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# On growth estimates for Fourier coefficients of Jacobi forms and an application

by

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## Introduction

In §1 of this note we shall briefly discuss some results and open questions about growth estimates for Fourier coefficients of Jacobi forms. In §2 we shall indicate an application to estimates for Fourier coefficients of Siegel modular forms of genus two.

## §1. Estimates for Fourier coefficients of Jacobi forms: results and open questions

Let $\phi(\tau,z)$ be a Jacobi cusp form of weight $k\in\mathbb{Z}$ and index $m\in\mathbb{N}$ on the Jacobi group $SL_2(\mathbb{Z})\ltimes\mathbb{Z}^2$, i.e. $\phi$ is a complex-valued holomorphic function on $H\times\mathbb{C}$ (where $H$ denotes the complex upper half-plane) satisfying the two transformation formulas

$$\phi\left(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d}\right) = (c\tau+d)^k\exp\left(2\pi i m\frac{cz^2}{c\tau+d}\right)\phi(\tau,z) \qquad \left(\forall \begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\Gamma_1\right)$$

and

$$(1)\qquad \phi(\tau,z+\lambda\tau+\mu) = \exp(-2\pi i m(\lambda^2\tau+2\lambda z))\phi(\tau,z) \qquad (\forall\,(\lambda,\mu)\in\mathbb{Z}^2)$$

and having a Fourier expansion

$$\phi(\tau,z) = \sum_{n,r\in\mathbb{Z},\,r^2<4mn} c(n,r)e^{2\pi i(n\tau+rz)}$$

with $c(n,r)\in\mathbb{C}$ [1,§1].

In the following we shall set

$D:=r^2-4mn.$

Formula (1) then implies that the Fourier coefficients $c(n,r)$ depend only on $D$ and the residue class $r \pmod{2m}$.

We shall be interested in estimates of the type

$$c(n,r) \ll_\phi |D|^c \qquad (|D| \to \infty),$$

where the exponent $c>0$ only depends on $k$.

Note that that there are natural linear maps defined on Fourier coefficients from Jacobi forms of weight $k$ to elliptic modular forms of (half-integral) weight $k-\frac{1}{2}$ [1,§5]. Although these maps in general are neither surjective nor injective, one may hope that most results or conjectures about estimates for Fourier coefficients of modular forms of half-integral weight should have natural "pendants" with essentially the same bounds for Jacobi forms.

Using the fact that the function $v^{k/2}\exp(-2\pi my^2/v)\,|\phi(\tau,z)|$ (where $v=\mathrm{Im}(\tau)$, $y=\mathrm{Im}(z)$) is bounded on $\mathcal{H}\times\mathbb{C}$ and applying the classical Hecke argument one immediately finds that

$$c(n,r) \ll_\phi |D|^{k/2}.$$

Note that here the Hecke argument applied to the Fourier coefficients $a(n)$ $(n\in\mathbb{N})$ of a cusp form $f$ of weight $k-\frac{1}{2}$ gives the better exponent $k/2-1/4$, i.e.

$$(2) \quad a(n) \ll_f n^{k/2-1/4}.$$

Spaces of cusp forms of weight >2 on congruence subgroups of $SL_2(\mathbb{Z})$ are generated by Poincaré series. As is well-known, the Fourier coefficients of the latter can be explicitly expressed as infinite sums whose general term is the product of an ordinary Bessel function and a Kloosterman-type sum. Using obvious bounds for the Bessel functions and bounds à la Weil for the Kloosterman sums one arrives in this way at bounds for the Fourier coefficients of elliptic cusp forms which in general are much better than those obtained from the Hecke argument.

The above method can also be applied in the context of Jacobi forms and leads to

$$(3) \quad c(n,r) \ll_{\phi,\varepsilon} |D|^{k/2-1/2+\varepsilon} \qquad (\varepsilon>0)$$

if k>2 (for references cf. §2). A similar result with the same exponent is known for cusp forms of weight $k-\frac{1}{2}$ [3,§1]. In both cases the Kloosterman-type sums which arise can explicitly be evaluated in terms of Salié sums. Note that Iwaniec in [3] used the latter fact together with some sophisticated estimates for sums of Salié sums to show that actually (2) for n squarefree and k>2 can be improved to

$$(4) \quad a(n) \ll_k \sigma_0(n)(\log 2n)^2 n^{k/2-15/28} \|f\|$$

where $\sigma_0(n)$ is the number of positive divisors of n, $\|f\|$ is the appropriately normalized Petersson norm of f and the constant implied in $\ll$ only depends on k. We expect that using Iwaniec's method one can prove a similar result as (4) for the coefficients c(n,r) of $\phi$ if D is a fundamental discriminant (i.e.

is the discriminant of a quadratic field) and $k>2$.

Let us mention that a classical theorem of Landau ([8], cf. also [9]) when applied to the Rankin-Dirichlet series of a cusp form, usually also implies estimates for its Fourier coefficients. In general, however, when used just like that it does not seem to give better bounds than the method of Poincaré series and Kloosterman sums, so we do not discuss it here.

Finally we would like to point out that there is the analogue of the Ramanujan-Petersson conjecture for modular forms of integral weight which predicts that

$$(5) \quad c(n,r) \ll_{\phi,\varepsilon} |D|^{k/2-3/4+\varepsilon} \qquad (\varepsilon>0)$$

provided that D is a fundamental discriminant (if $k=2$ one has to suppose that $\phi$ lies in the orthogonal complement of the space of theta functions). Note that by Waldspurger's results [11] applied in the context of Jacobi forms [2,Chap.II,§4], the inequality (5) for an appropriate Hecke eigenform $\phi$ in case $(D,m)=1$ is equivalent to

$$(6) \quad L_F(k-1,\chi_D) \ll_{F,\varepsilon} |D|^{\varepsilon} \qquad (\varepsilon>0)$$

where F is the normalized newform of weight $2k-2$ on $\Gamma_0(m)$ which corresponds to $\phi$ under the Skoruppa-Zagier lift [10] and $L_F(s,\chi_D)$ denotes the Hecke L-function of F twisted with the quadratic Dirichlet character $\chi_D$ of conductor D. Inequality (6) can be viewed as a generalization of a well-known conjecture on the

values $L(\frac{1}{2},\chi_D)$ (where $L(s,\chi_D)$ is the Dirichlet L-function attached to $\chi_D$) and -in fact- is the main motivation why one should expect (5) to be true (thus the motivation here is rather different from that in the integral weight case). Similar remarks as above, of course, apply in the context of modular forms of half-integral weight [3,§1].

## §2. An application

The proof that leads to the estimate (3) -when carefully analyzed- in fact, shows that

$$(7) \quad c(n,r) \ll_{\varepsilon,k} \left(m+|D|^{1/2+\varepsilon}\right)^{1/2} \frac{|D|^{k/2-3/4}}{m^{(k-1)/2}} \|\phi\| \qquad (\varepsilon>0)$$

where $\|\phi\|$ is the usual Petersson norm of $\phi$ [1,§2] and the constant implied in $\ll$ depends only on $\varepsilon$ and k [6,§1;7,§1]. The important fact here that we want to make use of in the context of cusp forms of genus two, -besides of the appearance of the norm of $\phi$ (compare (4))- is the appearance of an appropriate power of the index m in the *denominator* on the right-hand side of (7).

Let F be a Siegel cusp form of weight k on $Sp_2(\mathbb{Z})$ and denote by a(T) (T a positive definite symmetric half-integral (2,2)-matrix) its Fourier coefficients. Recall that F has a Fourier-Jacobi expansion

$$F(Z) = \sum_{m\geq 1} \phi_m(\tau,z)e^{2\pi i m\tau'} \qquad \left(Z=\begin{pmatrix}\tau & z\\ z & \tau'\end{pmatrix}\in H_2 := \text{Siegel upper half-space of genus two}\right),$$

where the coefficients $\phi_m$ are Jacobi cusp forms of weight k and index m [1,§6].

If we write $T=\begin{pmatrix} n & r/2 \\ r/2 & m \end{pmatrix}$, then by definition $a(T)$ is the $(n,r)$-th Fourier coefficient of $\phi_m$. By Theorem 2 in [6] we have

$$\|\phi_m\| \ll_{F,\varepsilon} m^{k/2-2/9+\varepsilon} \qquad (\varepsilon>0)$$

(the proof is based on Landau's method mentioned above applied to the Dirichlet series $\sum_{m\geq 1} \|\phi_m\|^2 m^{-s}$ which was introduced and studied in [5]). From (7) we therefore easily infer that

$$(8) \quad a(T) \ll_{F,\varepsilon} m^{5/18+\varepsilon}(m+|D|^{1/2+2\varepsilon})^{1/2}|D|^{k/2-3/4} \qquad (\varepsilon>0).$$

The estimate (8) implies that

$$(9) \quad a(T) \ll_{F,\varepsilon} (\det T)^{k/2-13/36+\varepsilon} \qquad (\varepsilon>0);$$

in fact, both sides of (9) are invariant if $T$ is replaced by $U'TU$ with $U\in GL_2(\mathbb{Z})$, hence we may assume that $m=\min(T)$, where $\min(T)$ is the least positive integer represented by $T$, and then in (8) use that $\min(T) \ll (\det T)^{1/2}$ by reduction theory.

Note that (9) is somewhat better than Kitaoka's bound

$$a(T) \ll_{F,\varepsilon} (\det T)^{k/2-1/4+\varepsilon} \qquad (\varepsilon>0)$$

(cf. [4]) obtained previously by the method of (Siegel-) Poincaré series and (matrix-argument) Kloosterman sums.

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