this article is to get the character formula of $\pi_\theta$ on regular elements in all compact modulo center Cartan subgroups of $A^\times$ when $[A : F] = l^2$, $l$ an odd prime. (For the case $l = 2$, see [HSY]). We note that, when $l$ is a prime, $A$ is isomorphic to the division algebra of dimension $l^2$ over $F$ or the algebra of $l \times l$ matrices over $F$. Our character formula is as follows.

THEOREM. Let $\theta$ be a regular quasi-character of $L^\times$ with $\min f(\theta \otimes (\eta \circ N_{L/F})) = m + 1$ and $\Gamma = \text{Gal}(L/F)$. ($f(\theta) = \min \{n | \text{Ker} \theta \supset 1 + P_{L}^{n}\}$). We denote by $\chi_{\pi_\theta}$ the character of $\pi_\theta$. Let $x$ be an elliptic regular element in $A^\times$.

1. If $F(x) = L$, then

$$\chi_{\pi_\theta}(x) = \begin{cases} q^{\frac{(l-1)j}{2}} \left( \sum_{\sigma \in \Gamma} \theta(x^\sigma) \right) & \text{if } x \in U_j^* \quad (0 \leq j < m) \\ q^{\frac{(l-1)m}{2}} \left( \sum_{\sigma \in \Gamma} \theta(x^\sigma) \right) & \text{if } x \in U_m. \end{cases}$$

where $U_0 = L^\times, U_i = F^\times(1 + P_{L}^i) \quad (i \geq 1)$ and $U_i^* = U_i - U_{i+1}$.

2. If $F(x) \not\simeq L$, then

$$\chi_{\pi_\theta}(x) = \begin{cases} 0 & \text{if } x \notin F^\times(1 + P_{F(x)}^{l^m+1}) \\ \theta(c)q^{\frac{(l-1)m}{2}} & \text{if } x = c(1 + y) \in F^\times(1 + P_{F(x)}^{l^m+1}). \end{cases}$$

Remark. (a) Any compact (mod center) Cartan subgroup of $A^\times$ is isomorphic to $E^\times$ for some extension $E/F$ of degree $n$. Therefore the above formula gives the complete information on the set of elliptic regular elements of $A^\times$.

(b) For the case $F(x) = L$, the above formula can be written as follows:

$$\chi_{\pi_\theta}(x) = \Delta(x)^{-1} \sum_{\sigma \in W(L^\times)} \theta(x^\sigma) \quad \text{if } x \in U_j^* \quad (0 \leq j < m).$$
where $\Delta(x) = |\text{det}(Ad(x) - 1)_{A/L}|_{F}^{\frac{1}{2}}$ and $W(L^{\times})$ is the Weyl group with respect to the Cartan subgroup $L^{\times}$. This is the analogy of the following formulas:

1. character formula for irreducible square-integrable representations of real semisimple Lie groups (see [HC]);
2. character formula for principal series induced from a regular character of a maximal split torus;
3. character formula for irreducible unitary representations of compact Lie groups (Weyl’s character formula).

In this article, we shall prove the formula when $A$ is a division algebra. For the matrix algebra case, we use the result of division algebra case and Deligne-Kazhdan abstract matching theorem ([BDKV]): there is a bijection between irreducible representations of $D_{n}^{\times}$ and essentially square-integrable representations of $GL_{n}(F)$ which preserves the characters up to $(-1)^{n-1}$ ($D_{n}$ is a division algebra of dimension $n^{2}$ over $F$). Then we have only to calculate the character only on the set of ‘very cuspidal’ elements. More precisely, see [T].

We denote by $O_{F}$, $P_{F}$, $\varpi_{F}$, $k_{F}$ and $v_{F}$ the maximal order of $F$, the maximal ideal of $O_{F}$, a prime element of $P_{F}$, the residue field of $F$ and the valuation of $F$ normalized by $v_{F}(\varpi_{F}) = 1$. We set $g$ be the number of elements in $k_{F}$. Hereafter we fix an additive character $\psi$ of $F$ whose conductor is $P_{F}$ i.e. $\psi$ is trivial on $P_{F}$ and not trivial on $O_{F}$. For an irreducible admissible representation $\pi$ of $A^{\times}$, the conductoral exponent of $\pi$ is defined to be the integer $f(\pi)$ such that the local constant $\epsilon(s, \pi, \psi)$ of Godement-Jacquet [GJ] is the form $aq^{-s(f(\pi)-n)}$ where $n^{2} = [A : F]$. We call $\pi$ minimal if

$$f(\pi) = \min_{\eta} f(\pi \otimes (\eta \circ N_{A/F}))$$

where $\eta$ runs through the quasi-characters of $F^{\times}$. For a quasi-character $\eta$ of $F^{\times}$, $\eta \circ N_{A/F}$ is denoted by simply $\eta$ when there is no risk of confusion. Let $G$ be a totally disconnected, locally compact group. We denote by $\hat{G}$ the set of (equivalence classes of) irreducible admissible representations of $G$.

1. Construction of the representation. Let $D$ be a division algebra of degree $l$ (dimension $l^{2}$) over $F$ with $l$ an odd prime. We denote by $O_{D}$, $P_{D}$, $\varpi_{D}$ and $v_{D}$ the maximal order of $D$, the maximal ideal of $O_{D}$, a prime element of $P_{D}$ and the valuation of $D$ normalized by $v_{D}(\varpi_{D}) = 1$.

Let $L$ be an unramified extension of $F$ of degree $l$. $L$ can be embedded into $D$ and up to conjugacy, the embedding is unique.

DEFINITION 1.1. Let $\theta$ be a quasi-character of $L^{\times}$.

1. $\theta$ is called regular if all its conjugates by the action of $\text{Gal}(L/F)$ are distinct. We denote by $\hat{L}_{\text{reg}}^{\times}$ the set of regular quasi-characters of $L^{\times}$.
2. Let $f(\theta) = \min\{n|\text{Ker} \theta \supset 1 + P_{L}^{n}\}$. $\theta$ is called generic if either
   (a) $f(\theta) = 1$ and $\theta$ is not written in the form $\eta \circ N_{L/F}$ where $\eta$ is a quasi-character of $F^{\times}$ or
   (b) $f(\theta) > 1$ and $k_{F}(\varpi_{F}^{f(\theta)-1}\gamma_{\theta}) = k_{L}$ where $\gamma_{\theta} \in P_{L}^{1-f(\theta)} - P_{L}^{2-f(\theta)}$ such that $\theta(1 + x) = \psi(t_{\theta \pi_{L}}(L_{\pi}(\gamma_{\theta}x)))$ for $x \in P_{L}^{f(\theta)-1}$.
We note that any regular quasi-character of \( L^\times \) is written in the form \((\eta \circ N_{L/F}) \otimes \theta\) where \( \eta \) is a quasi-character of \( F^\times \) and \( \theta \) is a generic quasi-character of \( L^\times \).

We construct an irreducible representation \( \pi_\theta \) from \( \theta \in \hat{L}_{\text{reg}}^\times \) according to [G]. At first we treat the case \( \theta \) is generic. If \( f(\theta) = 1 \), then \( \theta \) itself can be regarded as a quasi-character of \( F^\times O_D^\times \) since \( F^\times O_D^\times /1 + P_D \cong L^\times /1 + P_L \). Therefore we set

(1.2) \[ \pi_\theta = \text{Ind}_{F^\times O_D^\times}^{D^\times} \theta. \]

Then \( \pi_\theta \) is an irreducible representation of \( D^\times \) with \( f(\pi_\theta) = l \). If \( f(\theta) = m+1 > 1 \), then there exists an element \( \gamma_\theta \in P_L^{-m} - (F \cap P_L^{-m}) + P_L^{1-m} \) such that

(1.3) \[ \theta(1+x) = \psi(\text{tr}_{L/F}(\gamma_\theta x)) \quad \text{for} \quad x \in P_L^{[m+2]}. \]

where \([ \ ]\) is the greatest integer function. (Recall that the conductor of \( \psi \) is \( P_F \).) Let \( \psi_{\gamma_\theta}(1+x) = \psi(\text{tr}_{D/F}(\gamma_\theta x)) \) for \( x \in P_D^{[ml+2]} \). Then \( \psi_{\gamma_\theta} \) is a quasi-character of \( 1 + P_D^{[ml+2]} \).

Set \( H = L^\times (1 + P_D^{[ml+2]}) \subset D^\times \) and define a quasi-character \( \rho_\theta \) of \( H \) by

(1.4) \[ \rho_\theta(h \cdot g) = \theta(h) \psi_{\gamma_\theta}(g) \quad \text{for} \quad h \in L^\times, \ g \in 1 + P_D^{[ml+2]}. \]

We set

(1.5) \[ \pi_\theta = \text{Ind}_H^{D^\times} \rho_\theta. \]

Then \( \pi_\theta \) is an irreducible minimal representation of \( D^\times \) with \( f(\pi_\theta) = l(m+1) \). (cf. [H],IV).

For a regular quasi-character \( \theta \) written in the form \( \theta = (\eta \circ N_{L/F}) \otimes \theta' \) where \( \eta \) is a quasi-character of \( F^\times \) and \( \theta' \) is a non-trivial generic quasi-character of \( L^\times \), we set

(1.6) \[ \pi_\theta = \pi_{\theta'} \otimes \eta. \]

Now we get a correspondence \( \theta \in \hat{L}_{\text{reg}}^\times \mapsto \pi_\theta \in \hat{D}^\times \). The following result is known about this correspondence. (cf. [G],[H]).

**Proposition 1.7.** With the above notations, for any regular quasi-character \( \theta \) of \( L^\times \), \( \pi_\theta \) is an irreducible representation of \( D^\times \) such that:

(a) the representations \( \pi_\theta \) and \( \pi_{\theta'} \) associated two regular quasi-characters \( \theta \) and \( \theta' \) are equivalent if and only if \( \theta \) and \( \theta' \) are conjugate under \( \text{Gal}(L/F) \);
(b) the central quasi-character of \( \pi_\theta \) is the restriction of \( \theta \) to \( F^\times \);
(c) for any quasi-character \( \eta \) of \( F^\times \), the twisted representation of \( \pi_\theta \otimes \eta \) is equivalent to \( \pi_{\theta \otimes \eta \circ N_{L/F}} \);
(d) the contargredient representation of \( \pi_\theta \) is equivalent to \( \pi_{\theta^{-1}} \);
(e) the \( L \)-function of \( \pi_\theta \) is 1;
(f) the \( \epsilon \)-factor of \( \pi_\theta \) is \( \epsilon(\pi_\theta, \psi) = \epsilon(\theta, \psi \circ \text{tr}_{L/F}) \); in particular \( f(\pi_\theta) = l \cdot f(\theta) \);
(g) \( \{ \pi_\theta | \theta \in \hat{L}_{\text{reg}}^\times \} = \{ \pi \in \hat{D}^\times | f(\pi) \equiv 0 \pmod{l} \} \).
2. Character formula. In this subsection we compute the character of $\pi_\theta$. More precisely, for a separable extension $E/F$ of degree $l$ in $D/F$, we give the decomposition of $\pi_\theta$ as $E^\times$ module. First we treat the case $E$ is unramified. We can assume $E = L$ because $E$ is conjugate to $L$ in $D$. Let $U_0 = L^\times, U_i = F^\times(1 + P_L^j)$ ($i \geq 1$), $U_i^* = U_i - U_{i+1}$ and $X_i = \bigoplus \chi$. We set $\Gamma = \text{Gal}(L/F)$ and denote by $\chi_{\pi_\theta}$ the character of $\pi_\theta$.

**Theorem 2.1.** Let $\theta$ be a generic quasi-character of $L^\times$ with $f(\theta) = m + 1$ and $\pi_\theta$ as in (1.2) and (1.5).

1. (Decomposition of $\pi_\theta$ as $L^\times$-module)

$$\pi_\theta|_{L^\times} = (\bigoplus_{\sigma \in \Gamma} \theta \circ \sigma) \otimes \left( X_0 + (q - 1) \frac{q^{\frac{l(l-1)}{2}} - 1}{q^l - 1} \sum_{a=1}^{m} q^{\frac{(l-1)(l-2)(a-1)}{2}} X_a \right).$$

2. (Character formula of $\pi_\theta$ on $L^\times$)

$$\chi_{\pi_\theta}(x) = \begin{cases} q^{\frac{l(l-1)j}{2}} \left( \sum_{\sigma \in \Gamma} \theta(x^\sigma) \right) & \text{if } x \in U_j^* (0 \leq j < m) \\ \frac{l(l-1)m}{2} \left( \sum_{\sigma \in \Gamma} \theta(x^\sigma) \right) & \text{if } x \in U_m. \end{cases}$$

**Corollary 2.2.** Let $\theta$ be a regular quasi-character of $L^\times$ with $\min_{\eta} f(\theta \otimes (\eta \circ N_{L/F})) = m + 1$ and $\pi_\theta$ as in (1.6).

1. (Decomposition of $\pi_\theta$ as $L^\times$-module)

$$\pi_\theta|_{L^\times} = (\bigoplus_{\sigma \in \Gamma} \theta \circ \sigma) \otimes \left( X_0 + (q - 1) \frac{q^{\frac{l(l-1)}{2}} - 1}{q^l - 1} \sum_{a=1}^{m} q^{\frac{(l-1)(l-2)(a-1)}{2}} X_a \right).$$

2. (Character formula of $\pi_\theta$ on $L^\times$)

$$\chi_{\pi_\theta}(x) = \begin{cases} q^{\frac{l(l-1)j}{2}} \left( \sum_{\sigma \in \Gamma} \theta(x^\sigma) \right) & \text{if } x \in U_j^* (0 \leq j < m) \\ q^{\frac{l(l-1)m}{2}} \left( \sum_{\sigma \in \Gamma} \theta(x^\sigma) \right) & \text{if } x \in U_m. \end{cases}$$

**Proof of Corollary 2.2:** This follows immediately from Proposition 1.7 (c) and Theorem 2.1.

We need several steps to prove Theorem 2.1. Let us start with the structure of $D$. By Skolem-Noether theorem, there exists a prime element $\xi \in O_D$ such that

$$\xi^{-1} x \xi = x^\sigma \quad \text{for any } x \in L,$$
where \( \sigma \) is a generator of \( \text{Gal}(L/F) \). We set \( \varpi = \xi^l \). Then it follows that \( \varpi \) is a prime element of \( \mathcal{O}_{F} \) and

\[
\begin{align*}
D &= L \oplus \xi L \oplus \cdots \oplus \xi^{l-1}L \\
\mathcal{O}_D &= \mathcal{O}_L \oplus \xi \mathcal{O}_L \oplus \cdots \oplus \xi^{l-1}\mathcal{O}_L \\
P_D &= P_L \oplus \xi P_L \oplus \cdots \oplus \xi^{l-1}P_L \\
\mathcal{O}_D &= \mathcal{O}_L \oplus \xi \mathcal{O}_L \oplus \cdots \oplus \xi^{l-1}\mathcal{O}_L \\
\mathcal{O}_D &= \mathcal{O}_L \oplus \xi \mathcal{O}_L \oplus \cdots \oplus \xi^{l-1}\mathcal{O}_L \\
P_D^{i-1} &= P_L \oplus \xi P_L \oplus \cdots \oplus \xi^{l-1}P_L \\
\mathcal{O}_D &= \mathcal{O}_L \oplus \xi \mathcal{O}_L \oplus \cdots \oplus \xi^{l-1}\mathcal{O}_L.
\end{align*}
\]

Let \( \theta \) be a generic quasi-character of \( L^\times \) with \( f(\theta) = m + 1 \). If \( f(\theta) = 1 \), then \( \pi_\theta = \text{Ind}_{F^\times \mathcal{O}_D^\times}^{D^\times} \theta \). Since \( \{1, \xi, \xi^2, \cdots, \xi^{l-1}\} \) is a complete system of representatives of \( D^\times/F^\times \mathcal{O}_D^\times \), we get \( \chi_{\pi_\theta} = \sum_{\sigma \in \Gamma} (\theta 0 \sigma) \).

At first, we shall investigate \( L^\times \backslash D^\times /H \). We have only to consider \( L^\times \backslash F^\times \mathcal{O}_D^\times /H \) because

\[
L^\times \backslash D^\times /H = \bigcup_{i=0}^{l-1} \xi^i (L^\times \backslash F^\times \mathcal{O}_D^\times /H) \quad \text{(disjoint union)}.
\]

For convenience, we often use the following notation:

\[
n(i) = \begin{cases} 
\lfloor \frac{m+1}{2} \rfloor & (1 \leq i \leq \frac{l-1}{2}) \\
\lfloor \frac{m}{2} \rfloor & (\frac{l+1}{2} \leq i \leq l-1).
\end{cases}
\]

**Lemma 2.8.** Let \( a = 1 + \sum_{i=1}^{l-1} \xi^i \alpha_i \) and \( b = 1 + \sum_{i=1}^{l-1} \xi^i \beta_i \) (\( \alpha_i, \beta_i \in \mathcal{O}_L \)). Then \( aH = bH \) if and only if \( \alpha_i - \beta_i \in P_L^{n(i)} \) for \( 1 \leq i \leq l-1 \).

**Proof:** By (2.4), \( aH = bH \) implies that there exist \( \gamma_0 \in \mathcal{O}_L^\times \) and \( \gamma_1, \cdots, \gamma_{l-1} \in P_L^{n(i)} \) such that \( b = a (\sum_{i=0}^{l-1} \xi^i \gamma_i) \). Since \( \mathcal{O}_D = \mathcal{O}_L \oplus \xi \mathcal{O}_L \oplus \cdots \oplus \xi^{l-1}\mathcal{O}_L \) and \( \xi^{-1}x\xi = x^\sigma \) for \( x \in L \), we obtain:

\[
1 = \gamma_0 + \varpi \sum_{j=1}^{l-1} \gamma_j \alpha_{i-j}^{\sigma_j} \\
(*) \quad \beta_i - \alpha_i = (\gamma_0 - 1) + \gamma_i + \sum_{j=1}^{i-1} \gamma_j \alpha_{i-j}^{\sigma_j} \\
+ \varpi \sum_{j=i+1}^{l-1} \gamma_j \alpha_{i+j-i-j}^{\sigma_j} \quad (1 \leq i \leq l-1).
\]
Therefore we have $\gamma_0 \in 1 + P_L^{[\frac{m}{2}]^l+1}$ and $\beta_i - \alpha_i \in P_L^{n(i)}$ (1 $\leq$ i $\leq$ l - 1).

Conversely we assume $\beta_i - \alpha_i \in P_L^{n(i)}$ (1 $\leq$ i $\leq$ l - 1). By putting $\gamma_0 - 1 = -\varpi \sum_{j=1}^{l-1} \gamma_j \alpha_{l-j}^\sigma$ into (*), we get

$$\beta_i - \alpha_i = (1 - \varpi \alpha_{l-i}^\sigma) \gamma_i + \sum_{j=1}^{i-1} \gamma_j (\alpha_{i-j}^\sigma - \varpi \alpha_{l-j}^\sigma) + \varpi \sum_{j=i+1}^{l-1} \gamma_j (\alpha_{i+1-j}^\sigma - \alpha_{l-j}^\sigma) \quad (1 \leq i \leq l - 1).$$

Thus it follows that

$$v_L(\gamma_i) \geq \min([\frac{m+1}{2}], v_L(\gamma_1), \cdots, v_L(\gamma_{i-1}), v_L(\gamma_{i+1}) + 1, \cdots, v_L(\gamma_{l-1}) + 1)$$

for $1 \leq i \leq \frac{l-1}{2},$

$$v_L(\gamma_i) \geq \min([\frac{m}{2}], v_L(\gamma_1), \cdots, v_L(\gamma_{i-1}), v_L(\gamma_{i+1}) + 1, \cdots, v_L(\gamma_{l-1}) + 1)$$

for $\frac{l+1}{2} \leq i \leq l - 1.$

Hence our lemma follows from the following simple fact that there is no solution to the system of inequations:

$$x_i \geq \min(x_1, \cdots, x_{i-1}, x_{i+1} + 1, \cdots, x_{l-1} + 1) \quad (1 \leq i \leq l - 1).$$

**Lemma 2.9.** We put

$$M = \{(\alpha^\sigma \alpha^{-1}, \alpha^\sigma^2 \alpha^{-1}, \cdots, \alpha^\sigma^{l-1} \alpha^{-1})|\alpha \in L^\times\} \subset O_L^{(1)} \times \cdots \times O_L^{(1)} = (O_L^{(1)})^{l-1},$$

where $O_L^{(1)} = \text{Ker} N_{L/F}.$ Then the map $(\alpha_i) \in (O_L)^{l-1} \mapsto 1 + \sum_{i=1}^{l-1} \xi^i \alpha_i \in O_D^\times$ induces a bijection from $M \backslash (O_L)^{l-1} / (P_L^{[\frac{m+1}{2}]} \cap \frac{1}{2} \times (P_L^{[\frac{m}{2}]} \cap \frac{1}{2})$ to $L^\times \backslash F^\times O_D^\times / H.$

**Proof:** For $\alpha \in L^\times$ and $\beta_1, \cdots, \beta_{l-1} \in O_L,$

$$\alpha(1 + \sum_{i=1}^{l-1} \xi^i \beta_i)H = (1 + \sum_{i=1}^{l-1} \xi^i \alpha^\sigma \alpha^{-1} \beta_i)H.$$ 

Therefore our lemma is obtained from Lemma 2.8.

In order to prove Theorem 2.1, we need more information about $L^\times \backslash F^\times O_D^\times / H.$ We prepare some notations.
For $1 \leq i \leq l-1$ and $0 \leq \mu < n(i)$, we set

\[ I_{\mu,i} = \begin{cases} 
M(\mathcal{O}_L)^{i-1} \times \mathcal{O}_L^\times \times (\mathcal{O}_L)^{l-i-1} / (P_L^{\frac{m+1}{2}} - \mu - 1)^{i-1} \times (1 + P_L^{\frac{m+1}{2}} - \mu) \times \\
(P_L^{\frac{m+1}{2}} - \mu)^{-i-1} \times (P_L^{\frac{m+1}{2}})^{i-1} & \text{for } 1 \leq i \leq \frac{l-1}{2}, \\
M(\mathcal{O}_L)^{i-1} \times \mathcal{O}_L^\times \times (\mathcal{O}_L)^{l-i-1} / (P_L^{\frac{m+1}{2}} - \mu - 1)^{i-1} \times (1 + P_L^{\frac{m+1}{2}} - \mu) \times \\
(P_L^{\frac{m+1}{2}})^{i-1} & \text{for } \frac{l+1}{2} \leq i \leq l-1, \\
(\mathcal{O}_L/P_L^{\frac{m+1}{2}} - \mu)^{i-1} \times (\mathcal{O}_L^{\times}/1 + P_L^{\frac{m+1}{2}} - \mu) \times \\
(\mathcal{O}_L/P_L^{\frac{m+1}{2}})^{i-1} & \text{for } 1 \leq i \leq \frac{l-1}{2}, \\
(\mathcal{O}_L/P_L^{\frac{m+1}{2}} - \mu)^{i-1} \times (\mathcal{O}_L^{\times}/1 + P_L^{\frac{m+1}{2}} - \mu) \times \\
(\mathcal{O}_L/P_L^{\frac{m+1}{2}})^{i-1} & \text{for } \frac{l+1}{2} \leq i \leq l-1,
\end{cases} \]

and

\[ J_{\mu,i} = \{ 1 + \omega_\mu^{\overline{\alpha}_i} \sum_{j=1}^{i-1} \omega \xi_j^\sigma \beta_j + \sum_{j=i}^{l-1} \xi_j^\beta_j \} \mid (\beta_1, \cdots, \beta_{l-1}) \in I_{\mu,i} \} \].

We define $\varphi_i : (\mathcal{O}_L)^{i-1} \times \mathcal{O}_L^\times \times (\mathcal{O}_L)^{l-i-1} \to (\mathcal{O}_L)^{i-1} \times \mathcal{O}_L^\times \times (\mathcal{O}_L)^{l-i-1}$ as follows:

\[(2.10) \quad \varphi_i(\alpha_1, \cdots, \alpha_{l-1}) = (\beta_1, \cdots, \beta_{l-1}), \quad \beta_j = \alpha_j \alpha_i^{-j} \alpha_i^{-j} \cdots \alpha_i^{j},\]

where $k$ is determined by $0 \leq k < l$ and $-ki \equiv j \pmod{l}$. (In particular $\beta_i = N_{L/F} \alpha_i$).

**Lemma 2.11.** (1) A complete system of representatives of the double coset $L^\times \backslash F^\times \mathcal{O}_F^\times / H$ is given by $\bigcup_{0 \leq \mu < n(i)} K_{\mu,i} \cup \{1\}$.

(2) The map $\varphi_i$ induces a bijection from $I_{\mu,i}$ to $J_{\mu,i}$.

**Proof:** Part one follows immediately from Lemma 2.9. For part two, it suffices to see that $\varphi_i$ induces a bijection from $I_{0,1}$ to $J_{0,1}$. If $\beta_1, \gamma_1 \in \mathcal{O}_L^\times$ and $\beta_2, \cdots, \beta_{l-1}, \gamma_2, \cdots, \gamma_{l-1} \in \mathcal{O}_L$ satisfy $(\gamma_1, \cdots, \gamma_{l-1}) \in M(\beta_1, \cdots, \beta_{l-1})((1 + P_L^{\frac{m+1}{2}})^{l-3} \times (P_L^{\frac{m+1}{2}})^{l-1},$

then there exist $\alpha \in \mathcal{O}_L^\times$ and $y_i \in P_L^{n(i)}$ $(1 \leq i \leq l-1)$ such that

$\gamma_1 = \alpha^\sigma \alpha^{-1} \beta_1(1 + y_1),$

$\gamma_i = \alpha^\sigma \alpha^{-1} \beta_i + y_i$ $(2 \leq i \leq l-1)$.

This implies:

$N_{L/F}(\beta_1) \equiv N_{L/F}(\gamma_1) \mod 1 + P_L^{\frac{m+1}{2}}$ (multiplicative equivalence),

$\gamma_1^{\sigma^{-1}} \cdots \gamma_i^{\sigma^{-i}} \equiv \beta_i \beta_1^{\sigma^{-i}} \cdots \beta_1^i \mod P_L^{n(i)}$ for $2 \leq i \leq l-1$.

Therefore $\varphi_i$ induces a well-defined map from $I_{0,1}$ to $J_{0,1}$. The induced map's bijectivity follows from the bijectivity of the map $\mathcal{O}_L^{(1)} \backslash \mathcal{O}_L^\times / 1 + P_L^{n(i)} \mathcal{O}_F^\times / 1 + P_L^i$.

Next we consider the term $aHa^{-1} \cap L^\times$ in (2.5).
Lemma 2.12. If \( a \in K_{\mu,i} \), then \( aHa^{-1} \cap L^x = F^x(1 + P^{n(i)-\mu}) \).

Proof: Since \( F^x \subset aHa^{-1} \cap L^x \), we have only to see \( aHa^{-1} \cap O_L^x = O_F^x(1 + P^{n(i)-\mu}) \). If \( \alpha \in aHa^{-1} \cap O_L^x \), then there exist \( \gamma_0 \in O_L^x \) and \( \gamma_i \in P^{n(i)-\mu} \) \( (1 \leq i \leq l - 1) \) such that \( \alpha a = a \sum_{i=0}^{l-1} \xi^i \gamma_i \). Put \( a = 1 + \sum_{j=1}^{l-1} \xi^j \beta_j \). Then we have

\[
\gamma_0 = \alpha - \varpi \sum_{j=1}^{l-1} \gamma_j \beta^\sigma_{l-j},
\]

\[
(\alpha^\sigma^{-i} - \gamma_0) \beta_i = \gamma_i + \sum_{j=1}^{i} \beta^\sigma_{l-j} \gamma_j + \varpi \sum_{j=i+1}^{l-1} \beta^\sigma_{l+i-j} \gamma_j. \quad (1 \leq i \leq l - 1).
\]

By replacing \( \gamma_0 \) by \( \alpha - \varpi \sum_{j=1}^{l-1} \gamma_j \beta^\sigma_{l-j} \), we get

\[
(\alpha^\sigma^{-i} - \alpha) \beta_i \in P^{n(i)}_L \quad (1 \leq i \leq l - 1).
\]

Therefore \( \alpha \in O_F^x(1 + P^{n(i)-\mu}) \) and \( aHa^{-1} \cap O_L^x \subset O_F^x(1 + P^{n(i)-\mu}) \). As for \( aHa^{-1} \cap O_L^x \subset O_F^x(1 + P^{n(i)-\mu}) \), we can prove it by the same argument in the proof of Lemma 2.8.

Our next task is to compute \( \rho_\theta^a \) for \( a \in L^x \setminus D^x / H \). The above lemma tells us that \( \rho_\theta^a \in (F^x(1 + P^{n(i)-\mu}))^n \) if \( a \in K_{\mu,i} \). If \( a' = \xi^i a \), then \( a'Ha^{-1} \cap L^x = aHa^{-1} \cap L^x \) and \( \rho_\theta^a = \rho_\theta^{a'} \circ \sigma^j \). Therefore it suffices to consider \( \rho_\theta^a \) for \( a \in L^x \setminus (F^x(1 + P^{n(i)-\mu}) \cap D^x / H \).

Lemma 2.13. Let \( c \in F^x, y \in P^{n(i)-\mu}_L \) and \( a = 1 + \varpi^\mu (\varpi \sum_{j=1}^{i-1} \xi^j \alpha_j + \sum_{j=i}^{l-1} \xi^j \alpha_j) \in K_{\mu,i} \). Then

\[
(\rho_\theta^a \rho_\theta^{-1})(c(1 + y)) = \psi(\text{tr}_{L/F} \varpi^{n+1}(\varpi \sum_{j=1}^{i-1} (\gamma_\theta^\sigma^{-j} f_{l-j}(a) \alpha_j^{\sigma-j} - \gamma_\theta(f_{l-j}(a))^{\sigma-j} \alpha_j) + \sum_{j=i}^{l-1} (\gamma_\theta^\sigma^{-j} f_{l-j}(a) \alpha_j^{\sigma-j} - \gamma_\theta(f_{l-j}(a))^{\sigma-j} \alpha_j) y),
\]

where \( f_j(a) \in L \) is defined by \( a^{-1} = \sum_{i=0}^{l-1} \xi^i f_j(a) \).

Proof: Since \( (\rho_\theta^a \rho_\theta^{-1}) \) is trivial on \( F^x \), we can assume \( c = 1 \). Put \( g = 1 + x \), then

\[
a^{-1} g a^{-1} = (1 + a - 1)^{-1} g (1 + a - 1) g^{-1}
= (1 + a - 1)^{-1} (1 + g(a - 1) g^{-1})
= 1 + a^{-1} (g(a - 1) g^{-1} - (a - 1))
= 1 + a^{-1} \varpi^\mu (\varpi \sum_{j=1}^{i-1} \xi^j \alpha_j (g^{\sigma-j} g^{-1} - 1) + \sum_{j=i}^{l-1} \xi^j \alpha_j (g^{\sigma-j} g^{-1} - 1)).
\]
Since \( \varpi \sum_{j=1}^{i-1} \xi^j \alpha_j + \sum_{j=i}^{l-1} \xi^j \alpha_j \in P_D^{\left[ \frac{ml+2}{2} \right]} \), \( \rho_\theta(1 + x) = \psi(\text{tr}_{D/F} \gamma_\theta x) \) \( x \in P_D^{\left[ \frac{ml+2}{2} \right]} \)
and \( \text{tr}_{D/F} \gamma_\theta \xi^j L = 0 \) \( 1 \leq j \leq l - 1 \),
\[
(p_\theta^a p_\theta^{-1})(g) = \rho_\theta(a^{-1} g a^{-1})
= \psi(\text{tr}_{D/F} \gamma_\theta a^{-1} \varpi^\mu \sum_{j=1}^{i-1} \xi^j \alpha_j (g^\sigma g^{-1} - 1) + \sum_{j=i}^{l-1} \xi^j \alpha_j (g^\sigma g^{-1} - 1)))
\]
\[
= \psi(\text{tr}_{L/F} \gamma_\theta a^{-1} \varpi^\mu+1 \sum_{j=1}^{i-1} (f_{l-j}(a))^\sigma \alpha_j (g^\sigma g^{-1} - 1) + \sum_{j=i}^{l-1} (f_{l-j}(a))^\sigma \alpha_j (g^\sigma g^{-1} - 1)))
\]

In the last term of the above equations, \( \gamma_\theta \in P_{L^{-}} \), \( f_{i-j}(a) \in P_{L}^{\mu} \) and \( g^\sigma g^{-1} - 1 \equiv y^\sigma - y \mod P_{L}^{2(\alpha(i) - \mu)} \). Therefore
\[
(p_\theta^a p_\theta^{-1})(g) = \psi(\text{tr}_{L/F} \gamma_\theta a^{-1} \varpi^\mu+1 \sum_{j=1}^{i-1} (f_{l-j}(a))^\sigma \alpha_j (g^\sigma g^{-1} - 1) + \sum_{j=i}^{l-1} (f_{l-j}(a))^\sigma \alpha_j (g^\sigma g^{-1} - 1)))
\]

(We note \( \psi \) is trivial on \( P_{L} \)). Hence our lemma follows from the following property:
\[
\text{tr}_{L/F} uv^\sigma = \text{tr}_{L/F} u^\sigma v \quad \text{for any} \quad u, v \in L.
\]

We prepare the next lemma for the purpose of writing \( f_k(a) \) by \( (\alpha_j)_{1 \leq j \leq l-1} \).

**Lemma 2.14.** For \( a = \sum_{j=0}^{l-1} \xi^j \alpha_j \quad (\alpha_j \in L) \), put
\[
\Lambda(a) = \left( \varpi^{i+\frac{j-i}{l-i}} \alpha_{i-j \text{mod} l} \right)_{0 \leq i, j \leq l-1}
= \begin{pmatrix}
\alpha_0 & \varpi \alpha_1 & \cdots & \varpi \alpha_{l-1} \\
\alpha_1 & \alpha_0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_{l-1} & \cdots & \alpha_1 & \alpha_0 \\
\end{pmatrix}
\in M_l(L),
\]

and
\[ \Lambda_k(a) = (-1)^k \begin{vmatrix} \alpha_1 & \cdots & \omega \alpha_{l-k+1}^{\sigma^{k-1}} & \cdots & \omega \alpha_{l}^{\sigma^{1}} \\ \vdots & \cdots & \alpha_k & \cdots & \omega \alpha_{l-k}^{\sigma^{k+1}} \\ \vdots & \cdots & \alpha_{l-k+2}^{\sigma^{k-1}} & \cdots & \alpha_{0}^{\sigma^{l-1}} \end{vmatrix} \in L^\times \]

i.e. \( \Lambda_k(a) \) is the \((1,k+1)\)-cofactor of \( \Lambda(a) \). Then

\[ a^{-1} = \sum_{j=0}^{l-1} \xi^j \frac{\Lambda_j(a)}{|\Lambda(a)|}, \]

where \(|\Lambda(a)|\) is the determinant of \( \Lambda(a) \).

**Proof:** By the map \( \Lambda : D \to M_l(L) \), we can embed \( D \) into \( M_l(L) \). Then our lemma follows from the basic matrix theory.

We define \( L \)-valued functions \( R_{\mu,i} \) on \( \mathcal{O}_L^{i-1} \times \mathcal{O}_L^\times \times \mathcal{O}_L^{l-i-1} \) by:

\[
R_{\mu,i}(\beta_1, \cdots, \beta_{l-1}) = \sum_{j=1}^{i-1} (\gamma_\theta^j f_{l-j}(a) \alpha_j^{\sigma^j} - \gamma_\theta(f_{l-j}(a))^{\sigma^j} \alpha_j) + \sum_{j=i}^{l-1} (\gamma_\theta^j f_{l-j}(a) \alpha_j^{\sigma^j} - \gamma_\theta(f_{l-j}(a))^{\sigma^j} \alpha_j),
\]

where \( \varphi_i(\alpha_1, \cdots, \alpha_{l-1}) = (\beta_1, \cdots, \beta_{l-1}) \) and \( a = 1 + \omega^\mu (\omega \sum_{j=1}^{i-1} \xi^j \alpha_k + \sum_{j=i}^{l-1} \xi^j \alpha_k) \).

(As for the definition of \( \varphi_i \) and \( f_j(a) \), see 2.10 and Lemma 2.12 respectively.) It is easily seen that \( R_{\mu,i} \) is well-defined. In fact, we can show by virtue of Lemma 2.14 that \( R_{\mu,i}(\beta_1, \cdots, \beta_{l-1}) \) is a rational function of \( \{\beta_j^{\sigma^k}\}_{1 \leq j, k \leq l-1} \). We fix \( \beta_j(1 \leq j \leq l-i) \) for all \( j \) but \( l-i \) and define a function \( \tilde{R}_{\mu,i} \) on \( \mathcal{O}_L \) by:

\[
\tilde{R}_{\mu,i}(x) = R_{\mu,i}(\beta_1, \cdots, \beta_{l-i-1}, x, \beta_{l-i+1}, \cdots, \beta_{l-1}).
\]

The next lemma is the key point in this proof of Theorem 2.1.

**Lemma 2.15.** Let \( L^{(0)} = \{x \in L \mid \text{tr}_{L/F} x = 0\} \). Then \( \tilde{R}_{\mu,i} \) has the following property:

1. \( \tilde{R}_{\mu,i} \) induces a surjection from \( \mathcal{O}_L/P_{L}^{[m]} - \mu \) to \( P_{L}^{[m+1]-\mu} \cap L^{(0)} \), and each fiber of the induced map has \( q^{\frac{m+1}{2}} - \mu \) elements if \( 1 \leq i \leq \frac{l-1}{2} \).
2. \( \tilde{R}_{\mu,i} \) induces a surjection from \( \mathcal{O}_L/P_{L}^{[m+1]} - \mu - 1 \) to \( P_{L}^{[m+2]-\mu} \cap L^{(0)} \), and each fiber of the induced map has \( q^{\frac{m+3}{2}} - \mu - 1 \) elements if \( \frac{l+1}{2} \leq i \leq l-1 \).

**Proof:** We assume \( 1 \leq i \leq \frac{l-1}{2} \). By virtue of Lemma 2.14 and Lemma 2.15, we can show

\[
\tilde{R}_{\mu,i}(x) \equiv ax - (ax)^{\sigma^i} + b \mod P_{L}^{[m+2]}(x) + 2\mu + 1 - m,
\]
where \( a = \omega^{2\mu+1}(\gamma_\theta^{\sigma^i} - \gamma_\theta) \in P_L^{2\mu+1-m} - P_L^{2\mu+2-m} \) and \( b \) is a constant in \( P_L^{2\mu+1-m} \). Therefore we can get our lemma by induction on \([m/2] - \mu\) since \( \tilde{R}_{\mu,i}(x) \mod P_L^{\mu+1-[m+1/2]} \)

is a polynomial of \( \{x, x^\sigma, \cdots, x^\sigma^{l-1}\} \) whose coefficients belong to \( P_L^{2\mu+1-m} \). The case \( \frac{t-1}{2} \leq i \leq l - 1 \) is proved by the same way.

Summing up the above lemmas, we have the following result.

**Lemma 2.16.** (1) If \( 1 \leq i \leq \frac{t-1}{2} \),

\[
K_{\mu,i} \rightarrow (F^\times(1 + P_L^{[m+1/2]-\mu}))^\sim \\
\rho_\theta^a \rho_\theta^{-i}
\]

is a surjection to \( (F^\times(1 + P_L^{[m+1/2]-\mu})/F^\times(1 + P_L^{m-2\mu}))^\sim \) and each fiber of the map has

\[
(q-1)q^{(l-1)(l-2)(m-2\mu)/(2}-l(i-1)-1 \text{ elements.}
\]

(2) If \( \frac{t+1}{2} \leq i \leq l - 1 \),

\[
K_{\mu,i} \rightarrow (F^\times(1 + P_L^{[m/2]-\mu}))^\sim \\
\rho_\theta^a \rho_\theta^{-i}
\]

is a surjection to \( (F^\times(1 + P_L^{[m/2]-\mu})/F^\times(1 + P_L^{m-2\mu-1}))^\sim \) and each fiber of the map has

\[
(q-1)q^{(l-1)(l-2)(m-2\mu-1)/(2}-l(i-\frac{t+1}{2})-1 \text{ elements.}
\]

**Proof:** Let \( 1 \leq s < t \leq 2t, b \in P_L^s \cap L^{(0)}, c \in F^\times \) and \( y \in P_L^{1-t} \). Then the map \( b \mapsto \hat{b} = (c(1+y) \mapsto \psi(tr_{L/F}(by))) \) induces an isomorphism between \( P_L^s \cap L^{(0)} / P_L^t \cap L^{(0)} \)

and \( (F^\times(1 + P_L^{1-t})/F^\times(1 + P_L^{1-s}))^\sim \) since the conductor of \( \psi \) is \( P_L \) and \( L/F \) is unramified. Hence our lemma holds by virtue of Lemma 2.15 and 2.12.

**Proof of Theorem 2.1:** By Lemma 2.16,

\[
\bigoplus_{a \in K_{\mu,i}} \text{Ind}_{aH_{a-1}\cap L^\times}^L \rho_\theta^a = \theta \otimes \left\{ 
\begin{array}{ll}
(q-1)q^{(l-1)(l-2)(m-2\mu)/(2}-l(i-1)-1 X_{m-2\mu} & \text{if } 1 \leq i \leq \frac{t-1}{2}, \\
(q-1)q^{(l-1)(l-2)(m-2\mu-1)/(2}-l(i-\frac{t+1}{2})-1 X_{m-2\mu-1} & \text{if } \frac{t+1}{2} \leq i \leq t - 1,
\end{array}
\right.
\]

where \( X_j = \bigoplus_{\chi \in (L^\times/F^\times(1 + P_L^j))^\sim} \chi \). Thus by Lemma 2.11 and (2.5), we have:

\[
\pi_\theta|_{L^\times} = \bigoplus_{\sigma \in \Gamma} (\theta \circ \sigma) \otimes \left( X_0 + (q-1)q^{l(l-1)/2} q^l - 1 \sum_{a=1}^m q^{(l-1)(l-2)(a-1)/2} X_a \right).
\]

The rest of Theorem 2.1 follows immediately from the above formula.

Next we consider the case \( E \not\simeq L \). Then \( E \) is a totally ramified extension of \( F \) of degree \( l \). This case is very easy.
THEOREM 2.17. Let $\theta$ be a regular quasi-character of $L^\times$ with $m = \min f(\theta \otimes (\gamma o N_{L/F})) = m+1$ and $\pi_\theta$ as in (1.6).

1. (Decomposition of $\pi_\theta$ as $E^\times$-module)

$$\pi_\theta|_{E^\times} = \theta \otimes q^{\frac{(l-1)(l-2)m}{2}} \bigoplus_{\chi \in (E^\times/F^\times(1 + P_{E}^{lm+1}))^\wedge} \chi$$

2. (Character formula of $\pi_\theta$ on $E^\times$)

$$\chi_{\pi_\theta}(x) = \begin{cases} 
0 & \text{if } x \not\in F^\times(1 + P_{E}^{lm+1}) \\
\theta(c)q^{\frac{(l-1)m}{2}} & \text{if } x = c(1+y) \in F^\times(1 + P_{E}^{lm+1}). 
\end{cases}$$

PROOF: It suffices to say that $\chi_{\pi_\theta}(x) = 0$ if $\lfloor \frac{lm+2}{2} \rfloor \leq v_{E}(x-1) < lm$. (We note that $F^\times(1 + P_{E}^{lm}) = F^\times(1 + P_{E}^{lm+1})$). Set $r = v_{E}(x - 1)$. From the definition of $\pi_\theta$,

$$\chi_{\pi_\theta}(x) = \sum_{g \in D^\times/H} \rho_\theta(g^{-1}xg)$$

$$= \frac{1}{q^{l(lm+1-r-\lfloor \frac{lm+1-r}{2} \rfloor)}} \sum_{g \in D^\times/H} \sum_{k \in P_{D}^{[\frac{lm+1-r}{2}]}/P_{D}^{lm+1-r}} \rho_\theta((1+k)^{-1}g^{-1}xg(1+k)).$$

Set $g^{-1}xg = 1+h$. By virtue of $(1+k)^{-1}(1+h)(1+k) \equiv 1 + hk - kh \mod P_{D}^{lm+1}$, $\rho_\theta((1+\gamma)^{-1}(1+h)(1+k)) = \psi(tr_{D/F}(\gamma h - h \gamma))$. Since $h \in P_{D}$ and $h \not\in P_{L} + P_{D}^{r+1}$, the map $k \mapsto \psi(tr_{D/F}(\gamma h - h \gamma))$ is a non-trivial character of $P_{D}^{[\frac{lm+1-r}{2}]}/P_{D}^{lm+1-r}$. (cf. 6.7 [Ca]). Therefore $\chi_{\pi_\theta}(x) = 0$.

REFERENCES


