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Some Results on Jacobi Forms of Higher Degree

JAE-HYUN YANG

Abstract

In this article, the author gives some of his results on Jacobi forms of higher degree without proof. The proof can be found in the references [Y1] and [Y2].

1 Jacobi Forms

First of all, we introduce the notations. We denote by $Z$, $R$ and $C$ the ring of integers, the field of real numbers and the field of complex numbers respectively. We denote by $Z^+$ the set of all positive integers. $F^{(k,l)}$ denotes the set of all $k \times l$ matrices with entries in a commutative ring $F$. For any $M \in F^{(k,l)}$, $^tM$ denotes the transpose matrix of $M$. For $A \in F^{(k,l)}$, $\sigma(A)$ denotes the trace of $A$. For $A \in F^{(k,l)}$ and $B \in F^{(k,k)}$, we set $B[A] = ^tABA$. $E_n$ denotes the identity matrix of degree $n$. For any positive integer $g \in Z^+$, we let

$$H_g := \{ Z \in C^{(g,g)} \mid Z = ^tZ, \ Im Z > 0 \}$$

the Siegel upper half plane of degree $g$. Let $Sp(g,R)$ and $Sp(g,Z)$ be the real symplectic group of degree $g$ and the Siegel modular group of degree $g$ respectively.

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Let

\[(1.1) \quad O_g(R^+) := \{ M \in R^{(2g,2g)} \mid {}^tMJ_gM = \nu J_g \text{ for some } \nu > 0 \} \]

be the group of *similitudes* of degree $g$, where

\[J_g := \begin{pmatrix} 0 & E_g \\ -E_g & 0 \end{pmatrix}.\]

Let $M \in O_g(R^+)$. If ${}^tMJ_gM = \nu J_g$, we write $\nu = \nu(M)$. It is easy to see that $O_g(R^+)$ acts on $H_g$ transitively by

\[M < Z := (AZ + B)(CZ + D)^{-1},\]

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in O_g(R^+)$ and $Z \in H_g$.

For $l \in Z^+$, we define

\[(1.2) \quad O_g(l) := \{ M \in Z^{(2g,2g)} \mid {}^tMJ_gM = lJ_g \} \]

We observe that $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in O_g(l)$ is equivalent to the conditions

\[(1.3) \quad {}^tAC = {}^tCA, \quad {}^tBD = {}^tDB, \quad {}^tAD - {}^tCB = lE_g\]

or

\[(1.4) \quad A^tB = B^tA, \quad C^tD = D^tC, \quad A^tD - B^tC = lE_g.\]

For two positive integers $g$ and $h$, we consider the *Heisenberg group*

\[H_R^{(g,h)} := \{ [(\lambda, \mu), \kappa] \mid \lambda, \mu \in R^{(h,g)}, \quad \kappa \in R^{(h,h)}, \quad \kappa + \mu^t\lambda \text{ symmetric} \} \]

endowed with the following multiplication law

\[[(\lambda, \mu), \kappa] \circ [(\lambda', \mu'), \kappa'] := [(\lambda + \lambda', \mu + \mu'), \kappa + \kappa' + \lambda^t\mu' - \mu^t\lambda].\]
We define the semidirect product of $O_g(R^+)$ and $H^{(g,h)}_R$

\[(1.5)\quad O^{(g,h)}_R := O_g(R^+) \ltimes H^{(g,h)}_R\]

endowed with the following multiplication law

\[(1.6)\quad (M, [(\lambda, \mu), \kappa]) \cdot (M', [(\lambda', \mu'), \kappa']) := (MM', [(\nu(M')^{-1}\tilde{\lambda} + \lambda', \nu(M')^{-1}\tilde{\mu} + \mu'), \nu(M')^{-1}\kappa + \kappa' + \nu(M')^{-1}(\tilde{\lambda}'\mu' - \tilde{\mu}'\lambda')]),\]

with $M, M' \in O_g(R^+)$ and $(\tilde{\lambda}, \tilde{\mu}) := (\lambda, \mu)M'$. Clearly the Jacobi group $G^{(g,h)}_R := Sp(g, R) \ltimes H^{(g,h)}_R$ is a normal subgroup of $O^{(g,h)}_R$. It is easy to see that $O_g(R^+)$ acts on $H \times C^{(h,g)}$ transitively by

\[(1.7)\quad (M, [(\lambda, \mu), \kappa]) \cdot (Z, W) := (M < Z >, \nu(W + \lambda Z + \mu)(CZ + D)^{-1}) ,\]

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in O_g(R^+)$, $\nu = \nu(M)$, $(Z, W) \in H \times C^{(h,g)}$.

Let $\rho$ be a rational representation of $GL(g, C)$ on a finite dimensional complex vector space $V_\rho$. Let $M \in R^{(h,h)}$ be a symmetric half integral matrix of degree $h$. We define

\[(1.8)\quad (f|_{\rho,M}[(M, [(\lambda, \mu), \kappa)])](Z, W) := \exp\{-2\pi\nu i\sigma(M[W + \lambda Z + \mu](CZ + D)^{-1}C)\}
\times \exp\{2\pi\nu i\rho(M(\lambda Z'\lambda + 2\lambda W + (\kappa + \mu^t\lambda)))\}
\times \sigma(CZ + D)^{-1}f(M < Z >, \nu(W + \lambda Z + \mu)(CZ + D)^{-1}) ,\]

where $\nu = \nu(M)$.

**Lemma 1.1.** Let $g_i = (M_i, [(\lambda_i, \mu_i), \kappa_i]) \in O^{(g,h)}_R (i = 1, 2)$. For any $f \in C^\infty(H \times C^{(h,g)}, V_\rho)$, we have

\[(1.9)\quad (f|_{\rho,M}[g_1])|_{\rho,\nu(M_1)M}[g_2] = f|_{\rho,M}[g_1g_2].\]
**Definition 1.2.** Let $\rho$ and $\mathcal{M}$ be as above. Let

$$H_{Z}^{(g,h)} := \{[(\lambda, \mu), \kappa] \in H_{R}^{(g,h)} | \lambda, \mu \in \mathbb{Z}^{(h,g)}, \kappa \in \mathbb{Z}^{(h,h)} \}.$$ 

A *Jacobi form* of index $\mathcal{M}$ with respect to $\rho$ is a holomorphic function $f \in C^\infty(H, C^\langle h,g) , V_\rho)$ satisfying the following conditions (A) and (B):

(A) $f|_{\rho,\mathcal{M}}[\tilde{\gamma}] = f$ for all $\tilde{\gamma} \in \Gamma_g^{J} := \text{Sp}(g, \mathbb{Z}) \ltimes H_{Z}^{(g,h)}$.

(B) $f$ has a Fourier expansion of the following form:

$$f(Z, W) = \sum_{T \geq 0} \sum_{R \in \mathbb{Z}(gh)} C(T, R) \exp(2\pi i \sigma(TZ + RW))$$

with $c(T, R) \neq 0$ only if $\left( \frac{T}{2}, \frac{1}{2}R, \mathcal{M} \right) \geq 0$.

If $g \leq 2$, the condition (B) is superfluous by Koecher principle (see [Z] Lemma 1.6). We denote by $J_{\rho,\mathcal{M}}(\Gamma_g)$ the vector space of all Jacobi forms of index $\mathcal{M}$ with respect to $\rho$. In the special case $V_\rho = C$, $\rho(A) = (\text{det} A)^k (k \in \mathbb{Z}, A \in GL(g, C))$, we write $J_{k,\mathcal{M}}(\Gamma_g)$ instead of $J_{\rho,\mathcal{M}}(\Gamma_g)$ and call $k$ the weight of a Jacobi form $f \in J_{k,\mathcal{M}}(\Gamma_g)$.

Ziegler ([Z] Theorem 1.8 or [E-Z] Theorem 1.1) proves that the vector space $J_{\rho,\mathcal{M}}(\Gamma_g)$ is finite dimensional.

## 2 Singular Jacobi Forms

In this section, we define the concept of singular Jacobi forms and characterize singular Jacobi forms.

Let $\mathcal{M}$ be a symmetric positive definite, half integral matrix of degree $h$. A Jacobi form $f \in J_{\rho,\mathcal{M}}(\Gamma_g)$ admits a Fourier expansion (see Definition
A Jacobi form $f \in J_{\rho, \mathcal{M}}(\Gamma_g)$ is said to be *singular* if it admits a Fourier expansion such that the Fourier coefficient $c(T, R)$ is zero unless $\det(4T - R\mathcal{M}^{-1}R) = 0$.

**Example 2.1.** Let $\mathcal{M} = \mathcal{M}^t$ be as above. Let $S \in Z^{(2k, 2k)}$ be a symmetric positive definite integral matrix of degree $2k$ and $c \in Z^{(2k, h)}$. We consider the theta series

$$\theta_{S,c}^{(g)}(Z, W) := \sum_{\lambda \in Z^{(2k, g)}} e^{\pi i \sigma(S\lambda^t Z + 2S\lambda^t cW)}, \ Z \in H_g, \ W \in C^{(h,g)}.$$

We assume that $2k < g + \text{rank}(\mathcal{M})$. Then $\vartheta_{S,c}(Z, W)$ is a singular Jacobi form in $J_{k, \mathcal{M}}(\Gamma_g)$, where $\mathcal{M} = \frac{1}{2}c\mathcal{M}c$. We note that if the Fourier coefficient $c(T, R)$ of $\vartheta_{S,c}^{(g)}$ is nonzero, there exists $\lambda \in Z^{(2k, g)}$ such that

$$\frac{1}{2} \langle \lambda, c \rangle S(\lambda, c) = \left( \begin{array}{cc} T & \frac{1}{2}R \\ \frac{1}{2}R^t & \mathcal{M} \end{array} \right).$$

Thus

$$\text{rank} \left( \begin{array}{cc} T & \frac{1}{2}R \\ \frac{1}{2}R^t & \mathcal{M} \end{array} \right) \leq 2k < g + \text{rank}(\mathcal{M}).$$

Therefore $\det(4T - R\mathcal{M}^{-1}R) = 0$.

The following natural question arises:

**Problem:** Characterize the singular Jacobi forms.

The author ([Y1]) gives some answers for this problem. He characterizes singular Jacobi forms by the *differential equation* and the *weight* of the representation $\rho$. 
Now we define a very important differential operator characterizing singular Jacobi forms. We let

\[(2.3) \quad \mathcal{P}_g := \{ Y \in R^{(g,g)} | Y = {}^tY > 0 \} \]

be the open convex cone in the Euclidean space $R^{\frac{g(g+1)}{2}}$. We define the differential operator $M_{g,h,\mathcal{M}}$ on $\mathcal{P}_g \times R^{(h,g)}$ defined by

\[(2.4) \quad M_{g,h,\mathcal{M}} := det(Y) \cdot det\left( \frac{\partial}{\partial Y} + \frac{1}{8\pi} i \left( \frac{\partial}{\partial V} \right) \mathcal{M}^{-1} \left( \frac{\partial}{\partial V} \right) \right), \]

where $\frac{\partial}{\partial Y} = \left( \frac{(1+\delta_{\mu\nu})}{2} \frac{\partial}{\partial y_{\mu\nu}} \right)$ and $\frac{\partial}{\partial V} = \left( \frac{\partial}{\partial v_{kl}} \right)$.

**Definition 2.2.** An irreducible finite dimensional representation $\rho$ of $GL(g, C)$ is determined uniquely by its highest weight $(\lambda_1, \cdots, \lambda_g) \in Z^g$ with $\lambda_1 \leq \cdots \leq \lambda_g$. We denote this representation by $\rho = (\lambda_1, \cdots, \lambda_g)$. The number $k(\rho) := \lambda_g$ is called the weight of $\rho$.

**Theorem A.** Let $f \in J_{\rho,\mathcal{M}}(\Gamma_g)$ be a Jacobi form of index $\mathcal{M}$ with respect to $\rho$. Then the following are equivalent:

1. $f$ is a singular Jacobi forms.
2. $f$ satisfies the differential equation $M_{g,h,\mathcal{M}}f = 0$.

**Theorem B.** Let $2\mathcal{M}$ be a symmetric positive definite, unimodular even matrix of degree $h$. Assume that $\rho$ satisfies the following condition

\[(2.5) \quad \rho(A) = \rho(-A) \quad \text{for all} \quad A \in GL(g, C). \]

Then any nonvanishing Jacobi form in $J_{\rho,\mathcal{M}}(\Gamma_g)$ is singular if and only if $2k(\rho) < g + \text{rank} (\mathcal{M})$. Here $k(\rho)$ denotes the weight of $\rho$.

**Conjecture.** For general $\rho$ and $\mathcal{M}$ without the above assumptions on them, a nonvanishing Jacobi form $f \in J_{\rho,\mathcal{M}}(\Gamma_g)$ is singular if and only if
$2k(\rho) < g + \text{rank}(\mathcal{M})$.

**Remarks.** If $f \in J_{\rho,\mathcal{M}}(\Gamma_g)$ is a Jacobi form, we may write

$$(*) \quad f(Z, W) = \sum_{a \in \mathcal{N}} f_a(Z) \cdot \vartheta_{2,\mathcal{M},a,0}(Z, W), \quad Z \in H_g, \quad W \in C^{(h,g)},$$

where $\{ f_a : H_g \rightarrow V_\rho \mid a \in \mathcal{N} \}$ are uniquely determined holomorphic functions on $H_g$. A singular modular form of type $\rho$ may be written as a finite sum of theta series $\vartheta_{S,P}(Z)$'s with pluriharmonic coefficients (cf. [F]). The following problem is quite interesting.

**Problem.** Describe the functions $\{ f_a \mid a \in \mathcal{N} \}$ explicitly given by $(*)$ when $f \in J_{\rho,\mathcal{M}}(\Gamma_g)$ is a singular Jacobi form.

## 3 The Siegel-Jacobi Operators

In this section, we investigate the Siegel-Jacobi operator and the action of Hecke operator on Jacobi forms. The Siegel-Jacobi operator

$$\Psi_{g,r} : J_{\rho,\mathcal{M}}(\Gamma_g) \rightarrow J_{\rho^{(r)},\mathcal{M}}(\Gamma_r)$$

is defined by

$$(\Psi_{g,r}f)(Z, W) := \lim_{t \rightarrow \infty} f \left( \left( \begin{array}{cc} Z & 0 \\ 0 & itE_{g-r} \end{array} \right), (W, 0) \right), \quad f \in J_{\rho,\mathcal{M}}(\Gamma_g),$$

$Z \in H_r$, $W \in C^{(h,r)}$ and $J_{\rho,\mathcal{M}}(\Gamma_g)$ denotes the space of all Jacobi forms of index $\mathcal{M}$ with respect to an irreducible rational finite dimensional representation $\rho$ of $GL(g, \mathbb{C})$. We note that the above limit always exists because a Jacobi form $f$ admits a Fourier expansion converging uniformly on any set of the form

$$\{(Z, W) \in H_g \times C^{(h,g)} \mid Im Z \geq Y_0 > 0, \ W \in K \subset C^{(h,g)} \text{ compact} \}.$$
Here, the representation $\rho^{(r)}$ of $GL(r, C)$ is defined as follows. Let $V^{(r)}_{\rho}$ be the subspace of $V_{\rho}$ generated by $\{f(Z, W) \mid f \in J_{\varrho, \mathcal{M}}(\Gamma_g), (Z, W) \in H_g \times C^{(h, g)} \}$. Then $V^{(r)}_{\rho}$ is invariant under

$$\left\{ \begin{pmatrix} g & 0 \\ 0 & E_{g-r} \end{pmatrix} : g \in GL(r, C) \right\}.$$  

Then we have a rational representation $\rho^{(r)}$ of $GL(r, C)$ on $V^{(r)}_{\rho}$ defined by

$$\rho^{(r)}(g)v := \rho \left( \begin{pmatrix} g & 0 \\ 0 & E_{g-r} \end{pmatrix} \right)v, \quad g \in GL(r, C), \quad v \in V^{(r)}_{\rho}.$$ 

In the Siegel case, we have the so-called Siegel $\Phi-$operator

$$\Phi = \Phi_{g, g-1} : \left[ \Gamma_g, k \right] \longrightarrow \left[ \Gamma_{g-1}, k \right]$$

defined by

$$(\Phi f)(Z) := \lim_{t \to \infty} f \left( \begin{pmatrix} Z & 0 \\ 0 & it \end{pmatrix} \right), \quad f \in \left[ \Gamma_g, k \right], \quad Z \in H_{g-1},$$

where $\left[ \Gamma_g, k \right]$ denotes the vector space of all Siegel modular forms on $H_g$ of weight $k$.

Here $\left[ \Gamma_g, k \right]$ denotes the vector space of all Siegel modular forms on $H_g$ of weight $k$.

The following properties of $\Phi$ are known:

(S1) If $k > 2g$ and $k$ is even, $\Phi$ is surjective.

(S2) If $2k < g$, then $\Phi$ is injective.

(S3) If $2k + 1 < g$, then $\Phi$ is bijective.

H. Maass([M1]) proved the statement (1) using Poincaré series. E. Freitag ([F2]) proved the statements (2) and (3) using the theory of singular modular forms.
The author ([Y2]) proves the following theorems:

**Theorem C.** Let \( 2 \mathcal{M} \in Z^{(h,h)} \) be a positive definite, unimodular symmetric even matrix of degree \( h \). We assume that \( \rho \) satisfies the condition (3.1):

\[
(3.1) \quad \rho(A) = \rho(-A) \quad \text{for all } A \in GL(g, C).
\]

We also assume that \( \rho \) satisfies the condition \( 2k(\rho) < g + \text{rank}(\mathcal{M}) \). Then the Siegel-Jacobi operator

\[
\Psi_{g,g-1} : J_{\rho,\mathcal{M}}(\Gamma_g) \longrightarrow J_{\rho^{(g-1)},\mathcal{M}}(\Gamma_{g-1})
\]

is injective. Here \( k(\rho) \) denotes the weight of \( \rho \).

**Theorem D.** Let \( 2 \mathcal{M} \in Z^{(h,h)} \) be as above in Theorem A. Assume that \( \rho \) satisfies the condition (3.1) and \( 2k(\rho) + 1 < g + \text{rank}(\mathcal{M}) \). Then the Siegel-Jacobi operator

\[
\Psi_{g,g-1} : J_{\rho,\mathcal{M}}(\Gamma_g) \longrightarrow J_{\rho^{(g-1)},\mathcal{M}}(\Gamma_{g-1})
\]

is an isomorphism.

**Theorem E.** Let \( 2 \mathcal{M} \in Z^{(h,h)} \) be as above in Theorem A. Assume that \( 2k > 4g + \text{rank}(\mathcal{M}) \) and \( k \equiv 0(\text{mod } 2) \). Then the Siegel-Jacobi operator

\[
\Psi_{g,g-1} : J_{k,\mathcal{M}}(\Gamma_g) \longrightarrow J_{k,\mathcal{M}}(\Gamma_{g-1})
\]

is surjective.

The proof of the above theorems is based on the important Shimura correspondence, the theory of singular modular forms and the result of H. Maass.
We recall 
\[ O_g(l) := \{ M \in Z^{(2g,2g)} | {}^tMJ_gM = lJ_g \} \].

\( O_g(l) \) is decomposed into finitely many double cosets \( \text{mod} \ \Gamma_g \), i.e.,
\( (3.2) \quad O_g(l) = \bigcup_{j=1}^{m} \Gamma_g g_j \Gamma_g \) (disjoint union).

We define
\( (3.3) \quad T(l) := \sum_{j=1}^{m} \Gamma_g g_j \Gamma_g \in \mathcal{H}^{(g)}, \text{ the Hecke algebra.} \)

Let \( M \in O_g(l) \). For a Jacobi form \( f \in J_{\rho,M}(\Gamma_g) \), we define
\( (3.4) \quad f|_{\rho,M}(\Gamma_g M \Gamma_g) := \sum_{i} f|_{\rho,M}([M_i, [(0,0), 0]]) \),
where \( \Gamma_g M \Gamma_g = \bigcup_{i}^{m} \Gamma_g M_i \) (finite disjoint union) and \( k(\rho) \) denotes the weight of \( \rho \).

**Theorem F.** Let \( M \in O_g(l) \) and \( f \in J_{\rho,M}(\Gamma_g) \). Then
\[ f|_{\rho,M}(\Gamma_g M \Gamma_g) \in J_{\rho,l\lambda 4}(\Gamma_g). \]

For a prime \( p \), we define
\( (3.5) \quad O_{g,p} := \bigcup_{l=0}^{\infty} O_g(p^l). \)

Let \( \hat{\mathcal{L}}_{g,p} \) be the \( \mathbb{C} \)-module generated by all left cosets \( \Gamma_g M, M \in O_{g,p} \) and \( \hat{\mathcal{H}}_{g,p} \) the \( \mathbb{C} \)-module generated by all double cosets \( \Gamma_g M \Gamma_g, M \in O_{g,p} \). Then \( \hat{\mathcal{H}}_{g,p} \) is a commutative associative algebra. Since \( j(\hat{\mathcal{H}}_{g,p}) \subset \hat{\mathcal{L}}_{g,p} \), we have a monomorphism \( j : \hat{\mathcal{H}}_{g,p} \rightarrow \hat{\mathcal{L}}_{g,p} \).

In a left coset \( \Gamma_g M, M \in O_{g,p} \), we can choose a representative \( M \) of the form
\( (3.6) \quad M = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}, \quad {}^tAD = p^{k_0}E_g, \quad {}^tBD = {}^tDB, \)
(3.7) \[ A = \begin{pmatrix} a & \alpha \\ 0 & A^* \end{pmatrix}, \quad B = \begin{pmatrix} b & \beta_1^t \\ \beta_2 & B^* \end{pmatrix}, \quad D = \begin{pmatrix} d & 0 \\ \delta & D^* \end{pmatrix}, \]

where \( \alpha, \beta_1, \beta_2, \delta \in \mathbb{Z}^{g-1} \). Then we have

(3.8) \[ M^* := \begin{pmatrix} A^* & B^* \\ 0 & D^* \end{pmatrix} \in O_{g-1,p}. \]

For any integer \( r \in \mathbb{Z} \), we define

(3.9) \[ (\Gamma_g M)^* := \frac{1}{d^r} \Gamma_{g-1} M^*. \]

If \( \Gamma_g M \Gamma_g = \bigcup_{j=1}^{m} \Gamma_g M_j \) (disjoint union), \( M, M_j \in O_{g,p} \), then we define in a natural way

(3.10) \[ (\Gamma_g M \Gamma_g)^* = \frac{1}{d^r} \sum_{j=1}^{m} \Gamma_{g-1} M_j^*. \]

We extend the above map (3.9) linearly on \( \tilde{H}_{g,p} \) and then we obtain an algebra homomorphism

(3.11) \[ \tilde{H}_{g,p} \rightarrow \tilde{H}_{g-1,p} \]

\[ T \mapsto T^*. \]

It is known that the above map is a surjective map([ZH] Theorem 2).

**Theorem G.** Suppose we have

(a) a rational finite dimensional representation

\[ \rho : GL(g, \mathbb{C}) \rightarrow GL(V_{\rho}), \]

(b) a rational finite dimensional representation

\[ \rho_0 : GL(g - 1, \mathbb{C}) \rightarrow GL(V_{\rho_0}) \]
(c) a linear map $R: V_{\rho} \rightarrow V_{\rho_0}$ satisfying the following properties (1) and (2):

1. $R \circ \rho \left( \begin{array}{cc} 1 & 0 \\ 0 & A \end{array} \right) = \rho_0(A) \circ R$ for all $A \in GL(g - 1, C)$.

2. $R \circ \rho \left( \begin{array}{cc} a & 0 \\ 0 & E_{g-1} \end{array} \right) = a^*R$ for some $a \in Z$.

Then for any $f \in J_{\rho, M}(\Gamma_g)$ and $T \in \check{H}_{g,p}$, we have

$$(R \circ \Psi_{g,g-1})(f|T) = R(\Psi_{g,g-1}f)|T^*,$$

where $T^*$ is an element in $\check{H}_{g-1,p}$ defined by (3.11).

**Corollary.** The Siegel-Jacobi operator is compatible with the action of $T \mapsto T^*$. Precisely, we have the following commutative diagram:

$$
\begin{array}{ccc}
J_{\rho, M}(\Gamma_g) & \xrightarrow{\psi_{g,g-1}} & J_{\rho,(g-1),N}(\Gamma_{g-1}) \\
T \downarrow & & T^* \downarrow \\
J_{\rho, N}(\Gamma_g) & \xrightarrow{\psi_{g,g-1}} & J_{\rho,(g-1),N}(\Gamma_{g-1}).
\end{array}
$$

Here $N$ is a certain symmetric half integral semipositive matrix of degree $h$.

**Definition 3.2.** Let $f \in J_{\rho, M}(\Gamma_g)$ be a Jacobi form. Then we have a Fourier expansion given by (B) in Definition 1.2. A Jacobi form $f$ is called a cusp form if $c(T, R) \neq 0$ implies $\left( \frac{1}{2}T_{{}^tR}M \right) > 0$. We denote by $J_{\rho, M}^{cusp}(\Gamma_g)$ the vector space of all cusp forms in $J_{\rho, M}(\Gamma_g)$.

**Theorem H.** Let $1 \leq r \leq g$. Assume $k(\rho) > g + r + \text{rank}(M) + 1$ and $k(\rho)$ even. Then

$$J_{\rho, M}^{cusp}(\Gamma_r) \subset \Psi_{g,r}(J_{\rho, M}(\Gamma_g)).$$
4 Final Remarks

In this section we give some open problems which should be investigated and give some remarks.

Let

\[ G_{R}^{(g,h)} := Sp(g, R) \ltimes H_{R}^{(g,h)} \]

be the Jacobi group of degree \( g \). Let \( \Gamma_{g}^{J} := Sp(g, Z) \ltimes H_{Z}^{(g,h)} \) be the discrete subgroup of \( G_{R}^{(g,h)} \). For the case \( g = h = 1 \), the spectral theory for \( L^{2}(\Gamma_{1}^{J}\backslash G_{R}^{(1,1)}) \) had been investigated almost completely in [B1] and [B-B]. For general \( g \) and \( h \), the spectral theory for \( L^{2}(\Gamma_{g}^{J}\backslash G_{R}^{(g,h)}) \) is not known yet.

**Problem 1.** Decompose the Hilbert space \( L^{2}(\Gamma_{g}^{J}\backslash G_{R}^{(g,h)}) \) into irreducible components of the Jacobi group \( G_{R}^{(g,h)} \) for general \( g \) and \( h \). In particular, classify all the irreducible unitary or admissible representations of the Jacobi group \( G_{R}^{(g,h)} \) and establish the *Duality Theorem* for the Jacobi group \( G_{R}^{(g,h)} \).

**Problem 2.** Give the *dimension formulae* for the vector space \( J_{\rho,\mathcal{M}}(\Gamma_{g}) \) of Jacobi forms.

**Problem 3.** Construct Jacobi forms. Concerning this problem, discuss the *vanishing theorem* on the vector space \( J_{\rho,\mathcal{M}}(\Gamma_{g}) \) of Jacobi forms.

**Problem 4.** Develope the theory of L-functions for the Jacobi group \( G_{R}^{(g,h)} \). There are several attempts to establish L-functions in the context of the Jacobi group by Japanese mathematicians A. Murase and T. Sugano using so-called the Whittaker-Shintani functions.

**Problem 5.** Give applications of Jacobi forms, for example in algebraic geometry and physics. In fact, Jacobi forms have found some applications
in proving non-vanishing theorems for L-functions of modular forms [BFH],
in the theory of Heeger points [GKS], in the theory of elliptic genera [Za]
and in the string theory [C].

By a certain lifting, we may regard Jacobi forms as smooth functions on
the Jacobi group \( G^{(g,h)}_R \) which are invariant under the action of the discrete
subgroup \( \Gamma_g^J \) and satisfy the differential equations and a certain growth
condition.

**Problem 6.** Develope the theory of *automorphic forms* on the Jacobi
group \( G^{(g,h)}_R \). We observe that the Jacobi group is *not reductive*.

Finally for historical remarks on Jacobi forms, we refer to [B2].

**References**

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