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A generalization of Deligne-Grothendieck-MacPherson's natural transformation C_* and a conjecture

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Introduction

Let $\boldsymbol{V}$ be the category of compact complex algebraic varieties and $\boldsymbol{Ab}$ the category of abelian groups. Let $\boldsymbol{F}: \boldsymbol{V} \to \boldsymbol{Ab}$ be the "constructible function" covariant functor such that $\boldsymbol{F}(X)$ is the abelian group of **Q**-valued constructible functions on X, freely generated by characteristic functions 1_W's of subvarieties W of X, and for a morphism $f: X \to Y$ the pushforward $f_\# := \boldsymbol{F}(f)$ is defined by

$$(f_\# 1_W)(y) := \chi(f^{-1}(y) \cap W).$$

Let $H_*(\ ;\mathbf{Q}) : \boldsymbol{V} \to \boldsymbol{Ab}$ be the usual **Q**-homology covariant functor. Then we have

Theorem (conjectured by Deligne and Grothendieck and proved by MacPherson [6]) *There exists the unique natural transformation $C_* : F \to H_*(\ ;Q)$ satisfying the extra condition (which shall be called "normalization condition" or "smooth Chern condition") that*

$$(*) \qquad C_*(V)(1_V) = c(V) \cap [V] \text{ for any smooth variety } V,$$

where 1_V is the characteristic function of V and $c(V)$ is the total Chern cohomology class of the tangent bundle TV.

Outline of Proof: (The original statement is with **Z**-coefficients, but it also holds with **Q**-coefficients) For a variety X, $\boldsymbol{F}(X)$ is freely generated by local Euler constructible functions Eu_W, where W runs through all subvarites of X. Then the correspondence $C_* : \boldsymbol{F} \to H_*(\ ;\mathbf{Q})$ is defned by :

$$C_*(X)(\Sigma n_W Eu_W) = \Sigma\, n_W\, c^M(W) \quad \text{for any variety } X,$$

where $c^M(W)$ is the Chern-Mather class of W.

(i) Clearly the correspondence C_* satisfies the above extra condition (*).

(ii) The uniqueness of such a natural transformation C_* follows from resolution of singularities and the above "smooth Chern condition" (*).

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(iii) Thus it remains to show that the above correspondence C_* is actually natural, which is proved by MacPherson [6], using his graph construction method.

By the above construction of C_*, the target $H_*(;Q)$ of C_* can be replaced by $H_{2*}(;Q)$, the even part of $H_*(;Q)$. Let us call the natural transformation C_* the DGM-Chern natural transformation . In this note we address ourselves to the following

Problem (posed by C.McCrory): *Describe all natural transformations from* F *to* $H_{2*}(;Q)$.

Related to this problem is :

Question : *Is DGM-Chern transformation* $C_*: F \to H_{2*}(;Q)$ *universal ?*

Answering both is the following conjecture:

Conjecture (made by G.Kennedy): *If* $\tau : F \to H_{2*}(;Q)$ *is a natural transformation, then there exists a sequence* $\{r_i\}_{i\geq 0}$ *of rational numbers such that* $= \sum_{i\geq 0} r_i \, C_{*i}$, *where* C_{*i} *is the projection of* C_* *to the 2i-dimensional component.* (His original conjecture is for any natural transformation $\tau : F \to H_*(;Z)$.)

§1. Some supporting evidence for Kennedy's conjecture

First of all, McCrory's problem is perhaps hinted by the classical fact that Chern classes are "essential" for characteristic classes of complex vector bundles. Let K^0 be the contravariant functor of the Grothendieck groups of complex vector bundles and let $H^{2*}(;Q)^\wedge :=$

$1 + \sum_{i\geq 1} H^{2i}(;Q)$, where $H^{2*}(X;Q)^\wedge$ is a multiplicative group with the cup product. Then a

natural transformation $cl : K^0 \to H^{2*}(;Q)^\wedge$ is nothing but a multiplicative characteristic class of complex vector bundles, and cl can be uniquely expressed as a certain formal power series of individual Chern classes $\{c_i\}_{i\geq 0}$. To be more precise,

$$cl = 1 + \sum_{i\geq 1} T_i(c_1, c_2, ..., c_i) \in Q[c_1, c_2, c_3, ...],$$

where $T := \{1, T_1(x_1), T_2(x_1,x_2), ..., T_i(x_1, x_2,..., x_i),....\}$ is a Hirzebruch's multiplicative sequence [4, 7]. (If $cl : \mathbf{K}^o \to H^{2*}(\ ;Q)$ is just a characteristic class of complex vector bundles and not necessarily multiplicative, where $\mathbf{K}^o$ and $H^{2*}(\ ;Q)$ are considered as functors into the category of sets, then cl is any formal power series of $\{c_i\}_{i\geq 0}$, i.e.,

$$cl = \sum_{i\geq 0} P_i(c_1, c_2, ..., c_i) \in Q[c_1, c_2, c_3, ...],$$

where $P_i(c_1, c_2, ..., c_i)$ is homogeneous polynomial of degree i with $\deg(c_k) = k$ [7].)

This fact can also be put in the following fashion: The multiplicative sequence T gives rise to an endomorphism :

$$\Phi_T : H^{2*}(\ ;Q)^\wedge \to H^{2*}(\ ;Q)^\wedge ,$$

which is defined by $\Phi_T(1 + \sum_{i\geq 1} x_i) := 1 + \sum_{i\geq 1} T_i(x_1, x_2 , ..., x_i)$.

Then cl is nothing but the composite of the total Chern class $c = \sum_{i\geq 0} c_i$ and the endomorphism Φ_T ;

$$cl = \Phi_T \circ c .$$

$$\begin{tikzcd} & H^{2*}(\ ;Q)^\wedge \arrow[d, "\Phi_T"] \\ \mathbf{K}^o \arrow[ur, "c"] \arrow[r, "cl"'] & H^{2*}(\ ;Q)^\wedge \end{tikzcd}$$

Thus we are led to the naive question of whether one could get a similar result for the case of F and $H_{2*}(\ ;Q)$. Endomorphisms of $H_{2*}(\ ;Q)$ which we can think of without much thought are the following simple ones : Let $\{m_i\}_{i\geq 0}$ be a sequence of rational numbers and $\pi_i : H_{2*}(\ ;Q) \to H_{2i}(\ ;Q)$ the projection. Then

$$\Psi_{\{m_i\}} := \sum_{i=1}^{\infty} m_i\pi_i : H_{2*}(\ ;Q) \to H_{2*}(\ ;Q)$$

is an endomorphism of $H_{2*}(\ ;Q)$. Thus we get a natural transformation

$$\sum_{i=0}^{\infty} m_i C_{*i} = \Psi_{\{m_i\}} C_* : F \to H_{2*}(\ ;Q) \text{ ,where } C_{*i} := \pi_i C_*$$

It turns out that there are no endomorphisms of $H_{2*}(\ ;Q)$ other than the above linear one $\Psi_{\{m_i\}}$, i.e.,

Theorem (1.1) *If $\Psi: H_{2*}(;Q) \to H_{2*}(;Q)$ is a natural transformation, then there exists a unique sequence $\{r_i\}_{i\geq 0}$ of rational numbers such that $\Psi = \sum_{i\geq 0} r_i \pi_i$.*

The proof of this theorem is unexpectedly quite simple, thanks to the following "Identity Theorem":

Identity Theorem([2, §5]) *If $\alpha: H_{2*}(;Q) \to H_{2*}(;Q)$ is a natural transformation satisfying the property that for each projective space P^n, $n = 0, 1, 2,\ldots,$*

$$\alpha([P^n]) = [P^n] + \text{homology classes of lower degrees},$$

then α is an identity transformation.

Proof of Theorem (1.1): Let $\Psi: H_{2*}(;Q) \to H_{2*}(;Q)$ be a natural transformation and consider each projective space P^n, $n = 0, 1, 2, 3,\ldots.$ Then for each non-negative integer n there exists a rational number r_n such that

$$\Psi([P^n]) = r_n[P^n] + \text{homology classes of lower degrees.}$$

Then for the sequence $\{r_n\}_{n\geq 0}$ of these rational numbers r_n's the linear form $\sum_{n\geq 0} (1- r_n)\pi_n$ is a natural transformation and thus $\Psi' := \Psi + \sum_{n\geq 0} (1- r_n)\pi_n$ is a natural transformation. Now let us "evaluate" this new natural transformation Ψ' on each projective space P^n. It is easy to see that $\Psi'([P^n]) = [P^n]$ + homology classes of lower degrees. Hence by the above "identity Theorem" it follows that $\Psi' = I$, the identity, i.e., since $I = \sum_{n\geq 0} \pi_n$,

$$\Psi + \sum_{n\geq 0} (1- r_n)\pi_n = \sum_{n\geq 0} \pi_n \text{ , i.e.,}$$

$$\Psi + \sum_{n\geq 0} \pi_n - \sum_{n\geq 0} r_n\pi_n = \sum_{n\geq 0} \pi_n \text{ ,i.e.,}$$

$$\Psi = \sum_{n\geq 0} r_n\pi_n \text{ .} \qquad \text{Q.E.D.}$$

From Theorem (1.1) we can see the following

Corollary (1.2) Kennedy's conjecture and the universality of C_* are equivalent.

Another "characteristic homology class theory " of singular varieties is Baum-Fulton-MacPherson's Riemann-Roch for singular varieties:

BFM-Riemann-Roch theorem ([1]) *Let K_0 be the covariant functor of Grothendieck groups of coherent algebraic sheaves. Then there exists a unique transformation $Td_* : K_0 \to H_{2*}(;Q)$ such that ("smooth Todd condition") $Td_*(\mathcal{O}_X) = td(T_X) \cap [X]$ for any smooth variety X, where $\mathcal{O}_X$ is the structure sheaf of X and $td(T_X)$ is the total Todd class of the tangent bundle T_X.*

This BFM-R-R transformation Td_* has the following strengthened uniqueness:

Uniqueness Theorem ([1, Chap.III]) *The BFM-R-R transformation $Td_* : K_0 \to H_{2*}(;Q)$ is the only natural transformation τ satisfying the property that for each projective space P^n, $n = 0, 1, 2, \ldots$,*

$$\tau(\mathcal{O}_{P^n}) = [P^n] + \text{homology classes of lower degrees.}$$

This uniqueness theorem follows from the above "Identity Theorem" and the following fact [1, Chap.III] :

(1.3) Td_* induces an isomorphism $K_0(X) \times Q \xrightarrow{\cong} H_{2*}(X;Q)$ for any variety X.

Now, if we consider Kennedy's conjecture for BFM-Riemann-Roch transformation Td_*, then it turns out that the conjecture is correct and the proof is very simple thanks to the above "Uniqueness Theorem", as in the proof of Theorem (1.1):

Theorem (1.4) *(The universality of BFM- Riemann-Roch transformation Td_*) The BFM-R-R transformation $Td_* : K_0 \to H_{2*}(;Q)$ is universal. To be more precise, if $\tau : K_0 \to H_{2*}(;Q)$ is a natural transformation, then there exists a unique sequence $\{r_i\}_{i \geq 0}$ of rational numbers such that $\tau = \sum_{i \geq 0} r_i Td_{*i}$, where Td_{*i} is the projection of Td_* to the 2i-dimensional component.*

It is easy to see that "Uniqueness Theorem" and the universality of Td_* are equivalent to each other. So in an analogy with the above "Uniqueness Theorem" of Td_*, we would like to pose the following conjecture, which is equivalent to Kennedy's conjecture:

Conjecture (1.5)("C_*-version" of "Uniqueness Theorem" of Td_*) *The DGM-Chern transformation $C_* : F \to H_{2*}(;Q)$ is the only natural transformation τ satisfying the property that for each projective space P^n, $n = 0, 1, 2, \ldots$,*

$$\tau(1_{P^n}) = [P^n] + \textit{homology classes of lower degrees.}$$

Remark (1.6) If "C_*-version" of (1.3) were true, i.e., C_* induced an isomorphism $F(X) \times Q \xrightarrow{\sim} H_{2*}(X;Q)$ for any variety X, then we would be done. But in general it is not the case. So resolving Conjecture (1.5) would require some other arguments and techniques.

§2 Natural transformations from F to $H_{2*}(;Q)$

First we observe the following

Proposition (2.1) *Let $\tau, \tau' : F \to H_{2*}(;Q)$ be two natural transformations. Then the following are equivalent :*

(i) $\tau = \tau'$, i.e., for any variety X $\tau(X)$ and $\tau'(X) : F(X) \to H_{2*}(X;Q)$ are identical,

(ii) $\tau(V) = \tau'(V)$ for any smooth V,

(iii) $\tau(V)(1_V) = \tau'(V)(1_V)$ for any smooth V.

(The proof is the same as that of the uniqueness of the DGM-natural transformation C_*, by using resolution of singularities; cf.[3, 6])

Let $\tau : F \to H_{2*}(;Q)$ be a natural tansformation. Then to prove Kennedy's conjecture it suffices to show the following, putting aside an attempt to prove Conjecture (1.5), because of the above proposition:

There exists a unique sequence $\{r_i\}_{i \geq 0}$ of rational numbers such that

(2.2) $$\tau(1_V) = \left(\sum_{0 \leq i \leq \dim V} r_i \, c_{\dim V - i}\right)(T_V) \cap [V] \quad \text{for every smooth variety } V.$$

At the moment we can determine only 0-dimensional and top-dimensional components of $\tau(V)(\mathbf{1}_V)$, namely we have the following

Proposition (2.3) *Let* $\tau : F \to H_{2*}(\ ;Q)$ *be a natural transformation. Then there exists a unique integer* r_0 *such that for any variety* X

$$(\tau(X)(1_V))_0 = r_0 C_{*0}(V)(1_X) = r_0 \chi(V) = (r_0 c_{dim V})(T_V) \cap [V],$$

where $(.)_i$ *means the i-dimensional component of the total homology class.*

Proposition (2.4) *Let* $\tau : F \to H_{2*}(\ ;Q)$ *be a natural transformation. Then there exists a unique integer* $r_{dim V}$ *such that for any smooth variety* V

$$(\tau(V)(1_V))_{2dim V} = r_{dim V}[V] = r_{dim V} C_{*dim V}(V)(1_V) = (r_{dim V} c_0)(T_V) \cap [V].$$

These two propositions can be proved by considering a map $V \to \mathbf{P}^0 = \{\text{a point}\}$ and a Galois (branched) covering $V \to \mathbf{P}^{\dim V}$, respectively. This hints us that one could determine the other remaining components by considering "maps" of V to the other complex projective spaces $\mathbf{P}^1, \mathbf{P}^2, \ldots, \mathbf{P}^{\dim V-1}$. But we do not know how to do it. This problem seems to be heavily related to Conjecture (1.5).

Remark (2.5): If we assume that V is a projective variety, then we can consider linear projections from V to projective spaces, which are, however, rational maps, and so there arise some problems in dealing with the constructible function 1_V.

§3 Characteristic natural transformations from F to $H_{2*}(\ ;Q)$

In the case of the DGM-natural transformation C_*, for any dimension of varieties we consider a sole total characteristic class, i.e., the total Chern class c in the "smooth condition", so this kind of condition shall be called "universal smooth condition". As a matter of fact, the natural transformation satisfying a "universal smooth condition" is $r \circ C_*$ for some rational number r [9]. On the other hand, in the case of the linear natural transformation

$\Sigma_{i\geq 0} r_i\, C_{*i}$, as shown in (2.2) in the previous section, the characteristic classes which we consider for the "smooth condition" vary dimension-wisely, thus we shall this smooth condition "dimension-wise universal smooth condition". In fact, the converse of this holds, i.e., we can show the following "characterization" of the linear natural transformation $\Sigma_{i\geq 0} r_i\, C_{*i}$:

Theorem (3.1)([10]): *Let $\{cl^{(n)}\}_{n\geq 0}$ be a sequence of characteristic classes $cl^{(n)}$'s , where for each $n \geq 0$ $cl^{(n)} = \lambda^n{}_0 + \Sigma_{1\leq i\leq n} P^n{}_i(c_1, c_2, ..., c_i)$ is a degree-n polynomial of individual Chern classes with each i-th Chern class c_i being of weight i . If $\tau : F \to H_{2*}(\,;Q)$ is a natural transformation satisfying the "dimension-wise universal smooth condition" that $\tau(V)(1_V) = cl^{(dim V)}(V) \cap [V]$ for any smooth variety V, then there exists a unique sequence $\{r_i\}_{i\geq 0}$ such that each degree-n characteristic class $cl^{(n)}$ is a linear form $\Sigma_{0\leq i\leq n} r_{n-i}\, c_i$ and $\tau = \Sigma_{i\geq 0} r_i\, C_{*i}$.*

Our proof of this theorem uses Proposition (2.1) and the following two more propositions:

Proposition (3.2) (see [7, Theorem 16.7 and a remark right after it]) *The following $p(n)$ x $p(n)$ matrix M_n whose entries are $I_k(n)$-Chern numbes of $I_j(n)$-projective spaces $\mathbb{P}^{I_j(n)}$:*

$$M_n := \left(c_{I_k(n)}[\mathbb{P}^{I_j(n)}]\right)$$

is a non-singular matrix. (Here $I_r(n)$ is a partition of n . For $I_r(n) = \{r_1, r_2, ..., r_k\}$, $\mathbb{P}^{I_r(n)}$ is defined to be $\mathbb{P}^{r_1} \times \mathbb{P}^{r_2} \times ... \times \mathbb{P}^{r_k}$.)

Proposition (3.3)([11]) *Let $m \geq n$ and let X and Y be compact complex manifolds of dimension $m - n$ and n , respectively, and let $\pi : X \times Y \to X$ be the projection. Then*

$$\pi_*\left(c_{I_k(n)}[X \times Y]\right) = \left(c_{I_k(n)}[Y]\right)[X] ,$$

where $c_{I_k(n)}[X \times Y] := c_{I_k(n)}(T_{X \times Y}) \cap [X \times Y]$.

For a smooth variety V and a natural transformation τ: F -> $H_{2*}(\ ;Q)$, $\tau(V)(\mathbf{1}_V)$ is the Poincare dual of a certain unique cohomology clas of V , by the Poincare duality. In the case when $\tau = C_*$ and $\sum_{i\geq 0} r_i\, C_{*i}$, $\tau(V)(\mathbf{1}_V)$ is the Poincaré dual of certain conditioned or conditional characteristic classes of V discussed above. So it seems natural to ask oneself what if our "smooth condition" requires simply that $\tau(V)(\mathbf{1}_V)$ is the Poincaré dual of a characteristic cohomology class $cl^V(V)$ of V. (*Warning*: Even if V and W are of the same dimension cl^V and cl^W could be different.) Let us call such a smooth condition the "individual smooth condition", and a natural transformation τ: F -> $H_{2*}(\ ;Q)$ satisfying the "individual smooth condition" shall be called a "characteristic" natural transformation. At the moment we do not know how to determin characteristic natural transformations, but we would like to pose the following

Conjecture (3.4) *: Any characteristic natural transformation* τ*:* F *->* $H_{2*}(\ ;Q)$ *must be the linear natural transformation* $\sum_{i\geq 0} r_i\, C_{*i}$.

Remark (3.5) *Let* $\tau : F \text{->} H_{2*}(\ ;Q)$ *be a characteristic natural transformation. If we restrict this natural transformation* τ *to the subcategory of varieties of dimension* ≤ 1*, then there exist integers* m_0 *and* m_1 *such that*

$$\tau = m_0 C_{*0} + m_1 C_{*1}.$$

Remark (3.6) *Let* $\tau : F \text{->} H_{2*}(\ ;Q)$ *be a characteristic natural transformation with* $\tau(V)(1_V) = cl^V(V) \cap [V]$. *Then there exists a sequence* $\{m_i\}_{i \geqq 0}$ *of integers* m_i *such that for* *any smooth variety* V *of each dimension* n

$$(cl^V(V))_0 = m_n c_0(V) \text{ and } (cl^V(V))_{2n} = m_0 c_n(V).$$

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