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On Trinity of Parabolic Subgroups

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§0 Introduction

Let $f(Z)$ be a Siegel modular form of degree n and weight k with the Fourier expansion

$$f(Z)=\sum_{T\geq 0} a(T)\exp(2\pi\sqrt{-1}\mathrm{tr}(TZ)).$$

Maass [**Ma**] considered a Dirichlet series

$$D(s,f)=\sum_{0<T(\mathrm{mod} GL_n(\mathbb{Z}))}\frac{a(T)}{\varepsilon(T)|\det T|^s} \quad (\varepsilon(T)=|GL_n(\mathbb{Z})\cap O(T)|)$$

and showed that it has a meromorphic continuation to the whole s-plane and has a functional equation with respect to $s\mapsto k-s$. Let us consider the Dirichlet series of Maass associated with a theta series

$$\theta_S(Z)=\sum_{G\in M_{m,n}(\mathbb{Z})}\exp(\pi\sqrt{-1}\mathrm{tr}({}^tGSGZ))$$

where $S\in SL_m(\mathbb{Z})$ is symmetric positive definite even diagonals. Then $\theta_S(Z)$ is a Siegel modular form of degree n weight $m/2$, and

$$D(s,\theta_S)=\sum_{\substack{G\in M_{m,n}(\mathbb{Z})/GL_n(\mathbb{Z})\\ \mathrm{rank} G=n}}\frac{2^{ns}}{|\det({}^tGSG)|^s}$$

if $m>n$. The right hand side is called a Koecher's zeta function associated with S. It is a zeta function associated with a prehomogeneous vector space

$$(O(S)\times GL_n,\rho,M_{m,n}(\mathbb{C})) \quad (\rho(h,g)x=hxg^{-1}).$$

On the other hand, the theta series θ_S is the simplest case of theta lifting of automorphic form arising from a reductive dual pair $(Sp(n,\mathbb{R}),O(S,\mathbb{R}))$. One purpose of this note is to generalize this relation between the prehomogeneous vector spaces and the reductive dual pairs or theta lifting of automorphic forms (§4).

A Jacobi form is an automorphic form on a Jacobi group $SL_2(\mathbb{R}) \ltimes H_{\mathbb{R}}$ (c.f. [**EZ**]). The Heisenberg group $H_{\mathbb{R}}$ is isomorphic to

$$N = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1_2 & J_0 \cdot {}^t x \\ 0 & 0 & 1 \end{pmatrix} \in Sp(J, \mathbb{R}) \right\} \quad (J = \begin{pmatrix} & & 1 \\ & J_0 & \\ -1 & & \end{pmatrix}, \; J_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix})$$

which is the unipotent radical of a parabolic subgroup

$$P = \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \in Sp(J, \mathbb{R}) \right\}$$

of $Sp(J, \mathbb{R})$. On the other hand, $SL_2(\mathbb{R})$ is isomorphic to a subgroup

$$G_1 = \left\{ \begin{pmatrix} 1 & & \\ & g & \\ & & 1 \end{pmatrix} \in Sp(J, \mathbb{R}) \mid g \in SL_2(\mathbb{R}) \right\}$$

of the Levi part of P. The action of $SL_2(\mathbb{R})$ on $H_{\mathbb{R}}$ is compatible with the action of G_1 on N by conjugation (N is normal subgroup of P). The space of Jacobi forms is described by the "holomorphic discrete series" of Jacobi group which is a tensor product of a holomorphic discrete series of the 2-fold covering group of $SL_2(\mathbb{R})$ and a Weil representation [**Ta1**,Chap.3]. The other purpose of this note is to generalize these relations among Jacobi forms, parabolic subgroups and Weil representations (§5).

Any way, reductive dual pairs, prehomogeneous vector spaces (or zeta functions associated with them) and Jacobi groups (or Jacobi forms) arise naturally from parabolic subgroups and they are connected by Weil representations. I'd like to call this scheme a trinity of parabolic subgroups.

§1 and §2 are brief reviews of automorphic forms on locally compact groups and theta lifting of automorphic forms associated with reductive dual apirs respectively. In §3 we will show how parabolic subgroups produce all the reductive dual pairs (Theorem 3.1).

§1 Automorphic forms

Let G be a locally compact unimodular group, K a compact subgroup of G, Γ a closed unimodular subgroup of G and A a closed subgroup of $\Gamma \cap Z(G)$ where $Z(G)$ is the center of G. Let χ be a continuous unitary character of Γ. Let (π, H_π) (resp. (δ, V_δ)) be an irreducible unitary representation of G (resp. K) satisfying the conditions

1) the multiplicity of δ in $\pi|_K$ is equal to one,

2) $\pi(a) = \chi(a)$ for all $a \in A$.

Let $H_\pi(\delta)$ be the δ-isotypic component of $(\pi|_K, H_\pi)$, that is, $H_\pi(\delta) = \{u \in H_\pi \mid \pi(e_\delta)u = u\}$ where $e_\delta = (\dim\delta)\overline{\chi_\delta}$ with $\chi_\delta(k) = \mathrm{tr}\delta(k)$. The spherical function $\Psi_{\pi,\delta}$ of π with K-type δ is defined by $\Psi_{\pi,\delta}(x) = \pi(e_\delta) \circ \pi(x) \circ \pi(e_\delta) \in \mathrm{End}_{\mathbb{C}}(H_\pi(\delta))$ $(x \in G)$. The function $\psi_{\pi,\delta}(x) = \mathrm{tr}\Psi_{\pi,\delta}(x)$ $(x \in G)$ is called the spherical trace function of π with K-type δ.

Let $C_c(G/A, \chi, \delta)^0$ be a $\mathbb{C}$-vector space consisting of the $\mathbb{C}$-valued continuous functions φ on G such that

1) $\varphi(ax) = \chi(a)^{-1}\varphi(x)$ for all $a \in A$,

2) $\mathrm{supp}(\varphi)$ is compact modulo A,

3) $\varphi(kxk^{-1}) = \varphi(x)$ for all $k \in K$,

4) $\int_K e_\delta(k)\varphi(k^{-1}x)dk = \varphi(x)$.

$C_c(G/A, \chi, \delta)^0$ is an involutive $\mathbb{C}$-algebra with respect to the convolution product $\varphi * \psi(x) = \int_{G/A} \varphi(xy^{-1})\psi(y)dy$ and the involution $\varphi^*(x) = \overline{\varphi(x^{-1})}$. Put

$$\widehat{\psi}_{\pi,\delta}(\varphi) = (\dim\delta)^{-1} \int_{G/A} \varphi(x)\psi_{\pi,\delta}(x)dx$$

for $\varphi \in C_c(G/A, \chi, \delta)^0$. Then $\widehat{\psi}_{\pi,\delta} : C_c(G/A, \chi, \delta)^0 \to \mathbb{C}$ is a surjective involutive $\mathbb{C}$-algebra homomorphism.

Now we will define a space of automorphic forms og G (c.f. [**Ta**, Definition 5.1])

Definition 1.1. *We denote by $\check{\mathcal{A}}_\delta(\Gamma\backslash G, \chi, \pi)$ the complex vector space consisting of the locally integrable functions $f : G \to V_\delta$ such that*

1) $f(\gamma x) = \chi(\gamma^{-1})f(x)$ *for all* $\gamma \in \Gamma$,

2) $\int_{\Gamma\backslash G} |f(x)|^2 d(\dot{x}) < \infty$

3) $f(xk) = \delta(k^{-1})f(x)$ *for all* $k \in K$,

4) $\int_{G/A} f(xy^{-1})\varphi(y)d(y) = \widehat{\psi}_{\pi,\delta}(\varphi)f(x)$ *for all* $\varphi \in C_c(G/A, \chi, \delta)^0$,

endowed with an inner product $(f, g) = \int_{\Gamma\backslash G}(f(x), g(x))d(\dot{x})$ *with which* $\check{\mathcal{A}}_\delta(\Gamma\backslash G, \rho, \pi)$ *is a complex Hilbert space.*

Let $\check{\pi}$ (resp. $\check{\delta}$) be the contagredient representation of π (resp. δ). We will denote by $\mathcal{A}_{\check{\delta}}(\chi^{-1}, \check{\pi})$ the $\check{\delta}$-isotypic component of $\check{\pi}$-isotypic component of the induced representation $\mathrm{Ind}_\Gamma^G \chi^{-1}$. We have a $\mathbb{C}$-linear isometry

$$\check{\mathcal{A}}_\delta(\Gamma\backslash G, \chi, \pi) \otimes V_\delta^* \xrightarrow{\sim} \mathcal{A}_{\check{\delta}}(\chi^{-1}, \check{\pi})$$

defined by $f \otimes \alpha \mapsto (\dim \delta)^{1/2}\langle f, \alpha\rangle$. Here V_δ^* is the dual space of V_δ. Inparticular the dimensin of $\check{\mathcal{A}}_\delta(\Gamma\backslash G, \chi, \pi)$ is equal to the multiplicity of π in the induced representation $\mathrm{Ind}_\Gamma^G \chi$.

For the latter use, we will relax the conditions of Definotion 1.1 and define another space of automorphic forms on G.

Definition 1.2. *We denote by* $\check{\mathcal{M}}_\delta(\Gamma\backslash G, \chi, \pi)$ *the complex vector space consisting of the continuous functions* $f : G \to V_\delta$ *such that*

1) $f(\gamma x) = \chi(\gamma^{-1})f(x)$ *for all* $\gamma \in \Gamma$,

2) $f(xk) = \delta(k^{-1})f(x)$ *for all* $k \in K$,

4) $\int_{G/A} f(xy^{-1})\varphi(y)d_G(\dot{y}) = \widehat{\psi}_{\pi,\delta}(\varphi)f(x)$ *for all* $\varphi \in C_c(G/A, \chi, \delta)^0$.

It is easy to prove that $\check{\mathcal{A}}_\delta(\Gamma\backslash G, \chi, \pi)$ is a subspace of $\check{\mathcal{M}}_\delta(\Gamma\backslash G, \chi, \pi)$.

If $A = \{1\}$, the involutive $\mathbb{C}$-algebra $C_c(G/A, \chi, \delta)^0$ is denote by $C_c(G, \delta)^0$.

Example 1.3. Put $G = Sp(n, \mathbb{R})$, $\Gamma = Sp(n, \mathbb{Z})$ and $K = \{g \in G \mid g \cdot^t g = 1\}$ which is identified with the unitary group $U(n)$. Put also $A = \{1\}$. Let π be the holomorphic discrete series of $Sp(n, \mathbb{R})$ of minimal K-type $\delta = \det^k \in \widehat{K} = \widehat{U}(n)$. If $k > 2n$, that is, π is integrable, then $\check{\mathcal{A}}_\delta(\Gamma\backslash G, \mathbf{1}_\Gamma, \pi)$ is the space of the Siegel cusp forms of degree n and weight k (c.f. [**Ta1**,Chap.2]).

Example 1.4. Put $V = M_{m,2n}(\mathbb{R})$. Let $S_m(\mathbb{R})$ be the real symmetric matrices of size m. Define a $\mathbb{R}$-bilinear form $D : V \times V \to S_m(\mathbb{R})$ by $D(x,y) = xJ \cdot {}^t y - yJ \cdot {}^t x$ with $J = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}$. Define a group law on $\mathcal{H}(V) = V \times S_m(\mathbb{R})$ by $(x,t)\cdot(y,u) = (x+y, t+u+\frac{1}{2}D(x,y))$. $Sp(n,\mathbb{R})$ acts on $\mathcal{H}(V)$ as an automorphism group by $(x,t)\cdot\sigma = (x\sigma, t)$ and we have a semi-direct product $G = Sp(n,\mathbb{R}) \ltimes (V)$ which is a locally compact unimodular group. Put $\Gamma = Sp(n,\mathbb{Z}) \ltimes (M_{m,2n}(\mathbb{Z}) \times S_m(\mathbb{R}))$ which is a closed unimodular subgroup of G. Let $A = S_m(\mathbb{R}) = Z(G)$ be the center of G. Let K be the standard maximal compact subgroup of $Sp(n,\mathbb{R})$ which is considered as a compact subgroup of G. Take a positive integral symmetric matrix $0 < S \in S_m(\mathbb{Z})$ and define a unitary character χ_S of Γ by $\chi_S(\gamma, x, t) = \exp 2\pi\sqrt{-1}\mathrm{tr}(St)$. Let $\pi^{\ell,S}$ be the "holomorphic discrete series of G defined in [**Ta1**,§9] $(n + m/2 < \ell \in \mathbb{Z})$. Put $\delta = \det^\ell \in \widehat{K} = \widehat{U}(n)$. If $\ell > 2n + m$, that is $\pi^{\ell,S}$ is integrable modula A, then $\check{\mathcal{A}}_\delta(\Gamma\backslash G, \chi_S, \pi^{\ell,S})$ is the space of the cuspidal Jacobi forms of weight ℓ and index S (c.f. [**Ta1**,Chap.3]).

§2 Theta lifting of automorphic forms

Let V be a symplectic $\mathbb{R}$-space with symplectic $\mathbb{R}$-form D, and $Sp(V)$ the symplectic group of (V,D). The group $Sp(V)$ acts on V from right. Let W and W' be Lagrangian subspaces of V such that $V = W \oplus W'$. Any $\sigma \in Sp(V)$ is expressed by $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a \in \mathrm{End}_{\mathbb{R}}(W)$, $b \in \mathrm{Hom}_{\mathbb{R}}(W,W')$, $c \in \mathrm{Hom}_{\mathbb{R}}(W',W)$ and $d \in \mathrm{End}_{\mathbb{R}}(W')$. Fix a non-trivial unitary character $\chi(x) = \exp 2\pi\sqrt{-1}x$ of $\mathbb{R}$. There exists a non-trivial two-fold covering group $p : \widetilde{Sp}(V) \to Sp(V)$, and we have a unitary representation $(\omega_\chi, L^2(W))$ of $\widetilde{Sp}(V)$ called Weil representation.

Let $\mathcal{L} \subset W$ be a $\mathbb{Z}$-lattice. Put $\mathcal{L}' = \{Y \in W' \mid D(\mathcal{L}, Y) \subset \mathbb{Z}\}$ which is a $\mathbb{Z}$-lattice in W'. Let Γ be an arithmetic subgroup of $Sp(V)$ consisting of $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp(V)$ such that

1) $(\mathcal{L} \times \mathcal{L}')\sigma = \mathcal{L} \times \mathcal{L}'$,

2) $D(Xa, Xb) \in 2\mathbb{Z}$ for all $X \in \mathcal{L}$, and

3) $D(Yc, Yd) \in 2\mathbb{Z}$ for all $Y \in \mathcal{L}'$.

Put $\widetilde{\Gamma} = p^{-1}(\Gamma)$ which is a discrete subgroup of $\widetilde{Sp}(V)$ such that $\mathrm{vol}(\widetilde{\Gamma}\backslash\widetilde{Sp}(V)) < \infty$.

A theta series is defined by

$$\theta_\varphi(g) = \sum_{\ell \in \mathcal{L}} (\omega_\chi(g)\varphi)(\ell) \quad (g \in \widetilde{Sp}(V))$$

for a Schwartz function $\varphi \in \mathcal{S}(W)$ on W. There exists a unitary character ρ of $\widetilde{\Gamma}$, independent of φ, such that

$$\theta_\varphi(\gamma g) = \rho(\gamma)\theta_\varphi(g) \quad \text{for all } \gamma \in \widetilde{\Gamma}.$$

See [**We**] for the details.

Let (G_1, G_2) be a reductive dual pair in $Sp(V)$ such that G_1 is compact. Put $\widetilde{G}_j = p^{-1}(G_j)$. Then the elements of $\widetilde{G}_1$ and $\widetilde{G}_2$ are mutually commutative and we have a continuous group homomorphism $\imath(g,h) = gh$ from $\widetilde{G}_1 \times \widetilde{G}_2$ to $\widetilde{Sp}(V)$. Let $\mathcal{A}_j$ be the von Neumann algebra generated by $\omega_\chi(\widetilde{G}_j)$. The commutant of $\mathcal{A}_1$ is equal to $\mathcal{A}_2$ ([**Ho**,Th.6.1]). It means that

1) $(\omega_\chi \circ \imath, L^2(W))$ is multiplicity-free, and

2) for any $\sigma \in \widehat{\widetilde{G}}_1$, there exists at most one $\tau \in \widehat{\widetilde{G}}_2$ such that $\sigma \otimes \tau$ is a subrepresentation of $\omega_\chi \circ \imath$.

Because $\widetilde{G}_1$ is compact, the restriction $\omega_\chi|_{\widetilde{G}_1 \times \widetilde{G}_2} = \omega_\chi \circ \imath$ decomposes discretely. So put

$$\omega_\chi|_{\widetilde{G}_1 \times \widetilde{G}_2} = \bigoplus_{\lambda \in \Lambda} (\pi_\lambda \otimes \pi'_\lambda)$$

and

$$L^2(W) = \overline{\bigoplus_{\lambda \in \Lambda} H_\lambda}$$

with a unitary intertwining operator $U_\lambda : H_{\pi_\lambda} \otimes H_{\pi'_\lambda} \xrightarrow{\sim} H_\lambda$.

Let K_j be a compact subgroup of G_j, and put $\widetilde{K}_j = p^{-1}(K_j)$. Take a $\pi \in \widehat{\widetilde{G}}_1$ and $\delta \in \widehat{\widetilde{K}}_1$ (resp. $\pi' \in \widehat{\widetilde{G}}_2$ and $\delta' \in \widehat{\widetilde{K}}_2$) such that the multiplicity of δ in $\pi|_{\widetilde{K}_1}$ (resp. δ'

in $\pi'|_{\widetilde{K}_2}$) is equal to one. Assume that $\pi \otimes \pi'$ is a subrepresentation of $\omega_\chi|_{\widetilde{G}_1 \times \widetilde{G}_2}$. Let $U : H_\pi \otimes H_{\pi'} \to L^2(W)$ be a unitary intertwining operator. Put $\widetilde{\Gamma}_j = \widetilde{\Gamma} \cap \widetilde{G}_j$ and $\rho_j = \rho|_{\widetilde{\Gamma}_j}$.

Take a $f \in \mathcal{A}_{\check{\delta}}(\rho_1^{-1}, \check{\pi})$ and a Schwartz function $\varphi \in \mathcal{S}(W)$. Here $\check{\pi}$ (resp. $\check{\delta}$) denotes the contragredient representation of π (resp. δ). Put

$$F_{f,\varphi}(h) = \int_{\widetilde{\Gamma}_1 \backslash \widetilde{G}_1} f(g)\theta_\varphi(gh)d(\dot{g}) \quad (h \in \widetilde{G}_2).$$

which has the following properties;

1) $F_{f,\varphi}(\gamma h) = \rho_2(\gamma)F_{f,\varphi}(h)$ for all $\gamma \in \widetilde{\Gamma}_2$,

2) $F_{f,\omega_\chi(\psi)\varphi} = \widehat{\psi}_{\pi,\delta}(\psi)F_{f,\varphi}$ for all $\psi \in C_c(\widetilde{G}_1, \delta)^0$,

3) $\int_{\widetilde{G}_2} F_{f,\varphi}(hy)\psi(y)dy = F_{f,\omega_\chi(\psi)\varphi}(h)$ for all $\psi \in C_c(\widetilde{G}_2)$.

Then we have

Proposition 2.1.

1) If $F_{f,\varphi} \neq 0$, then $\varphi \in U(H_\pi(\delta) \otimes H_{\pi'})$,

2) If $\varphi = U(u \otimes v)$ for $u \in H_\pi(\delta)$ and $v \in H_{\pi'}$, then

$$\int_{\widetilde{G}_2} F_{f,\varphi}(hy)\psi(y)dy = \begin{cases} \widehat{\psi}_{\pi',\delta'}(\psi)F_{f,\varphi}(h), & \text{if } v \in H_{\pi'}(\delta') \\ 0, & \text{if} \notin H_{\pi'}(\delta') \end{cases}$$

for all $\psi \in C_c(\widetilde{G}_2, \delta')^0$.

Now we will define a theta lifting of automorphic forms. Suppose that $U(H_\pi(\delta) \otimes H_{\pi'}(\delta')) \subset \mathcal{S}(W)$. Take an ortho-normal $\mathbb{C}$-base $\{u_1, \cdots, u_d\}$ of $H_\pi(\delta)$. For any $v \in H_{\pi'}(\delta')$, put

$$\Theta_v(g) = \sum_{j=1}^{d} \overline{\theta_{U(u_j \otimes v)}(g)} v \in H_\pi(\delta) \quad (g \in \widetilde{Sp}(V))$$

which is independent of the choice of the orthonormal basis $\{u_1, \cdots, u_d\}$ of $H_\pi(\delta)$. Then we have the following theorem which descrive the theta lifting of automorphic forms associated with the reductive dual pair (G_1, G_2);

Theorem 2.2. *Take an automorphic form* $f \in \check{\mathcal{A}}_\delta(\widetilde{\Gamma}_1\backslash\widetilde{G}_1, \rho_1, \pi)$. *Define a function* $F_f : \widetilde{G}_2 \to H_{\pi'}(\delta')^*$ *by*

$$\langle F_f(h), v\rangle = \int_{\widetilde{\Gamma}_1\backslash\widetilde{G}_1} (f(g), \Theta_v(gh))d\dot{g} \quad (h \in \widetilde{G}_2,\ v \in H_{\pi'}(\delta')).$$

Then $F_f \in \check{\mathcal{M}}_{\check{\delta}}(\widetilde{\Gamma}_2\backslash\widetilde{G}_2, \rho_2^{-1}, \check{\pi}')$.

We have

$$\langle F_f(h), v\rangle = \sum_{j=1}^{d} F_{f_j, U(u_j\otimes v)}(h)$$

where $f_j(g) = (f(g), u_j)$ with an ortho-normal $\mathbb{C}$-base $\{u_1, \cdots, u_d\}$ of V_δ.

§3 Parabolic subgroups and reductive dual pairs

Let A be a simple $\mathbb{R}$-algebra with a $\mathbb{R}$-involution ι, and put

$$G = \{\alpha \in \mathrm{Aut}_{\mathbb{R}}(A) \mid \alpha\circ\iota = \iota\circ\alpha,\ \alpha|_{Z(A)} = id\}.$$

Then G is a semi-simple simple classical real Lie group of adjoint type. Let P be a parabolic subgroup of G such that its unipotent radical N is 2-step-nilpotent. Let $P = L \ltimes N$ be the Levi decomposition of P. Then the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ has a canonical decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{l} \oplus \mathfrak{n}$ where $\mathfrak{l} = \mathrm{Lie}(L)$ and $\mathfrak{n} = \mathrm{Lie}(N)$. The real dual space of $\mathfrak{n}$ is identified with $\mathfrak{n}^-$ via a non-degenerate pairing $\langle X, Y\rangle = B_{\mathfrak{g}}(X, Y)$ ($X \in \mathfrak{n},\ Y \in \mathfrak{n}^-$) with the Killing form $B_{\mathfrak{g}}$ of $\mathfrak{g}$. Let

$$Ad : P \to GL_{\mathbb{R}}(\mathfrak{n}) \quad (\text{resp. } Ad^* : P \to GL_{\mathbb{R}}(\mathfrak{n}^-))$$

be the adjoint (resp. coadjoint) representation of P ($\langle X, Ad^*(g)Y\rangle = \langle Ad(g^{-1})X, Y\rangle$). Let $\Omega \subset \mathfrak{n}^-$ be a $Ad^*(N)$-orbit. Then Ω is a symplectic manifold. In other word, the tangent space $T_F(\Omega) = \mathfrak{n}/\mathfrak{n}_F$ of Ω at $F \in \Omega$ has a symplectic $\mathbb{R}$-form $D_F(X, Y) = \langle [X, Y], F\rangle$. Here

$$\mathfrak{n}_F = \{X \in \mathfrak{n} \mid \langle [X, Y], F\rangle = 0 \text{ for all } Y \in \mathfrak{n}\}$$

is the Lie algebra of $N_F = \{g \in N \mid Ad^*(g)F = F\}$. Put

$$L_\Omega = \{g \in L \mid Ad^*(g)\Omega = \Omega\}, \quad L_F = \{g \in L \mid Ad^*(g)F = F\}.$$

Then $L_F \subset L_\Omega$ and there exist some examples such that $L_F \subsetneqq L_\Omega$.

Take a $F \in Z(\mathfrak{n}^-)$ such that $\mathfrak{n}_F = Z(\mathfrak{n})$, and put $\Omega = Ad^*(N)F$. Then we have $L_\Omega = L_F$, and L_Ω acts on Ω fixing $F \in \Omega$. So L_Ω acts on $T_F(\Omega) = \mathfrak{n}/\mathfrak{n}_F$ (the action is given by Ad). The action induces a group homomorphism

$$Ad_F : L_\Omega \to Sp(T_F(\Omega)) \quad (X Ad_F(g) = Ad(g^{-1})X)$$

which is injective because of $\mathfrak{n}_F = Z(\mathfrak{n})$. Here we assume that $Sp(T_F(\Omega))$ acts on $T_F(\Omega)$ from right. Put

$$G_1 = \{g \in L_\Omega \mid Ad^*(g)X = X \text{ for all } X \in Z(\mathfrak{n}^-)\}$$
$$G_2 = \{h \in L_\Omega \mid [h, G_1] = 1\}.$$

They are considered as subgroups of $Sp(T_F(\Omega))$ via Ad_F. Then we have

Theorem 3.1.

1) (G_1, G_2) is a reductive dual pair in $Sp(T_F(\Omega))$ which is irreducible and type I.

2) All irreducible type I reductive dual pairs are obtained in this way.

Let us give a more explicit description. Put $A = M_n(\mathbb{D})$ with a division $\mathbb{R}$-algebra $\mathbb{D}$. Fix a standard $\mathbb{R}$-involution $X^* = {}^t\bar{X}$ on A. Here $\bar{\ }$ denotes the identity map for $\mathbb{D} = \mathbb{R}$, the identity map or the complex conjugation for $\mathbb{D} = \mathbb{C}$, and the main involution for $\mathbb{D} =$ the Hamilton quaternions. Then there exists a $J \in GL_n(\mathbb{D})$ such that $X^\iota = JX^*J^{-1}$ and $J^* = \varepsilon J$ ($\varepsilon = \pm 1$). The group G is isomorphic to $GU(J, \mathbb{D})/Z(\mathbb{D})$ where

$$GU(J, \mathbb{D}) = \{g \in GL_n(\mathbb{D}) \mid gJg^* = \nu(g)J,\ \nu(g) \in Z(\mathbb{D})\}.$$

For the latter use, we will put $U(J, \mathbb{D}) = \{g \in GU(J, \mathbb{D}) \mid \nu(g) = 1\}$. Choosing a suitable $\mathbb{R}$-basis of A, we can assume that

$$J = \begin{pmatrix} & & 1_p \\ & J_0 & \\ \varepsilon 1_p & & \end{pmatrix}, \quad J_0 \in GL_q(\mathbb{D}) \text{ s.t. } J_0^* = \varepsilon J_0$$

and

$$P = \{\begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \in GU(J, \mathbb{D})/Z(\mathbb{D})\}.$$

Then

$$Z(\mathfrak{n}^-) = \{\begin{pmatrix} & & 0 \\ & 0 & \\ S & & \end{pmatrix} \mid S \in M_p(\mathbb{D}),\ S^* = -\varepsilon S\}.$$

For a $F = \begin{pmatrix} & & 0 \\ & 0 & \\ S & & \end{pmatrix} \in Z(\mathfrak{n}^-)$, we have $\mathfrak{n}_F = Z(\mathfrak{n})$ if and only if $S \in GL_p(\mathbb{D})$. For such a $F = \begin{pmatrix} & & 0 \\ & 0 & \\ S & & \end{pmatrix} \in Z(\mathfrak{n}^-)$, we have

$$T_F(\Omega) = \mathfrak{n}/Z(\mathfrak{n}) = \left\{ \begin{pmatrix} 0 & D & 0 \\ 0 & 0 & -(DJ_0)^* \\ 0 & 0 & 0 \end{pmatrix} \mid D \in M_{p,q}(\mathbb{D}) \right\},$$

and

$$G_1 = \left\{ \begin{pmatrix} 1_p & & \\ & e & \\ & & 1_p \end{pmatrix} \mid e \in U(J_0, \mathbb{D}) \right\}, \quad G_2 = \left\{ \begin{pmatrix} a^{*-1} & & \\ & 1_q & \\ & & a \end{pmatrix} \mid a \in U(S, \mathbb{D}) \right\}.$$

Identifying $T_F(\Omega)$ with $M_{p,q}(\mathbb{D})$ by $\begin{pmatrix} 0 & D & 0 \\ 0 & 0 & -(DJ_0)^* \\ 0 & 0 & 0 \end{pmatrix} = D$, the symplectic $\mathbb{R}$-form $D_F(X,Y)$ is equal to

$$D(X,Y) = \mathrm{Tr}(XJ_0)(SY)^* + \mathrm{Tr}\{(XJ_0)(SY)^*\}^* \quad (X, Y \in M_{p,q}(\mathbb{D}))$$

up to a constant multiple, where $\mathrm{Tr} : M_*(\mathbb{D}) \to Z(\mathbb{D})$ is the reduced trace. The group G_1 (resp. G_2) is identified with $U(J_0, \mathbb{D})$ (resp. $U(S, \mathbb{D})$) by $\begin{pmatrix} 1_p & & \\ & e & \\ & & 1_p \end{pmatrix} = e$ (resp. $\begin{pmatrix} a^{*-1} & & \\ & 1_q & \\ & & a \end{pmatrix} = a$). Under these identification, we have $G_1 \hookrightarrow Sp(T_F(\Omega))$ by $g = [X \mapsto Xg]$ and $G_2 \hookrightarrow Sp(T_F(\Omega))$ by $h = [X \mapsto h^* X]$.

Remark 3.2. The irreducible reductive dual pairs of type II are obtained in a similar way. In this case, we should start from a semi-simple $\mathbb{R}$-algebra $A = A_1 \oplus A_2$ with isomorphic simple factors A_j, and a $\mathbb{R}$-involution $\imath$ on A such that $\imath(A_1) = A_2$.

§4 Zeta functions associated with prehomogeneous vector spaces

We will use the identification at the end of §3. That is, let $V = M_{p,q}(\mathbb{D})$ be a $\mathbb{R}$-vector space of the matrices of size $p \times q$ with entries in a $\mathbb{R}$-division algebra $\mathbb{D}$. Take a $J_0 \in GL_q(\mathbb{D})$ and $S \in GL_p(\mathbb{D})$ such that $J_0{}^* = \varepsilon J_0$ and $S^* = -\varepsilon S$ respectively ($\varepsilon = \pm 1$). Define a symplectic $\mathbb{R}$-form D on V by

$$D(x, y) = \mathrm{Tr}(xJ_0)(Sy)^* + \mathrm{Tr}\{(xJ_0)(Sy)^*\}^* \quad (x, y \in V).$$

The groups $G_1 = U(J_0, \mathbb{D})$ and $G_2 = U(S, \mathbb{D})$ are embedded in $Sp(V)$ by

$$G_1 \hookrightarrow Sp(V) \ by \ g = [x \mapsto xg]$$

$$G_2 \hookrightarrow Sp(V) \ by \ h = [x \mapsto h^* x].$$

Here $Sp(V)$ acts on V from right. Then (G_1, G_2) is a reductive dual pair in $Sp(V)$.

We will assume that

1) G_1 is compact, that is, $\varepsilon = 1$ and $J_0{}^* = J_0 > 0$,
2) S is of hyperbolic type, that is, $S = \begin{pmatrix} 0 & 1_r \\ -1_r & 0 \end{pmatrix}$ $(p = 2r)$.

Let W and W' be Lagrangian subspaces of V defined by

$$W = \{\begin{pmatrix} x \\ 0 \end{pmatrix} \in V \mid x \in M_{r,q}(\mathbb{D})\}$$

$$W' = \{\begin{pmatrix} 0 \\ y \end{pmatrix} \in V \mid y \in M_{r,q}(\mathbb{D})\}$$

which are stable under the action of G_1, and $V = W \oplus W'$. We will use the notations and the convension of §2. The Weil representation $(\omega_\chi, L^2(W))$ of $\widetilde{Sp}(V)$ $(\chi(x) = \exp 2\pi\sqrt{-1}x)$ splits over

$$P^+ = \left\{\begin{pmatrix} a & b \\ 0 & {}^t a^{-1} \end{pmatrix} \in Sp(V) \mid \det a > 0\right\}.$$

That is, there exists a continuous group homomorphism $r : P^+ \to \widetilde{Sp}(V)$ such that

1) $p \circ r = \mathrm{id}$ $(p : \widetilde{Sp}(V) \to Sp(V)$ the covering mapping)
2) $(\omega_\chi(r\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix})\varphi)(x) = \chi(\frac{1}{2}D(x, xb))\varphi(x)$,

3) $(\omega_\chi(r\begin{pmatrix} a & 0 \\ 0 & {}^t a^{-1}\end{pmatrix})\varphi)(x) = (\det a)^{1/2}\varphi(xa)$.

Put

$$G_2(W') = \{h \in G_2 \mid W'h = W'\}$$

$$L_2 = \{h \in G_2 \mid W'h = W',\ Wh = W\}.$$

Let G_1^+, $G_2^+(W')$ and L_2^+ be the connected component of G_1, $G_2(W')$ and L_2 respectively. They are subgroups of P^+.

A $\mathbb{Z}$-lattice $\mathcal{L}$ in W defines an arithmetic subgroup Γ of $Sp(V)$ as in §2. Put $\Gamma_j = \Gamma \cap G_j$ and $\Lambda_2 = \Gamma \cap L_2$. We have

$$\Gamma_1 = \{g \in G_1 \mid \mathcal{L}g = \mathcal{L}\}, \quad \Lambda_2 = \{h \in L_2 \mid \mathcal{L}h = \mathcal{L}\}.$$

Put $\Gamma_1^+ = \Gamma_1 \cap G_1^+$ and $\Lambda_2^+ = \Lambda_2 \cap L_2^+$.

Take a compact subgroup K_1 of G_1. Let π (resp. δ) be an irreducible unitary representation of $\widetilde{G}_1 = p^{-1}(G_1)$ (resp. $\widetilde{K}_1 = p^{-1}(K_1)$) such that the multiplicity of δ in $\pi|_{\widetilde{K}_1}$ is one. Take an automorphic form $f \in \check{\mathcal{A}}_\delta(\widetilde{\Gamma}_1\backslash\widetilde{G}_1, \mathbf{1}_{\widetilde{\Gamma}_1}, \pi)$ ($\widetilde{\Gamma}_1 = p^{-1}(\Gamma_1)$) and put $f_u = (f(*), u) \in \mathcal{A}_{\check{\delta}}(\mathbf{1}_{\widetilde{\Gamma}_1}, \check{\pi})$ ($u \in V_\delta$). If $G_1 = \Gamma_1 \cdot G_1^+$, the theta lifting

$$F_{f_u,\varphi}(h) = \int_{\widetilde{\Gamma}_1\backslash\widetilde{G}_1} f_u(g)\theta_\varphi(gh)d\dot{g} \quad (h \in \widetilde{G}_2 = p^{-1}(G_2),\ \varphi \in \mathcal{S}(W))$$

has a Fourier expansion

$$F_{f_u,\varphi}(r(h)) = \sum_T a(f_u, \varphi; T, a)\chi(\mathrm{Tr}(Tb))$$

for $h = \begin{pmatrix} 1 & b \\ 0 & 1\end{pmatrix}\begin{pmatrix} a^{*-1} & 0 \\ 0 & a\end{pmatrix} \in G_2^+(W')$. Here T runs over a $\mathbb{Z}$-lattice in $\{T \in M_r(\mathbb{D}) \mid T^* = T\}$, and $\mathrm{Tr} : M_r(\mathbb{D}) \to Z(\mathbb{D})$ is a reduced trace. Put

$$F^+_{f_u,\varphi}(r(h)) = \sum_{\mathrm{N}(T)\neq 0} a(f_u, \varphi; T, a)\chi(\mathrm{Tr}(Tb))$$

with a reduced norm $\mathrm{N} : M_r(\mathbb{D}) \to Z(\mathbb{D})$.

Define a representation σ of $G_1 \times L_2$ on V by $\sigma(g,h) = gh \in Sp(V) \subset GL_{\mathbb{R}}(V)$. Then the complexifications $(G_1 \times L_2, \sigma, W)_{\mathbb{C}}$ and $(G_1 \times L_2, \sigma, W')_{\mathbb{C}}$ are prehomogeneous vector spaces which are mutually dual with respect to a non-degenerate pairing

$$W \times W' \ni (x,y) \mapsto D(x,y) \in \mathbb{R}.$$

If $q > r$, the relative invariants of $(G_1 \times L_2, \sigma, W)_{\mathbb{C}}$ are the integral power of $P(X) = N(xJ_0x^*)$ $(X = \begin{pmatrix} x \\ 0 \end{pmatrix} \in W)$. The character of $P(X)$ is $\chi_P(g, h) = N(aa^*)^{-1}$ for $g \in G_1$ and $h = \begin{pmatrix} a^{*-1} & 0 \\ 0 & a \end{pmatrix} \in L_2$.

Zeta functions with automorphic forms associated with prehomogeneous vector spaces are studied by F.Sato (c.f. his article in this volume and its references). In our case, such a zeta function comes from a zeta integral

$$Z(f_u, \varphi, s) = \int_{(\Gamma_1^+ \times \Lambda_2^+)\backslash(G_1^+ \times L_2^+)} |\chi_P(g,h)|^{-s} f_u(g) \sum_{\ell \in \mathcal{L}, P(\ell) \neq 0} \varphi(\ell\sigma(g,h)) d(g,h).$$

Now we have

Theorem 4.1. *Suppose* $G_1 = \Gamma_1 \cdot G_1^+$ *and* $q > r$. *Then*

$$\begin{aligned} &\int_{\Lambda_2^+ \backslash L_2^+} N(aa^*)^{-s} F^+_{f_u,\varphi}(r(h)) dh \qquad && (h = \begin{pmatrix} a^{*-1} & 0 \\ 0 & a \end{pmatrix} \in L_2^+) \\ &= Z(f_u, \varphi, s + q') && (q' = \frac{1}{2}(Z(\mathbb{D}) : \mathbb{R})). \end{aligned}$$

Let K_2 be a maximal compact subgroup of G_2. Then, in our case, $G_2/K_2 = \mathcal{V} \oplus \sqrt{-1}\mathcal{C} \subset \mathcal{V}_{\mathbb{C}}$ is a tube domain with an open convex cone $\mathcal{C}$ in a $\mathbb{R}$-vector space $\mathcal{V}$, and $L_2^+ K_2/K_2 = \sqrt{-1}\mathcal{C} \subset \mathcal{V}_{\mathbb{C}}$. So the left hand side of Theorem 4.1 is in fact an integral over $\Lambda_2^+ \backslash \sqrt{-1}\mathcal{C}$. Hence it gives a Dirichlet series of Maass type associated with an automorphic form $F_{f_u,\varphi}$ on G_2. In this way, Theorem 4.1 gives a relation between a Dirichlet series of Maass type and a zeta function associated with a prehomogeneous vector space.

§5 Jacobi forms

Let us start with general nonsense on representations of Jacobi groups. Let (V, D) be a symplectic $\mathbb{R}$-space, G a locally compact unimodular group and $\sigma : G \to Sp(V)$ a continuous group homomorphism. The symplectic group $Sp(V)$ acts on V from right. Let W and W' be Lagrangian subspaces of V such that $V = W \oplus W'$. Define a $\mathbb{R}$-bilinear form $B : V \times V \to \mathbb{R}$ by $B((x, y), (x', y')) = D(x, y')$ $(x, x' \in W,\ y, y' \in W')$. A group law on $H(V) = V \times \mathbb{R}$ is defined by

$$(x, t) \cdot (y, u) = (x + y, t + u + B(x, y)),$$

and $Sp(V)$ acts on $H(V)$ as an automorphism group by

$$(x, t) \cdot \sigma = (x\sigma, t + \frac{1}{2}B(x\sigma, x\sigma) - \frac{1}{2}B(x, x)) \quad (\sigma \in Sp(V)).$$

With this action, we have a semi-direct product $G_J = G \ltimes H(V)$ which we will call a generalized Jacobi group. Let χ be a non-trivial unitary character of $\mathbb{R}$ such that $\{a \in \mathbb{R} \mid \chi(a\mathbb{Z}) = 1\} = \mathbb{Z}$. Let $\mathcal{L} \subset W$ be a $\mathbb{Z}$-lattice, and put $\mathcal{L}' = \{y \in W' \mid D(\mathcal{L}, y) \subset \mathbb{Z}\}$ which is a $\mathbb{Z}$-lattice in W'. Let Γ be a closed unimodular subgroup of G satisfying the conditions

1) $(\mathcal{L} \oplus \mathcal{L}')\gamma = \mathcal{L} \oplus \mathcal{L}'$ for all $\gamma \in \Gamma$

2) $B(x\gamma, x\gamma) \equiv B(x, x) \pmod{2\mathbb{Z}}$ for all $\gamma \in \Gamma$, $x \in \mathcal{L} \oplus \mathcal{L}'$.

Put $\Lambda = (\mathcal{L} \oplus \mathcal{L}') \times \mathbb{R}$ which is a closed unimodular subgroup of $H(V)$. Then the semi-direct product $\Gamma_J = \Gamma \ltimes \Lambda$ is a closed unimodular subgroup of G_J. Define a unitary character χ_J (resp. χ_Λ) of Γ_J (resp. Λ) by $\chi(\gamma, x, t) = \chi(t)$ (resp. $\chi_\Lambda(x, t) = \chi(t)$). Let us denote by

$$(\pi_\chi, E_\chi) = \mathrm{Ind}_{\Gamma_J}^{G_J} \chi_J \quad (\text{resp. } (\pi^\chi, E^\chi) = \mathrm{Ind}_\Lambda^{H(V)} \chi_\Lambda)$$

the induced representation. More explicitly, the representation space E_χ consists of (the equivalence classes of) the $\mathbb{C}$-valued locally integrable functions φ on G_J such that

1) $\varphi(\gamma x) = \chi_J(\gamma)\varphi(x)$ for all $\gamma \in \Gamma_J$,

2) $\int_{\Gamma_J \backslash G_J} |\varphi(\dot{x})|^2 d\dot{x} < \infty$,

and $(\pi_\chi(g)\varphi)(x) = \varphi(xg)$ for $g \in G_J$ and $\varphi \in E_\chi$. The representation space E^χ is defined similarly.

The Weil representation ω_χ of $\widetilde{Sp}(V)$ associated with the character χ is realized on E^χ. Here $\widetilde{Sp}(V)$ is a non-trivial 2-fold covering group of $Sp(V)$ with projection mapping p. Define a fibre product

$$\widetilde{G} = G \times_{Sp(V)} \widetilde{Sp}(V) = \{(g,\tau) \in G \times Sp(V) \mid \sigma(g) = p(\tau)\}$$

which is a 2-fold covering group of G with projection mapping $p(g,\tau) = g$. The group homomorphism $\sigma : G \to Sp(V)$ is lifted to $\widetilde{\sigma} : \widetilde{G} \to \widetilde{Sp}(V)$ defined by $\widetilde{\sigma}(G,\tau) = \tau$. Being connected with $\widetilde{\sigma}$, the Weil representation ω_χ defines a unitary representation of $\widetilde{G}$ on E^χ which is also denoted by ω_χ. Put $\widetilde{\Gamma} = p^{-1}(\Gamma)$. Then there exists a unitary character ε of $\widetilde{\Gamma}$ such that $(\omega_\chi(\gamma)\varphi)(h) = \varepsilon(\gamma)\varphi(h\gamma)$ for $\gamma \in \widetilde{\Gamma}$, $\varphi \in E^\chi$ and $h \in H(V)$. We will denote by $(\pi^{\bar{\varepsilon}}, E^{\bar{\varepsilon}}) = \mathrm{Ind}_{\widetilde{\Gamma}}^{\widetilde{G}} \bar{\varepsilon}$ the induced representation.

The group $\widetilde{G}$ acts on $H(V)$ via $p : \widetilde{G} \to G$, and we have a semi-direct product $\widetilde{G}_J = \widetilde{G} \ltimes H(V)$. Define a projection mapping $p_J : \widetilde{G}_J \to G_J$ (resp. $\widetilde{q} : \widetilde{G}_J \to \widetilde{G}$) by $p_J(g,h) = (p(g),h)$ (resp. $\widetilde{q}(g,h) = g$). Put $\widetilde{\Gamma}_J = p^{-1}(\Gamma_J)$. Then the unitary representation $\pi_\chi \circ p_J$ is equivalent to $\mathrm{Ind}_{\widetilde{\Gamma}_J}^{\widetilde{G}_J}(\chi_J \circ p_J) = (\pi_{\tilde{\chi}}, E_{\tilde{\chi}})$. We have an irreducible unitary representation $\omega_{\chi,J}$ of $\widetilde{G}_J$ on E^χ defined by $\omega_{\chi,J}(g,h) = \omega_\chi(g) \circ \pi^\chi(h)$. Then

Proposition 5.1. *We have a unitary equivalence*

$$(\pi^{\bar{\varepsilon}} \circ \widetilde{q}) \otimes \omega_{\chi,J} \xrightarrow{\sim} \pi_\chi \circ p_J \quad \text{via} \quad \varphi \otimes \psi \mapsto \varphi \boxtimes \psi$$

where $\varphi \boxtimes \psi \in E_{\tilde{\chi}}$ *is defined by*

$$(\varphi \boxtimes \psi)(g,h) = \varphi(g)(\omega_\chi(g)\psi)(hp(g)^{-1}) \quad ((g,h) \in \widetilde{G}_J)$$

for $\varphi \in E^{\bar{\varepsilon}}$ *and* $\psi \in E^\chi$.

Now we will recall a result of Satake [**Sa**,Prop.2]. Let $\widehat{\widetilde{G}}_J(\chi)$ be the unitary equivalence classes $\pi \in \widehat{\widetilde{G}}_J$ such that $\pi|_{\mathbb{R}} = \chi$ $(\mathbb{R} = Z(H(V)) \subset Z(\widetilde{G}_J))$. Then

Proposition 5.2. *We have a bijection $\sigma \mapsto (\sigma \circ \tilde{q}) \otimes \omega_{\chi,J}$ from $\hat{\tilde{G}}$ to $\hat{\tilde{G}}_J(\chi)$.*

These two propositions give

Theorem 5.3. *Take a $\pi \in \widehat{G}_J$ such that $\pi|_{\mathbb{R}} = \chi$. Then*

1) *there exists uniquely a $\sigma \in \hat{\tilde{G}}$ such that $\pi \circ p_J = (\sigma \circ \tilde{q}) \otimes \omega_{\chi,J}$,*

2) *the multiplicity of π in $\mathrm{Ind}_{\Gamma_J}^{G_J} \chi_J$ is equal to the multiplicity of σ in $\mathrm{Ind}_{\widetilde{\Gamma}}^{\widetilde{G}} \bar{\varepsilon}$.*

Now let us suppose that (G, H) is a reductive dual pair in $Sp(V)$ and $\sigma : G \to Sp(V)$ is the inclusion. Let K be a maximal compact subgroup of G. What is remarkable in this case is that, taking the centralizer M of K in $Sp(V)$, the pair (K, M) is a reductive dual pair in $Sp(V)$ [**Ho**]. If H is compact, then M is compact, G/K is a Hermitian symmetric space and (K, M) is a reductive dual pair of complex unitary groups. In this case, the irreducible decomposition of $\omega_\chi|_{\widetilde{K}}$ ($\widetilde{K} = p^{-1}(K)$) is given by [**KV**,Theorem 7.2]. The result is that $\omega_{\chi,J}|_{\widetilde{K}}$ has the minimal $\widetilde{K}$-type δ with multiplicity one and $\dim \delta = 1$. Take a non-zero vector θ in the δ-isotypic component of $\omega_{\chi,J}|_{\widetilde{K}}$. Then the correspondence

$$\varphi \mapsto \varphi \boxtimes \theta \tag{*}$$

gives a correspondence from the space of the automorphic forms on $\widetilde{G}$ (in other words, the half-integral weight modular forms on G) to the space of the automorphic forms on G_J (in other words, the Jacobi forms).

In the case of $(G, H) = (Sp(n, \mathbb{R}), O(m, \mathbb{R}))$, the correspondence $(*)$ gives the correspondence given by [**EZ**] or [**Ib**]. Confer [**Ta1**,Chap 3] for the representation theoretic treatment of Jacobi forms.

The above arguments work also on adeles. In other words, the correspondence $(*)$ is compatible with the Hecke operators.

The details will be treated in the forthcoming paper [**Ta2**].

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