数理解析研究所講究録
807巻 1992年 163-187

FP(AUG/92):

Rank two reflexive sheaves which are constructed from the prime field F_p

Nobuo Sasakura Yoichi Enta and Masatake Kagesawa
(Tokyo Metropolytan University)

The purpose of this manuscript is to construct rank two reflexive sheaves on the projective spaces from the prime field F_p ($p \equiv 1 (\mathrm{mod}.4)$). The central part is some combinatorics which are related to a certain graph, which is obtained from the theory of quadratic residue. This manuscript may be regarded also an introduction to our theory of '*construction of reflexive sheaves from divisor configrations*, *cf.* [Sa-1,2,3] .

The present note requires some refinements for the final publication. Also references are far from adequate.

1. Let p be a prime number of the form $p = 4q + 1$ with a positive integer $q (p = 5,13,17,29,37,...)$, and let $F_p = Z/pZ$ be the prime field of characteristic p. Moreover, let $F_p^* = F_p - \{0\}$ denote the multiplicative group of F_p and let ρ be a generator (primitive root) of F_p^*. For an element $i \in F_p$ we set:

$$S_i^+ (\mathrm{resp}.S_p^-) = \{j \in F_p - \{i\} : j \sim i (\mathrm{resp}.j \not\sim i)\},$$

where $j \sim i$ means that $(i-j) = \rho^{2a}$ with an element $a \in Z/(p-1)Z$, i.e.,$(i-j)$ is a quadratic residue). Note that '$\sim$' is not, in general, an equivalence relation.

Let $P_{p-1} = P_{p-1}(C)$ be the $(p-1)$-dimensional projective space

with homogeneous coordinates z_i, $i \in F_p$. Writing $X = P_{p-1}$ we set $X_i^1 = Z(z_i)$, $i \in F_p$, and $X^1 = \cup_{i \in F_p} X_i^1$. We write Ω for the line bundle $\mathcal{O}[X^1] \simeq \mathcal{O}(p)$, where $\mathcal{O}$ denote the structure sheaf of X and $\mathcal{O}(p)$ is the p-times twist of $\mathcal{O}$ by the hyperplane bundle $\mathcal{O}(1)$.

Now to each hyperplane X_i^1, $i \in F_p$, we attach a two term vector $\in \Gamma(\mathcal{O}_i(2q))^{\oplus 2}$, $\mathcal{O}_i$ being the structure sheeaf of X_i^1, as follows:

$g_i = {}^t(\ {}_j z_j\ ,\ {}_k z_k)$, where j(resp.k) satisfies $j \sim/ i$(resp.$k \sim i$),

Writing $\mathcal{G}_i$ for $\mathcal{O}_i(2q+1)$, the multiplication by g_i defines an $\mathcal{O}_i$-morphism:

$$g_i : \mathcal{G}_i \xrightarrow{\otimes g_i} \Omega_i^{\oplus 2}, \text{ with } \Omega_i = \text{the restriction of } \Omega \text{ to } X_i^1 .$$

Let $\delta = \oplus_{i \in F_p} \delta_i : \Omega^{\oplus 2} \to C^1(\Omega^{\oplus 2}) := \oplus_{i \in F_p} \Omega_i$ be the coboundary morphism, with δ_i = the restriction morphism: $\Omega \to \Omega_i$. Moreover, we set $g = \oplus_i g_i : \mathcal{G}^1 := \oplus_i \mathcal{G}_i \to C^1(\Omega^{\oplus 2}) = \oplus_i \Omega_i^{\oplus 2}$, with $i \in F_p$. Now define an $\mathcal{O}$-submodule of $\Omega^{\oplus 2}$ as follows:

$$\mathfrak{E} = \delta^{-1}(g(\mathcal{G}^1)).$$

Clearly $\mathfrak{E}_{|X-X^1} = \Omega^{\oplus 2}_{|X-X^1}$. Moreover, we have:

Lemma 1. *$\mathfrak{E}$ is reflexive(i.e.,$\mathfrak{E} \simeq \mathfrak{E}^{**}$). Moreover $\mathfrak{E}$ is locally free on* $(X - X^2)$,cf.[Sa-1].

Here we set:

$$X^2 = \cup_{i \neq j} X_{ij}^2 \text{ with } X_{ij}^2 = X_i^1 \cap X_j^1 .$$

The main task of the remainder of the present paper is to examine the structure of $\mathfrak{E}$ on X^2. Concerning this we remark the following:

An element $\zeta \in \Omega_p^{\oplus 2}$, $p \in X$, is in $\mathfrak{E}_p$, if and only if there is an element $\sigma_i \in \mathcal{G}_{i,p}$ satisfying

(1) $\qquad \zeta_{|i} = g_i \otimes \sigma_i$ for each $i \in F_p$.

If $p \in/ X_i^1$, then the both sides = 0. Note that, by restricting (1) to

X^2_{ij} we have the coboundary relation:

(2) $\quad (g_i \otimes \sigma_i)|_{ij} = (g_j \otimes \sigma_j)|_{ij}$.

Conversely, if a collection $\{\sigma_i\}_i$, $\sigma_i \in \mathfrak{S}_{i,p}$ satisfies (2), then there is an element $\zeta \in \mathfrak{L}^{\oplus 2}$ satisfying (1). Thus the determination of $\mathfrak{O}_p$-module $\mathfrak{E}_p$ is reduced to that of a collection $\{\sigma_i\}_i$ satisfying (2).

2. **Multiplier**. Take elements $i \neq j \in F_p$. We set:

S^{++}_{ij}(resp.S^{--}_{ij}) = $\{k \in F_p - \{i,j\} : k \sim i$ and $k \sim j$(resp.$k \not\sim i$, $k \not\sim j$)$\}$.

Thus subsets S^{+-}_{ij} and S^{-+}_{ij} of F_p is defined similarly to the above. Now we restrict the vectors g_i and g_j on X^2_{ij} . Clearly we have:

Proposition 2.1. $\#S^{++}_{ij}$, $\#S^{--}_{ij}$, $\#S^{+-}_{ij}$ *and* S^{-+}_{ij} *are indepedent of* i,j. *Actually*

$\#S^{-+}_{ij} = \#S^{+-}_{ij} = q$, *and* $(\#S^{++}_{ij}, \#S^{--}_{ij}) = (q-1,q)$ *or* $(q,q-1)$,

according to whether $i \sim j$ *or* $i \not\sim j$.

Proof. This follows from an elementary consideration. Note that if $k \neq l \in F_p$ satisfies: $(i-k) = (j-l) + \rho^{2a}$, $a \in Z/(p-1)Z$, then $\#S^{\alpha\beta}_{ij} = S^{\alpha\beta}_{jk}$, where $\alpha, \beta = +$ or $-$. From this it suffices to consider the pair $(i,j) = (0,1), (0,\rho)$.

Now define $\mathfrak{O}_{ij}$-invertible sheaves as follows:

$\mathfrak{M}_{i,j} = \mathfrak{O}_{ij}(q)$, $\mathfrak{M}_{j,i} = \mathfrak{O}_{ij}(q)$ *and* $\mathfrak{G}_{ij} = \mathfrak{O}_{ij}(q)$.

These three sheaves voincide. But the role of them differ,cf.[Sa-1]. Define ellemtns $g_{ij} \in \Gamma(\mathfrak{G}_{ij})^{\oplus 2}$ and $\psi_{i,j} \in \Gamma(\mathfrak{M}_{i,j})$, $\psi_{j,i} \in \Gamma(\mathfrak{M}_{j,i})$ as follows:

$g_{ij} = {}^t(\coprod_k z_k, 0)$, $\psi_{i,j} = \coprod_l z_l$ and $\psi_{j,i} = \coprod_m z_m$, if $i \sim j$,

where according to whether $i \sim j$ or $i \not\sim j$, k, l and m run through those elements satisfying

$k \in S^{--}_{ij}$, $\quad l \in S^{-+}_{ij}$ and $m \in S^{+-}_{ij}$ or $k \in S^{++}_{ij}$, $l \in S^{+-}_{ij}$ and $m \in S^{-+}_{ij}$.

Now we restrict g_i and g_j to X^2_{ij} Then, according to whether $i \sim j$ or $i \not\sim j$, the second or first component of $g_{ij} = 0$. From the definition of g_{ij} and $\psi_{i,j}, \psi_{j,i}$, we have:

Lemma 2.2. *The vectors* g_i *and* g_j *are decomposed as follows:*

$$g_{i|ij} = g_{ij} \otimes \psi_{i,j} \quad \textit{and} \quad g_{j|ij} = g_{ij} \otimes \psi_{j,i} .$$

Next, take a point $x \in X^2_{ij}$, and take elements $\sigma_\sigma \in \mathfrak{S}_{\alpha,x}$, $\alpha = i,j$, satisfying the coboundary relation (2), § 1. Then, from **Lemma 2.2**

$$\sigma_{i|ij} \otimes \psi_{i,j} = \sigma_{j|ij} \otimes \psi_{j,i} .$$

Thus we obviously have:

Lemma 2.3. *The elements* σ_i *and* σ_j *are decomposed as follows:*

$\sigma_{i|ij} = \sigma_{ij} \otimes \psi_{j,i}$ *and* $\sigma_{j|ij} = \sigma_{ij} \otimes \psi_{i,j}$ *with a unique element* $\sigma_{ij} \in \mathfrak{S}_{ij,p}$, *where we set:* $\mathfrak{S}_{ij} = \mathfrak{D}_{ij}[q+1]$.

Next we introduce the following

Definition 2.2. (i) We say the '*matrix*' $\Psi := (\psi_{i,j})_{i,j \in F_p}$ is the *multiplier of* $\mathfrak{E}$.

Here we understand that $\psi_{i,i} = \phi$(empty set), $i \in F_p$.

The multiplier Ψ is regarded as a *boundary value of* $\mathfrak{E}$ at X^2. In our general frame work,cf.[Sa], the notion of the multiplier plays a central role. Theoretically the multiplier Ψ should be takn as the starting point, and the vectors $g_i, i \in F_p$ should be regarded as its invariant.cf.[Sa-3]. Next we set $\varphi_{i,j} = \psi_{j,i}$, and form a matix

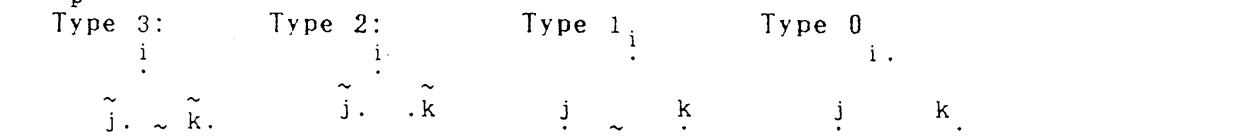
$$\Phi^2 = (\varphi_{i,j})_{i,j \in F_p} \text{ with } \varphi_{i,i} = 0.$$

The matrix Φ is essentially a transpose of Ψ. But it is good to regard that the role of Φ is independent of Ψ.

3. Inductive structure. The purpose here is to give a similar decomposition law to (3) for codimension three and four. First

we discuss it at codimension three. Take a subset $I = (i,j,k)$ of F_p with $\# I = 3$. We form a graph by using $\sim$, cf. § 2.

Type 3: Type 2: Type 1 Type 0



Clearly, the above exhausts all possible graphes(by arranging indices). We use the above normal form. Note that Type 3 and Type 0 (resp.Type 2 annd Type 1) are dual in the sense that the later is obtained from the former by replacing '$\sim$ ' by ' $\sim/$'. The arguments are done dually. We will concern Type 3 and Type 2.

Now, for each type, the foloowing 3x3-matrices play important roles:

$$(\gamma\Psi)_I = \begin{array}{c} \\ i \\ j \\ k \end{array} \begin{array}{c} \begin{array}{ccc} ij & ik & jk \end{array} \\ \begin{pmatrix} \psi_{i,j|I} & -\psi_{i,k|I} & 0 \\ \psi_{j,i|I} & 0 & -\psi_{j,k|I} \\ 0 & \psi_{k,i|I} & -\psi_{k,j|I} \end{pmatrix} \end{array}$$

The matrix $(\delta\Phi^2)_I$ is defined similarly, by changing ψ to φ. The above may be ragarded as a coboundary of Ψ and Φ, cf.[Sa-], and the symbol 'δ ' is used. These matrices are regarded ,respectively, as $\mathfrak{O}_I$-morphism: $(\mathfrak{S}_{ij|I} \oplus \mathfrak{S}_{ik|I} \oplus \mathfrak{S}_{jk|I}) \to (\mathfrak{S}_{i|I} \oplus \mathfrak{S}_{j|I} \oplus \mathfrak{S}_{k|I})$ (resp. $(\mathfrak{S}_{ij|I} \oplus \mathfrak{S}_{ik|I} \oplus \mathfrak{S}_{jk|I}) \to (\mathfrak{S}_{i|I} \oplus \mathfrak{S}_{j|I} \oplus \mathfrak{S}_{k|I})$). We examine the kernel of these morphisms.

Proposition 3.1. *For type* 3, *no element* $\psi_{\alpha,\beta|I}$ (*resp.* $\varphi_{\alpha,\beta|I}$) , $\alpha \neq \beta \neq 0$. *For type 2,* $\psi_{j,i|I}$, $\psi_{k,i|I}$ and $\varphi_{i,j|I}$ and $\varphi_{i,k|I} = 0$.

Proof. This is checked combinatorially. For type 3, the indices i,j,k are treated evenly.

$|\psi_{i,j}| = S^{-+}_{ij}$ (resp. $|\varphi_{i,j}| = S^{+-}_{ij}$) *and does not contain* k.

For type 2,

$|\psi_{j,i}| = |\varphi_{i,j}| = S^{+-}_{ij}$ k(resp. $|\psi_{k,i}| = |\varphi_{i,k}| = S^{+-}_{ik}$ j), and

$|\psi_{i,j}| = |\varphi_{j,i}| = S^{-+}_{ij}$ k(resp. $|\psi_{i,k}| = |\varphi_{k,i}| = S^{+-}_{ik}$ j).

$|\psi_{j,k}| = |\varphi_{k,j}| = S^{-+}_{jk}$ i(resp. $|\psi_{k,j}| = |\psi_{j,k}| = S^{+-}_{kj}$ i) q.e.d.

Proposition 3.2. *det*$(\delta\Psi)_I$ = *det*$(\delta\Phi)_I$ = 0.

Proof. For type 2, this is clear. For type 3, we have:

$$|\psi_{i,j}|\cup|\psi_{j,k}|\cup|\psi_{k,i}| = S^{-+}_{ij}\cup S^{-+}_{jk}\cup S^{-+}_{ki} = S^{+--}_{ijk}\cup S^{-++}_{ijk}\cup S^{-+-}_{ijk}\cup S^{+-+}_{ijk}\cup S^{++-}_{ijk}$$
$$\cup S^{--+}_{ijk} = |\psi_{j,i}|\cup|\psi_{i,k}|\cup|\psi_{k,j}| .$$

Note that $\det(\delta\Phi)_I = -\det(\delta\Psi)_I$. q.e.d.

The following matrices are obtained from $(\delta\Psi)_I$,.., by replacing $\psi_{i,j|I}$ by $|\psi_{i,j|I}|$.(When the former = 0, we set the later is 0.)

Type 3

	ij	ik	jk
i	S^{-+}_{ij}	S^{-+}_{ik}	
j	S^{+-}_{ji}		S^{-+}_{jk}
k		S^{+-}_{ik}	S^{-+}_{kj}

$(\delta\Psi)_I$

Type 2:

	ij	ik	jk
i	S^{-+}_{ij}	S^{-+}_{ik}	
j	0	0	S^{+-}_{jk}
k		0	S^{-+}_{jk}

$(\delta\Psi)_I$

	ij	ik	jk
i	0	0	
j	S^{-+}_{ij}		S^{-+}_{jk}
k		S^{-+}_{ik}	S^{+-}_{jk}

$(\delta\Phi)_I$

For type 3, the corresponding fact to $(\delta\Phi)_I$ is obtained by replacing + and -. Next we consider the following set theoretical equation for each $\alpha \in I$:

$$|\psi_{\alpha,\beta|I}|\cup S_{\alpha\beta,I}(\Psi) = |\psi_{\alpha,\gamma|I}|\cup S_{\alpha\gamma,I}(\Psi)|, \text{ with } \beta,\gamma \in I - \{\alpha\} .$$
$$|\varphi_{\alpha,\beta|I}|\cup S_{\alpha\beta,I}(\Phi) = |\varphi_{\alpha,\gamma|I}|\cup S_{\alpha\gamma,I}(\Phi) .$$

where the set $S_{\alpha\beta,I}(I)$ is a set whose element is in F_p,see below for the precise meaning. We find $(S_{ij,I}(\Psi),S_{ik}(\Psi),S_{jk}(\Psi))$ satisfying the equation and is minimal concering the incluison relation(and similarly for Φ). For type 2, we understand that

$S_{jk,I}(\Psi) = \infty$ and we regard that the equation for i-row holds automatically.

Proposition 3.3. *For type* 3, *take a pair* $(\alpha,\beta) \subset I$. *Then*

$$S_{\alpha\beta,I}(\Psi) = S^{--+}_{\alpha\beta\gamma} \quad \textit{and} \quad S_{\alpha\beta,I}(\Phi) = S^{++-}_{\alpha\beta\gamma} \quad \textit{with} \ \{\gamma\} = I-[\alpha,\beta].$$

For type 2,

$$S_{i\alpha,I}(\Psi) = S^{--+}_{i\alpha\beta}, \ (\alpha,\beta) = (j,k) \ \textit{or} \ (k,j) \ ,$$

$$S_{i\alpha,I}(\Phi) = 2S^{--+}_{i\alpha\beta} \cup S^{-+-}_{i\alpha\beta} = S^{-+}_{\alpha\beta} \cup S^{--+}_{i\alpha\beta}, \text{ with } (\alpha,\beta) = (j,k) \ \textit{or} \ (k,j),$$

$$S_{jk,I}(\Phi) = S^{-++}_{ijk} \cup S^{-+-}_{ijk} \cup S^{--+}_{ijk} = S^{-}_{i} - S^{---}_{ijk} \ .$$

Here

$$S^{+++}_{ijk}(\text{resp. } S^{+++}_{ijk}) = \{l \in F_p - \{i,j,k\} : l \sim i,j,k(\textit{resp. } l \sim / i,j,k)\}.$$

The sets S^{+--}_{ijk}, S^{--+}_{ijk},... are understood similarly.

Proof. For type 3, let $S_\alpha(\Psi) = |\psi_{\alpha.\beta|I}| \cap |\psi_{\beta,\alpha|I}|$, and we replace $|\psi_{\alpha,\beta|I}|$ by $|\psi_{\alpha,\beta|I}| - S_\alpha(\Psi)$. The equation is not changed. Then we have $S_{\alpha\beta,I}(\Psi) = (|\psi_{\alpha,\gamma|I}| - S_\alpha(\Psi)) \cup (|\psi_{\beta,\gamma|I}| - S_\beta(\Psi))$, cf. Appendix. The case for type 2 is a modification of the above. q.e.d.

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Now we define $\mathcal{O}_I$-invertible sheves as follows:

$$\mathfrak{M}_{\alpha\beta,I} = \mathcal{O}_I(m_{\alpha\beta,I}), \text{ with } m_{\alpha\beta,I} = \# S_{\alpha\beta,I}(\Psi)$$

$$\mathfrak{N}_{\alpha\beta,I} = \mathcal{O}_I(n_{\alpha\beta}), \text{ with } n_{\alpha\beta} = \#S_{\alpha\beta,I}(\Phi)$$

where $(\alpha,\beta) \subset I$ and $\{\gamma\} = I - \{\alpha,\beta\}$.

(If I is of type 2, we understnd that $\mathfrak{N}_{jk,I} = 0$.

Moreover, we set:

$$\mathfrak{G}_I = \mathcal{O}_I(m_I) \simeq \mathfrak{G}_{\alpha\beta|I} \otimes \mathfrak{M}^*_{\alpha\beta,I} \quad \text{for any } (\alpha,\beta) \subset I,$$

$$\mathfrak{G}_I = \mathcal{O}_I(n_I) \simeq \mathfrak{G}_{\alpha\beta|I} \otimes \mathfrak{N}^*_{\alpha\beta,I} \quad \text{for any } (\alpha,\beta) \subset I,$$

(If I is of type 2, then we omit $(\alpha\beta)=(jk)$ from the first definition.)

That the right hand sises in independent of the choice of indices follow from the definition of $S_{\alpha\beta,I}(\Psi)$,...,cf.~~also Appendix~~. The

explicit form of m_I and n_I are computed from the above equation:

$$m_I = q - \#S_{\alpha\beta,I}(\Psi) = q - S^{--+}_{\alpha\beta\gamma}(=S^{--}_{\alpha\beta} - S^{--+}_{\alpha\beta\gamma}) = S^{---}_{\alpha\beta\gamma} .$$

$$n_I = (q+1) - S^{++-}_{\alpha\beta\gamma} \quad \text{for type 3}$$

$$n_I = (q+1) - \#(S^{-}_{i} - S^{---}_{ijk}) = (q+1) - (2q-\#S^{---}_{ijk})) = \#S^{---}_{ijk} - (1-q).$$

Define sections $\psi_{\alpha\beta,I} \in \Gamma(\mathfrak{M}_{\alpha\beta,I})$ and $\varphi_{\alpha\beta,I} \in \Gamma(\mathfrak{N}_{\alpha\beta,I})$ as follows:

$\psi_{\alpha\beta,I} = \coprod_{\mu} z_{\mu}$ and $\varphi_{\alpha\beta,I} = \coprod_{\nu} z_{\nu}$, with $\mu \in S_{\alpha\beta,I}(\Psi)$ and $\nu \in S_{\alpha\beta,I}(\Phi)$.

(If I is of type 2, then we understand that $\psi_{jk,I} = 0$.)

From the above defintion we have:

Lemma 3.4. *The morphisms*

$$(\psi_{ij,I} \oplus \psi_{ik,I} \oplus \psi_{jk,I}) : \mathfrak{G}_I \to (\mathfrak{G}_{ij|I} \oplus \mathfrak{G}_{ik} \oplus \mathfrak{G}_{jk|I})$$

$$(\varphi_{ij,I} \oplus \varphi_{ik,I} \oplus \varphi_{jk,I}) : \mathfrak{S}_I \to (\mathfrak{S}_{ij|I} \oplus \mathfrak{S}_{ik|I} \oplus \mathfrak{S}_{jk|I})$$

gives isomorphism between $\mathfrak{G}_I$ *and ker* $(\delta\Psi)_I$ *(resp.* $\mathfrak{S}_I$ *and* $(\delta\Phi)_I$*).*

Decomposition law . Note that we have the following coboundarey relation:

$$g_{\alpha|I} = (g_{\alpha\beta}\psi_{\alpha,\beta})_{|I} = (g_{\alpha\gamma}\psi_{\alpha,\gamma})_{|I}, \ \alpha \in I \text{ and } \{\beta,\gamma\} = I - \{\alpha\}$$

and we have:

$$(\delta\Psi)_I \, {}^t(g_{ij|I}, g_{ik|I}, g_{jk|I}) = 0.$$

Lemma 3.5. *The vectors* $g_{\alpha\beta}$ *are decomposed as follows:*

$g_{\alpha\beta|I} = g_I \otimes \psi_{\alpha\beta,I}$ *for any* $(\alpha,\beta) \subset I$, *where* g_I *is a (unique) element of* $\Gamma(\mathfrak{G}_I^{\oplus 2})$.

Note that, in the case of type 2, $g_{jk|I} = 0$. *In the other cases,* $g_{\alpha\beta|I} \neq 0$.

Explicitly, regardless of type 3 or type 2, we have:

$$g_I = {}^t(g_I, 0) \quad \text{where } |g_I| = S^{---}_{ijk} .$$

Next take a point $x \in X_I$ and elements $\sigma_\alpha \in \mathfrak{S}_{\alpha,x}$, $\alpha \in I$, satisfying the coboundary relation,cf.(2),§ 2.

$$(g_\alpha \otimes \sigma_\alpha)|_{\alpha\beta} = (g_\beta \otimes \sigma_\beta)|_{\alpha\beta} \text{ for any } (\alpha,\beta) \in I.$$

By the decomposition law at codimension two, we write:

$$\sigma_{\alpha|\alpha\beta} = \sigma_{\alpha\beta} \otimes \varphi_{\alpha,\beta} \text{ with a unique element } \sigma_{\alpha\beta} \in \mathfrak{S}_{\alpha\beta,p}.$$

Thus we see that ${}^t(\sigma_{ij|I}, \sigma_{ik|I}, \sigma_{jk|I})$ is in the kernel of $(\delta\Phi)_I$, and, for any $(\alpha,\beta) \subset I$.

Lemma 3.5. *The elements $\sigma_{\alpha\beta}$ are decomposed as follows:*

$$\sigma_{\alpha\beta|I} = \sigma_I \otimes \varphi_{\alpha\beta,I}, \text{ with a unique element } \sigma_I \in \mathfrak{S}_{I.x}.$$

4. Inductive structure 2. Here we are concerned with the case of codimension four. The situation is much subtle then the case of codimension three. We fix a subset $I = (i,j,k,l)$ of F_p with $\# I = 4$. As in § 3, we first consider the graph formed by '$\sim$ ':

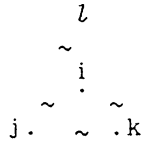
Type 6:

$\alpha \sim \beta$ for any pair $(\alpha,\beta) \subset I$

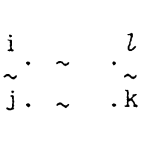
Type 5:

$\alpha \sim \beta$ for any pair $(\alpha,\beta) \subset I$ except one pair, (k,l)

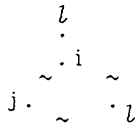
Type 4a:



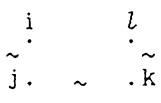
Type 4b:



Type 3a:



Type 3c:



We see easily that the above graphes and their dual(obtained by replacing $\sim$ and $\not\sim$) exhaust all graphs with arrangements of indices. We use the above normal forms. (Type 3b is used for the dual of **type** 3a,cf.[Sa-].) The following matrices are central subjects in § 4:

$$(\delta\Psi)_I = \begin{array}{c|cccc} & ijk & ijl & ikl & jkl \\ \hline ij & \psi_{ij,ijk|I} & -\psi_{ij,ijl|I} & 0 & 0 \\ ik & \psi_{ik,ijk|I} & 0 & -\psi_{ik,ikl|I} & 0 \\ il & 0 & \psi_{il,ijl|I} & 0 & 0 \\ jk & \psi_{jk,jkl|I} & 0 & 0 & -\psi_{jk,jkl|I} \\ jl & 0 & \psi_{jl,ijl|I} & 0 & -\psi_{jl,jkl|I} \\ kl & 0 & 0 & \psi_{kl,ikl|I} & -\psi_{kl,jkl|I} \end{array}$$

The matrix $(\delta\Phi)_I$ is defined similarly by changing ψ to φ.
The behaviour of such matrices are various. The discussion is done according to the type of the graph. We make some preparations. First we list all coefficents $\psi_{\alpha,\beta}$ which vanish on X_I.(Or equivalently $\gamma \in |\psi_{\alpha,\beta}|$ with an element $\gamma \in I - \{\alpha,\beta\}$.)

Proposition 4.1. (i) *For type* 6, *no element* $\psi_{\alpha,\beta}$ *vanishes on* X_I.
(ii) *For type* 5, $\psi_{\alpha,\beta}$, *with* $\alpha \in \{k,l\}$ *and* $\beta \in \{i,j\}$, *vanish on* X_I.
(iii) *For type* 4b, *all* $\psi_{\alpha,\beta}$ *vanish, where* $\alpha \sim \beta$.
(iv) *For type* 4a, $\psi_{\alpha,i}$, $\alpha = j,k,l$, *and* $\psi_{\beta,l}$, $\beta = j,k$, *vanish.*
(v) *For type* 3c, $\psi_{i,l}$, $\psi_{l,i}$, $\psi_{j,k}$, $\psi_{k,i}$ *and* $\psi_{i,j}$, $\psi_{l,k}$ *vanish.*
(vi) *For type* 3a, $\psi_{\alpha,l}$, $\alpha = i,j,k$, *vanish.*

Type 5:

	i	j	k	l
i				
j				
k	l	l		
l	k	k		

Type 4a:

	i	j	k	l
i				
j	l			k
k	l			j
l	jk			

Type 4b:

	i	j	k	l
i		k		k
j	l		l	
k		i		i
l	j		j	

Type 3a:

	i	j	k	l
i			jk	
j			ik	
k			ij	
l				

Type 3c:

	i	j	k	l
i		k		j
j			l	i
k	l	i		
l	k		j	

This table indicates all the cases where '$\gamma \in |\psi_{\alpha,\beta}|$' *occur*. Also in such a case, we list all elements $\gamma \in I - \{\alpha,\beta\}$ satisfying the relation. Thus, for type 3a, $(\beta,\gamma) \in |\psi_{\alpha,l}|$ *for* any permutation (α,β,γ) of (j,k,l).

Proof. Take elements α,β and $\gamma \in I$ satisfying $\gamma \in |\psi_{\alpha,\beta}|$. According to whether $\alpha \sim \beta$ or not, the graph for (α,β,γ) is as follows:

$$\begin{tikzcd} \alpha \arrow[r, dash, "\sim"] \arrow[dr, dash, "\sim"'] & \beta \\ & \gamma \end{tikzcd} \qquad \begin{tikzcd} \alpha \arrow[dr, dash, "\sim"'] & & \beta \\ & \gamma & \end{tikzcd}$$

In each case check all (α,β,γ) satisfying these graphes. q.e.d.

Next we check the coefficients of $(\delta\Psi)_I$ and $(\delta\Psi)_I$ which = 0.

Next we examine $(\alpha\beta,\alpha\beta\gamma)$-coefficients of $(\delta\Psi)_I$ and $(\delta\Phi)_I$ which vanish on X_I.

Proposition 4.2. *The following lists all zero terms of the* $\psi_{\alpha\beta,\alpha\beta\gamma|I}$ *of the matrix* $(\delta\Psi)_I$:

(i) *No element for type* 6, *the elements in* kl-*row for type* 5, *the elements* ik- *and* jl-*rows for type* 4b.

(ii) (jk,jkl)-, (jl,ijl)-*and* (kl,ikl)-*part for type* 4a.

(ij,ijl)-, (ik,ikl)- *and* (jk,jkl)-*parts for type* 3a.

(iii) *All elements except for the ones in* il *and* jk-*rows for type* 3c.

Proof. By chaking the cases.

Type 4b:

	ijk	jkl	kli	lij
ij	.			.
jk	.	.		
kl		.	.	
li			.	.

Type 3c:

	ijk	ijl	ikl	jkl
il		.	.	
jk	.			.
ij	l	0		
ik	0		j	
jl		k		0
kl			0	i

Type 4a:

	ijk	ijl	ikl	jkl
ij	.	.		
ik	.		.	
il		.	.	
jk	l			0
jl		0		.
kl			0	.

Type 3a:

	ijk	ijl	ikl	jkl
ij	.	0		
ik	.		0	
jk	.			0
il		.	.	
jl		.		.
kl			.	.

(*For type* 5, $\psi_{kl,\alpha kl}=0$, α = i,j. *Also, for type* 4b, $\psi_{ik,\alpha ik}$,α=j,l, *and* $\psi_{jl,jl\beta}$ = 0, β = i,k.

Proof. Take $(\alpha,\beta,\gamma) \subset$ I. Assume that the graph of them is of type 3. Then $\delta\neq\alpha,\beta,\gamma \in$ I is contained in $|\psi_{\alpha\beta,\alpha\beta\gamma}| = S^{--+}_{\alpha\beta\gamma}$ if and only if

(i) $\overset{\alpha.}{\underset{\beta.}{\sim}}\ \underset{\sim}{\overset{\sim}{\gamma.}}\ \underset{\sim}{\delta.}$ (This occurs only for type 4a: with $\delta = l,\gamma$ = i and $(\alpha\beta)$ is a permutation of (jk).

This case occurs only for Type 4a: $\delta \sim l$, γ = i and $(\alpha\beta)$ =(jk).

If (α,β,γ) is of type 2: $\beta \sim \alpha \sim \gamma$, then:

(ii) $\psi_{\beta\gamma,\alpha\beta\gamma} = 0$. This happens for type 5, type 4b, type 4a and type 3c.

Moreover, $\delta \in |\psi_{\alpha\beta,\alpha\beta\gamma}| = S^{--+}_{\alpha\beta\gamma}$ if and only if

(iii) $\begin{smallmatrix} & .\alpha & \\ .\beta\sim & & \sim \\ & .\gamma & \end{smallmatrix} \sim \delta.$ (This happens for only type 3c.)

Thirdly assume that (α,β,γ) is of type 1. $\beta \sim \gamma$. Then

(iv) $\psi_{\beta\gamma,\alpha\beta\gamma} = 0$. (This happens for type 4a, type 3a and type 3c.)

Moreover, $\delta \in |\psi_{\alpha\beta,\alpha\beta\gamma}| = S^{++-}_{\alpha\beta\gamma}$ if and only if

(v) $\delta. \begin{smallmatrix} \sim.\alpha & \\ \sim.\beta & \sim .\gamma \end{smallmatrix}$ (This occurs for type 3c.)

The proposition is shown by checking (i) $\sim$ (v). q.e.d.

Propsittion 4.3. *The rank of* $(\delta\Psi)_I \leqq 3$. *The rank* $= 2$ *if and only if* I *is of type* 3c.

Proof. The last part is clear. The first part is checked using arguments of Appendix. Except type 4b, it suffices to check the vanishing of the determinant of 3x3-matrix(α-matrix),cf.Apeendix. Such a matrix is of the similar form in § 3, and the proof itself is reduced to the one in § 3. For type 4b, it suffices to check:

$$S^{--+}_{ijk} \dot\cup S^{--+}_{jkl} \dot\cup S^{--+}_{kli} \dot\cup S^{--+}_{lij} = S^{+--}_{ijk} \dot\cup S^{+--}_{jkl} \dot\cup S^{+--}_{kli} \dot\cup S^{+--}_{lij}$$

since $|\psi_{ij,ijk}| = S^{--+}_{ijk}, \ldots$ The check of this is an exicise. q.e.d.

As in § 3 we regard $(\delta\Psi)_I$ as an $\mathfrak{O}_I$-morphism:

$$(\oplus_{\alpha\beta\gamma}\mathfrak{G}_{\alpha\beta\gamma|I}) \to (\oplus_{\alpha\beta}\mathfrak{G}_{\alpha\beta|I})$$

and we analyze the kernel of this morphism. The case of type 3c is dealt independetly. We first discuss the other cases. As in § 3, we define $S_{\alpha\beta\gamma,I}(\Psi)$ to be the subset(with multiplicity) of F_p, which is characterized to be the smallest one among those satisfying

$|\psi_{\alpha\beta,\alpha\beta\gamma|I}| \cup S_{\alpha\beta\gamma,I}(\Psi) = |\psi_{\alpha\beta,\alpha\beta\delta|I}| \cup S_{\alpha\beta\delta}(\Psi)$ *for any* $\alpha,\beta \in I$, *with* $\{\gamma,\delta\} = I - \{\alpha,\beta\}$.

(if the $\alpha\beta$-row = 0, then we understand that this holds automatically. For type 4a and type 3a we understand that

$$S_{jkl}(\Psi) = \infty \quad \textit{and} \quad S_{ijk}(\Psi) = \infty .$$

For type 6 and type 5 we consider similar equations for the ij- ik- and il-rows, and it is seen that the solution for such rows is also the solution for the original $(\delta\Psi)_I$. For type 4a and trype 3a, the problem is reduced to the type treated in § 3. The case of type 4b should be treated newly.The following is checked easily by the procedure as above:

Proposition 4.4. (i) *For type* 6, *type* 5 *and type* 4b,

$S_{\alpha\beta\gamma,I}(\Psi) = S^{---+}_{\alpha\beta\gamma\delta}$ *for any permutation* $(\alpha,\beta,\gamma,\delta)$ *of* (i,j,k,l).

(ii) *For type* 4a,

$S_{\alpha\beta\gamma,I}(\Psi) = S^{---+}_{\alpha\beta\gamma\delta}$ *if* $(\alpha\beta\gamma) \neq (jkl)$.

(iii) *For type* 3a,

$S_{\alpha\beta\gamma,I} = S^{+++-}_{\alpha\beta\gamma\delta}$ *if* $(\alpha\beta\gamma) \neq (ijk)$.

Simialarly to § 3 we define $\mathcal{O}_I$-invertible sheaves as follows:

$\mathfrak{M}_{\alpha\beta\gamma,I} = \mathcal{O}_I(m_{\alpha\beta\gamma,I})$ *with* $m_{\alpha\beta\gamma,I} = \# S_{\alpha\beta\gamma,I}(\Psi)$

for any $(\alpha\beta\gamma) \subset I$, *except the cases*: $(\alpha\beta\gamma) = (jkl)$ *for type* 4a *and* $(\alpha\beta\gamma) = (ijk)$ *for type* 3a.

In these two exceptional cases

$\mathfrak{M}_{\alpha\beta\gamma,I} = 0$. *Also we set:*

$\mathfrak{G}_I = \mathfrak{G}_{\alpha\beta\gamma,I} \otimes \mathfrak{M}^*_{\alpha\beta\gamma,I}$ *for any* $(\alpha\beta\gamma) \subset I$ *such that* $\mathfrak{M}_{\alpha\beta\gamma,I} \neq 0$.

Moreover, define an element $\psi_{\alpha\beta\gamma,I} \in \Gamma(\mathfrak{M}_{\alpha\beta\gamma,I})$ *by*

$\psi_{\alpha\beta\gamma,I} = {}_\mu z_\mu$ *with* $\mu \in S_{\alpha\beta\gamma,I}(\Psi)$.

(In the exceptional two cases we set $\psi_{\alpha\beta\gamma,I} = 0$.)

FPIII(Aug/92):

As in the case of § 3 we have:

Lemma 4.5. *The vector* $g_{\alpha\beta\gamma}$ *is decomposed as follows:*

$$g_{\alpha\beta\gamma|I} = g_I \otimes \psi_{\alpha\beta\gamma,I} \text{ with a unique element } g_I \in \Gamma(\mathfrak{G}_I^{\oplus 2}).$$

(In the exceptional two cases, $g_{\alpha\beta\gamma|I} = 0$.*)*

The element g_I *is explicitly*

$$g_I = (g_I, 0) \quad \text{where } |g_I| = S^{----}_{ijkl} \text{ for type } 6,5,4b, \text{and } 4a.$$

$$g_I = (0, g_I) \quad \text{where } |g_I| = S^{++++}_{ijkl}$$

Here we set:

$$S_I^-(=S^{----}_{ijkl}) = \{m \in F_p - I : m \not\sim \alpha \text{ for any } \alpha \in I\},$$

$$S_I^+(=S^{++++}_{ijkl}) = \{m \in F_p - I : m \sim \alpha \text{ for any } \alpha \in I\}.$$

The sets F^{+++-}_{ijkl}, F^{++--}_{ijkl},... are defined similarly.

In the case of type 3c, the kernel of $(\delta\Psi)_I$ is of rank two.

It is written as $ker_1 \oplus ker_2$ *where* ker_1 and ker_2 are the submodules of $(\mathfrak{G}_{ijk|I} \otimes \mathfrak{G}_{jkl|I})$ and $(\mathfrak{G}_{ijl|I} \oplus \mathfrak{G}_{ikl|I})$. We define $\mathfrak{O}_I$-invertiable sheaves as follows:

$$\mathfrak{M}_{\alpha\beta\gamma,I} := \mathfrak{O}_I(m_{\alpha\beta\gamma,I}), \text{ where } m_{\alpha\beta\gamma,I} = \# S^{---+}_{\alpha\beta\gamma\delta}.$$

Define two $\mathfrak{O}_I$-invertible sheaves:

$$\mathfrak{G}_I(1) \simeq \mathfrak{G}_{ijk|I} \otimes \mathfrak{M}^*_{ijk,I} \simeq \mathfrak{G}_{jkl|I} \otimes \mathfrak{M}^*_{jkl,I}$$

$$\mathfrak{G}_I(2) \simeq \mathfrak{G}_{ijl|I} \otimes \mathfrak{M}^*_{ijl,I} \simeq \mathfrak{G}_{ikl|I} \otimes \mathfrak{M}^*_{ikl}$$

Also define an element $\psi_{\alpha\beta\gamma,I} \in \Gamma(\mathfrak{M}_{\alpha\beta\gamma,I})$ in a similae manner to the previous cases.

Lemma 4.6. *Setting* $g_I(1) = {}^t(g_I, 0)$ *and* $g_I(2) = {}^t(0, g_I)$, with $|g_I(1)| = S_I^-$ *and* $|g_I(2)| = S_I^+$, *the vectors* g_{ijk},...*are decomposed as follows:*

$$g_{ijk|I} = g_I(1) \otimes \psi_{ijk,I}, \quad g_{jkl|I} = g_I(1) \otimes \psi_{jkl,I}, \quad \text{and}$$

$$g_{ijl|I} = g_I(2) \otimes \psi_{ijl,I}, \quad g_{ikl|I} = g_I(2) \otimes \psi_{ikl,I}.$$

Inductive structure for Φ: The arguments for Φ are more involved. This seems to stem because $|\varphi_{\alpha\beta,\alpha\beta\gamma}|$ is of more complicated form than $|\psi_{\alpha\beta,\alpha\beta\gamma}|$ if $(\alpha\beta\gamma)$ is of type 2 or type 1.

Here we only state the results. (The present proof requires a refinement.) First, a very few terms of $(\delta\Phi)_I = 0$. They are:

ij-row for type 5, and ik- and jl- rows for type 4b.

il-row for type 4a.

Moreover, in the case of type 3c,

all rows except il- and jk-rows.

(In these cases, the treatments of $(\delta\Phi)_I$ are done parallely to those of $(\delta\Psi)_I$ except the complication mentioned just above. For type 6, the argument is completely dual.) In any case one can check:

The rank of $(\delta\Phi)_I \leqq 3$ and $= 2$ *only for type* 3c.

One can define $S_{\alpha\beta\gamma,I}(\Phi)$ to be the minimal solution of the identity:

$$|\varphi_{\alpha\beta,\alpha\beta\gamma}| \cup S_{\alpha\beta\gamma,I} = |\varphi_{\alpha\beta,\alpha\beta\delta}| \cup S_{\alpha\beta\delta,I} \text{ for any } (\alpha,\beta) \subset I,$$

where $\alpha\beta$-row $\neq 0$. (*Here* $(\gamma,\delta) = I - (\alpha,\beta)$.)

We set (*except the case of type* 3c):

$$\mathfrak{N}_{\alpha\beta\gamma,I} \simeq \mathfrak{O}_I(n_{\alpha\beta\gamma,I}) \textit{ with } n_{\alpha\beta\gamma,I} = \# S_{\alpha\beta\gamma,I}(\Phi)$$

$$\mathfrak{S}_I \simeq \mathfrak{S}_{\alpha\beta\alpha|I} \otimes \mathfrak{N}_{\alpha\beta\gamma,I} \textit{ for any } (\alpha\beta\gamma) \subset I \textit{ (and independent of the choice of } (\alpha\beta\gamma)).$$

Take elements $\sigma_\alpha \in \mathfrak{S}_{\alpha,x}$, $\alpha \in I$ and $x \in X_I$. Then, by § 2 and § 3, we have (unique) elements $\sigma_{\alpha\beta} \in \mathfrak{S}_{\alpha\beta,x}$ and $\sigma_{\alpha\beta\gamma} \in \mathfrak{S}_{\alpha\beta\gamma,x}$ such that

$$\sigma_{\alpha|\alpha\beta} = \sigma_{\alpha\beta} \oplus \varphi_{\alpha,\beta} \textit{ and } \sigma_{\alpha\beta} = \sigma_{\alpha\beta\gamma} \otimes \varphi_{\alpha\beta,\alpha\beta\gamma} .$$

Thus $\oplus_{\alpha\beta\gamma}\sigma_{\alpha\beta\gamma|I}$ is in the kernel of $(\delta\Phi)_I$. Thus we have:

Lemma 4.6. *The elements* $\sigma_{\alpha\beta\gamma}$ *is decomposed as follows:*

$$\sigma_{\alpha\beta\gamma|I} = \sigma_I \otimes \varphi_{\alpha\beta\gamma,I} \textit{ with a unique element } \sigma_I \in \mathfrak{S}_{I,x}.$$

The explicit form of $S_{\alpha\beta\gamma,I}(\Phi)$ is as follows:

Type 6: $S_{\alpha\beta\gamma,I}(\Phi) = S^{++++-}_{\alpha\beta\gamma\delta}$.

In the other cases we write only the typical terms:

Type 5: $S_{ijk,I}(\Phi) = 2S^{-+}_{kl} - S^{++-+}_{ijkl}$, $S_{ikl,I}(\Phi) = S^{+-}_{ij} - S^{+---}_{ijkl}$

Type 4b: $S_{ijk,I}(\Phi) = S^{--+}_{ijl} \cup S^{--+}_{jkl} \cup S^{+-++}_{ijkl} = S^{-+}_{jl} \cup S^{---+}_{ijkl}$

Type 4a: $S_{ijk,I}(\Phi) = 2(S^{-+}_{il} - S^{-+++}_{ijkl}) \cup (S^{++}_{il} - S^{+}_{I})$

$S_{ijl,I}(\Phi) = 2(S^{-+-+}_{ijkl} \cup S^{++-+}_{ijkl}) \cup S^{++--}_{ijkl}$

$S_{jkl,I}(\Phi) = S^{-+}_{il} \cup S^{-++-}_{ijkl}$

Type 3a: $S_{ijk,I}(\Phi) = S^{+}_{l} - S^{+}_{I}$

$S_{ijl,I}(\Phi) = 2S^{++-+}_{ijkl} \cup S^{++--}_{ijkl}$

In the case of type 3c, the kernel of $(\delta\Phi)_I$ is of rank two. We form $\mathfrak{N}_{\alpha\beta\gamma,I}(\Phi)$ by

$$\mathfrak{N}_{\alpha\beta\gamma,I} = \Omega_I(m_{\alpha\beta\gamma,I}) \textit{ with } m_{\alpha\beta\gamma,I} = \# S_{\alpha\beta\gamma,I}(\Phi).$$

Here $S_{\alpha\beta\gamma,I}(\Phi)$ are the minimal solution of

$$|\varphi_{il,ijl}| \cup S_{ijl,I}(\Phi) = |\varphi_{il,ikl}| \cup S_{ikl}(\Phi)$$

$$|\varphi_{jk,ijk}| \cup S_{ijk,I}(\Phi) = |\varphi_{jk,jkl}| \cup S_{jkl}(\Phi).$$

We set:

$$\mathfrak{S}_I(1) \simeq \mathfrak{S}_{ijl|I} \otimes \mathfrak{N}^*_{ijl,I} \simeq \mathfrak{S}_{ikl|I} \otimes \mathfrak{N}^*_{ikl,I}$$

$$\mathfrak{S}_I(2) \simeq \mathfrak{S}_{ijk|I} \otimes \mathfrak{N}^*_{ijk,I} \simeq \mathfrak{S}_{jkl|I} \otimes \mathfrak{N}^*_{jkl,I}$$

Lemma 4.7. *The elemnts $\sigma_{\alpha\beta\gamma}$ are decomposed as follows:*

$$\sigma_{i\alpha l|I} = \sigma_I(1) \otimes \psi_{i\alpha l,I}, \ \alpha = j,k,$$

$$\sigma_{\beta jk|I} = \sigma_I(2) \otimes \psi_{\beta jk,I} \ , \ \beta = i,l \ ,$$

with unique elements $\sigma_I(\alpha) \in \mathfrak{S}_I(\alpha)_x$, $\alpha = 1,2$.

The expl;icit form of $S_{\alpha\beta\gamma,I}(\Phi)$ is:

$$S_{ijl}(\Phi) = 2S^{++-+}_{ijkl} \cup S^{-+-+}_{ijkjl} \cup S^{---+}_{ijkl}$$

$$S_{ikl}(\Phi) = 2S^{+-++}_{ijkl} \cup S^{+-+-}_{ijkl} \cup S^{---+}_{ijkl}$$

and

$$S_{ijl,I}(\Phi) = 2S^{+---}_{ijkl} \cup S^{+-+-}_{ijkl} \cup S^{++--}_{ijkl}$$

$$S_{ijk,I}(\Phi) = 2S^{---+}_{ijkl} \cup S^{+-++}_{ijkl} \cup S^{--++}_{ijkl}$$

FPIV(Aug/92):

§ 5. **Local freeness condition.** Here we discuss the local freeness of $\mathfrak{E}$ up tp codimension four. The decomposition laws in the previous sections are foundamental for this purpose.

First we give a system of generators of $\mathfrak{E}_x$, $x \in X^2$. First assume that $x \in X^2_{ij}$ with $i \neq j \in I$. Take elements $\sigma_\alpha \in \mathfrak{S}_{\alpha,p}$, $\alpha = i,j$ satisfyi[illegible] $(\sigma_i \otimes \mu_i)_{|ij} = (\ _j \otimes \mu_j)_{|ij}$, cf. (2), § 1. By Lemma 2.3, the elements μ_i and μ_j are decomposed as follows:

(1.1) $\mu_{i|ij} = \mu_{ij} \otimes \varphi_{i,j}$ and $\mu_{j|ij} = \mu_{ij} \otimes \varphi_{j,i}$ with a (unique) $\mu_{ij} \in \mathfrak{S}_{ij,p}$.

This step is conveersed. Assume that $x \in \dot{X}^2 := X^2 - X^3$. Then take an arbitrary element $\mu_{ij} \in \mathfrak{S}_{ij,x}$. We extend $\mu_{ij} \otimes \varphi_{i,j}$ and $\mu_{ij} \otimes \varphi_{j,i}$ to $\mathfrak{S}_{i,x}$ and $\mathfrak{S}_{j,x}$, and we have elements $\mu_\alpha \in \mathfrak{S}_{\alpha,x}$, $\alpha = i,j$, satisfying (1). Now start with an element $\mu_{ij} \in \mathfrak{S}_{ij,x}$ such that

(1.2) the value of μ_{ij} at $x \neq 0$.

We have elements μ_α, $\alpha = i,j$, by the above procedure. We then have an element $g(ij) \in \mathfrak{E}_x$ satisfying $g(ij)_{|\alpha} = g_\alpha \otimes \mu_\alpha$, $\alpha = i,j$,cf.(1),§ 1. Moreover, take a frame l of $\mathfrak{O}(1)_x$ and elements $x_\alpha \in \mathfrak{O}_x$, $\alpha = 1,2$, such that $Z(x_\alpha) = X^1_\alpha$. Moreover, we extend $g_{\alpha,x} \in \mathfrak{G}^{\oplus 2}_{\alpha,x} (= \mathfrak{O}_\alpha(2q)^{\oplus 2}_x)$ to $\mathfrak{O}(2q)^{\oplus 2}_x$, and write the extension also as $g_{\alpha,x}$. Setting $g(\alpha) = g_\alpha \otimes l^{\otimes(q+1)}$ we see that

$$x_\beta g(\alpha) \in \mathfrak{E}_x, \text{ with } (\alpha,\beta) = (i,j) \text{ or } (j,i).$$

Furthermore, let e^1 and e^2 denote ${}^t(\ _\nu z_\nu, 0)$ and ${}^t(0,\ _\nu z_\nu)$ with $\nu \in F_p$. These elements are in $\Gamma(\mathfrak{E})$.

Lemma 5.1. *The $\mathfrak{O}_x$-module $\mathfrak{E}_p$ is generated by*

(1.3) e^1, e^2, $x_i g(j)$, $x_j g(i)$, $g(ij)$.

Proof. Take an element $\zeta \in \mathfrak{E}_x$. Then we have elements $\mu'_\alpha \in \mathfrak{S}_{\alpha,x}$, $\alpha = i,j$, $\mu'_{ij} \in \mathfrak{S}_{ij,x}$ satisfying the decomposition law: $\zeta_{|\alpha} = g_{\alpha,x} \otimes \mu'_\alpha$ and $\mu'_{\alpha|ij} = \mu'_{ij} \otimes \varphi_{\alpha,\beta}$. One can write $\mu'_{ij} = c_{ij}\mu_{ij}$ with an element $c_{ij} \in \mathfrak{I}_{ij,p}$ and extend c_{ij} to $\mathfrak{O}_x$. Using 'c_{ij}' to the extension, we set $\zeta' = \zeta - c_{ij}g(ij))$. Then $\zeta'_{|ij} = 0$. Letting μ''_α be the 'decomposing element" for ζ', we see that $\mu''_\alpha \equiv O(X^2_{ij,x})$ .Thus $\zeta'_{|\alpha} = c_\alpha g_\alpha$ with an element $c_\alpha \in \mathfrak{O}_{\alpha,x}$, and $\zeta' - (c_i g(i) + c_j g(j)) \equiv O(X^1_x)$. q.e.d.

Next take a subset $J = (i,j,k) \subset F_p$ with $\# J = 3$ and assume that $x \in X^3_J$. Take elements $\tau_\alpha \in \mathfrak{S}_{\alpha,x}$, $\alpha \in J$, satisfying (2),1: $(g_\alpha \otimes \tau_\alpha)_{|\alpha\beta} = (g_\beta \otimes \tau_\beta)_{|\alpha\beta}$ for any $(\alpha,\beta) \subset J$. By Lemma 2.3 we have (unique) elements $\tau_{\alpha\beta} \in \mathfrak{S}_{\alpha\beta,x}$. By Lemma 3.2 we have a (unique) element $\sigma_J \in \mathfrak{S}_{J,x}$. The following series of decomposition holds:

(2.1) $\tau_{\alpha|\alpha\beta} = \tau_{\alpha\beta} \otimes \varphi_{\alpha,\beta}$ and $\tau_{\alpha\beta|J} = \tau_J \otimes \varphi_{\alpha\beta,J}$ for any $(\alpha,\beta) \subset J$.

Assume that $x \in \dot{X}^3_J = X^3_J - X^4$. The step (2.1) is reversed. Take an element $\tau_J \in \mathfrak{S}_{J,p}$ arbitrarily. By extending $\tau_J \otimes \varphi_{\alpha\beta,J}$ to $\mathfrak{S}_{\alpha\beta,J}$ we have elements $\tau_{\alpha\beta}$ satisfying the second relstion in (2.1). Next, for each $\alpha \in J$,

(2.2) $(\tau_{\alpha\beta} \otimes \varphi_{\alpha,\beta})_{|J} = (\tau_{\alpha\gamma} \otimes \varphi_{\alpha,\gamma})_{|J} = (\varphi_{\alpha,\beta} \otimes \varphi_{\alpha\beta,J}) \otimes \tau_J = (\varphi_{\alpha,\gamma} \otimes \varphi_{\alpha\gamma,J}) \otimes \tau_J$

and this is extended to an element $\tau_\alpha \in \mathfrak{S}_{\alpha,x}$. Now assume that

(2.3) the value of τ_J at $x \neq 0$.

We form an element $g(J) \in \mathfrak{E}_x$ satisfying $g(J)_{|\alpha} = g_{\alpha,x} \otimes \tau_\alpha$. Moreover, for each $(\alpha,\beta) \subset J$ we apply the previous procedure and we have an element $g(\alpha\beta)$. Also we have an element $g(\alpha)$, $\alpha \in J$. Note that

(2.4) $x_\gamma g(\alpha\beta)$, $\gamma \in I - \{\alpha,\beta\}$, and $x_\beta x_\gamma g(\alpha)$, $\beta,\gamma \in J - \{\alpha\}$, $\in \mathfrak{E}_x$.

Lemma 5.2. *The* $\mathfrak{O}_x$*-module* $\mathfrak{E}_x$ *is generated by*

(2.5) e^1, e^2, $x_\gamma g(\alpha\beta)$, $x_\beta x_\gamma g(\alpha)$ *and* $g(I)$.

Here (α,β,γ) *runs through a permutation of* (i,j,k).

The proof of this lemma is similar to Lemma 5.1.

Thirdly, take a subset $I = (i,j,k,l) \subset F_p$ and assume that $x \in X$. Take elements $\sigma_\alpha \in \mathfrak{S}_{\alpha,p}$, $\alpha \in I$, satisfying (2),§ 1:$(g_\alpha \otimes \sigma_\alpha)|_{\alpha\beta} = (g_\beta \otimes \sigma_\beta)|_{\alpha\beta}$ for any pair $(\alpha,\beta) \subset I$. By Lemma 2.3, Lemma 3.5 and Lemma 4.6, we have (unique) elements $\sigma_{\alpha\beta} \in \mathfrak{S}_{\alpha\beta,x}$, $\sigma_{\alpha\beta\gamma} \in \mathfrak{S}_{\alpha\beta\gamma,x}$ and $\sigma_I \in \mathfrak{S}_{I,\alpha}$ satisfying the following series of decomposition laws:

(3.1) $\sigma_{\alpha|\alpha\beta} = \sigma_{\alpha\beta} \otimes \varphi_{\alpha,\beta}$, $\sigma_{\alpha\beta|\alpha\beta} = \sigma_{\alpha\beta\gamma} \otimes \varphi_{\alpha\beta\gamma,I}$ and $\sigma_{\alpha\beta\gamma|I} = \sigma_I \otimes \varphi_{\alpha\beta\gamma,I}$

(for any permutation $(\alpha,\beta,\gamma,\delta)$ of (i,j,k,l).

Remark 5.3. When I is of type 3c, the last step should be replaced as:

(3.2) $\sigma_{il\alpha|I} = \sigma_I(1) \otimes \varphi_{il\alpha,I}$, $\alpha = j,k$, and $\sigma_{jk\beta|I} = \sigma_I(2) \otimes \varphi_{jk\beta,I}$, $\beta = i,l$.

Assume that $x \in \dot{X}^4_I := X^4_I - X^5$. The step is conversed. Namely, taking an element $\sigma_I \in \mathfrak{S}_{I,x}$ arbitrarily, we get $\sigma_{\alpha\beta\gamma}$, $\sigma_{\alpha\beta}$ and σ_α satisfying (3.2). Assume that

(3.3) the value of σ_I at $x \neq 0$.

Then we form an element $g(I) \in \mathfrak{E}_x$ satisfying $g(I)_{|x} = g_{\alpha,x}\sigma_\alpha$ for any $\alpha \in I$. As before we form elements $g(\alpha\beta\gamma)$, $g(\alpha\beta)$ and $g(\alpha)$.

Lemma 5.4. *If* I *is of not type* 3c, *then* $\mathfrak{E}_x$ *is generated by*

(3.4) e^1, e^2, $x_{\beta\gamma\delta}g(\alpha)$, $x_{\gamma\delta}g(\alpha\beta)$, $x_\delta g(\alpha\beta\gamma)$ *and* $g(I)$.

where $\{\alpha,\beta,\gamma,\delta\}$ *is a permutation of* (i,j,k,l) *and we write* $x_{\alpha\beta\gamma} = x_\alpha x_\beta x_\gamma$ *and* $x_{\gamma\delta} = x_\gamma x_\delta$.

The proof is similar to the previous ones.

Remark 5.5. For type 3c, we have two vectors $g(I:1)$ and $g(I:2)$ at codiumension four level.

Local freeness condition. Take a point $x \in X^1$.

Proposition 5.5. (i) *Take two elements* ξ_1 *and* $\xi_2 \in \mathfrak{E}_x(\subset \Omega_x^{\oplus 2})$. *Then*

$\det(\xi_1 \wedge \xi_2) = y.u$, *where* y *and* u *are elements of* $\mathcal{O}_x$ *and* y *satisfies*: $Z(y) = X_x^1$.

(ii) $\mathfrak{E}_x$ *is* $\mathcal{O}_x$*-free if and only if there are elements* ξ_1 *and* ξ_2 *of* $\mathfrak{E}_x$ *such that*

$(1/y)\det(\xi_1 \wedge \xi_1)$ *is a unit* $\in \mathcal{O}_x$.

Proof. First note that at each point of $X_i^1 - X^2$, two elemwnts $(e^1, g(i))$ or $(e^2, g(i))$ forms a frame of $\mathfrak{E}$. Obviously,

$\det(e^1 \wedge g(i))$ and $\det(e^2 \wedge g(i))$ satisfy the condition in (ii) .

Since ξ_1 and ξ_2 are linear combination of $(e^1, g(i))$, $\det(\xi_1 \wedge \xi_2)$ $\equiv O(X_{i,x}^1 - X_x^2)$. Thus we have (i). Concerning (ii), we remark $\mathfrak{E} = i_*(\mathfrak{E}_{|X-X^2})$ with the inclusion $i:(X-X^2) \quad X$.

Thus $\mathfrak{E}$ is locally free if and only if there are elements ξ_1, ξ_2, which spans $\mathfrak{E}$ on $(X-X^2)_x$. This is equivalent to the condition in (ii). q.e.d.

Now take a point $x \in \dot{X}_{ij}^2$. Then according to whether $i \sim j$ or not, we have:

$\det(g(ij) \wedge e^2) = y$(the first component of $g(ij)$) or

$\det(g(ij) \wedge e^1) = y$(the second component of $g(ij)$

On the otherhand

$g(ij)_{|ij} = g_{ij} \otimes (\psi_{i,j} \otimes \psi_{j,i}) \otimes \sigma_{ij}$ and $g_{ij} = {}^t(g_{ij}, 0)$ or ${}^t(0, g_{ij})$, and $|g_{ij}| = S_{ij}^{--}$ or $= S_{ij}^{++}$.

Lemma 5.6. $\mathfrak{E}$ *is locally free at* x, *and* $g(ij), e^2$ *or* $g(ij), e^1$ *form a frame at* x.

Next take a subset $J = (i,j,k) \subset F_p$ and take a point $x \in \dot{X}_{ijk}$.

Here we consider the case where J is of type 3 or type 2. In the case of tyoe 0 and type 1 is treated dually(by changing + and - and also changing the first row and the second row of the vectors in question.) First note that

Lemma 5.7. *$\mathfrak{E}$ is lcally free at x. According to whether I is of type 3 or type 2, $(g(J),e^2)$(resp. $(g(J),g(i))$ form a frame of $\mathfrak{E}$ at x.*

Proof. First, in the case of type 3, we restrict $(g(J),(z_{ij})^{-1}e^2)$ on X_I^4:

(*) $\quad (g(J),(z_{ij})^{-1}e_2)|_I = (g_J\kappa(J)\sigma_J, {}^t(0, {}_{\nu\in F_p-(i,j)}))$, *where*

$g_J = {}^t(g_J,0)$ with $|g_J| = S_J^-(:=S_{ijk}^{---})$ and

$\kappa(I) = \psi_{\alpha,J}\varphi_{\alpha,J}$ for any α, where $\psi_{\alpha,J} = \psi_{\alpha,\beta}\psi_{\alpha\beta,J}$ and

$\varphi_{\alpha,J} = \varphi_{\alpha,\beta}\varphi_{\alpha\beta,J}$, $\beta \neq \alpha \in J$.

(The elements $\varphi_{\alpha,I}$, $\psi_{\alpha,I}$ and $\kappa(\alpha,I)$ sre well defined.) Note that

$|\psi_{\alpha,I}| = S_\alpha^- - S_J^-$ and $|\varphi_{\alpha,I}| = S_\alpha^+ - S_J^+ - \{j,k\}$, and

$|\kappa(J)| = F_p - \{i,j,k\} - (S_J^+ \cup S_J^-)$.

Thus the determinant of (*) $\neq 0$. Next we restrict $(g(J),g(i))$ on X_{jk}:

$$g(J)|_{jk} = g_{jk}\psi_{j,k}\psi_{k,j}\tau_{jk},\ g(i)|_{jk} = g_J\psi_{i,J}.$$

Thus

$$\det(g(J)|_{jk}\wedge g(i)) = -(g_{jk,2}g_J)(\psi_{i,j}\psi_{j,i}\psi_{i,I}).$$

where $g_{jk,2}$ is the second component of g_{jk} and $|g_{jk,2}| = S_{jk}^{++}$ i.

Thus the above deterimant is of the form $x_i x$(unit). q.e.d.

By the above lemma we have:

$\mathfrak{E}$ is locally free over $(X - X^4)$.

Now we consider the case of codimension four. Take a subset $I = (i,j,k,l)$ of F_p with $\# I = 4$ and a point $x \in \dot{X}_I^4 := X_I^4 - X^5$.

Theorm 5.8. *Except the case of type 4b, $\mathfrak{E}$ is locally free at x.*

he following two vectors form a frame at $x \in X$.

$(g(I), e_2)$ *for type* 6, $(g(I), x_{kl}g(il))$ *for type* 5,

$(g(I), x_{jk}g(il))$ *for type* 4a, $(g(I), x_{ijk}g(i))$ *for type* 3a,

$(g(I:1), g(I:2))$ *for type* 3c.

roof. The case of type 6 and type 3a are treated similarly to Lemma 5.7. We restrict $g(I)$ on X_I^4 and $(g(I), g(l))$ on X_{ijk}^3 :

$g(I)_{|I} = g_I \kappa(I)\sigma_I$, *where* $\kappa(I) = \psi_{\alpha,I}\varphi_{\alpha,I}$ for any $\alpha \in I$, with

$\psi_{\alpha,I} = \psi_{\alpha,\beta}\psi_{\alpha\beta,\alpha\beta\gamma}\psi_{\alpha\beta\gamma,I}$ and $\varphi_{\alpha,I} = \varphi_{\alpha,\beta}\varphi_{\alpha\beta,\alpha\beta\gamma}\varphi_{\alpha\beta\gamma,I}$.

(These elements are well defined.) As before

$|\psi_{\alpha,I}| = S_\alpha^- - S_I^-(=S_{ijkl}^{----})$, $|\varphi_{\alpha,I}| = S_\alpha^+ - S_I^+(=S_{ijkl}^{++++}) - \{\beta,\gamma,\delta\}$,

$\alpha\nu\varphi\ \kappa(I) = F_p - \{i,j,k,l\} - \{S_I^- \cup S_I^+\}$.

$(g(I), g(l))_{|ijk} = (g_{ijk}\kappa(ijk)\sigma_{ijk},\ g_I\psi_{i,I})$

Note that $g_{ijk} = {}^t(g_{ijk}, 0)$ with $|g_{ijk}|$ l and $g_I = (0, g_I)$ with $|g_I| = S_I^+$. Moreover, $|\kappa(ijk)| = F_p - \{\{i,j,k\} \cup S_{ijk}^{---} \cup S_{ijk}^{+++}\}$ l , and $|\psi_{i,I}| = S_i^- - S_{ijk}^{---}$ l, and we have the assertion for the above two cases. Thirdly consider the case of type 5, and we restrict $(g(I), g(ij))$ to X_{kl}^2:

$(g(I), g(ij))_{|kl} = (g_{kl}\psi_{k,l}\psi_{l,k}\sigma_{kl}, g_{ij}\psi_{i,j}\psi_{j,i}\mu_{ij})$, where

$g_{kl} = {}^t(0, g_{kl})$ with $|g_{kl}| = S_{kl}^{++}$ i,j, and $g_{ij} = {}^t(g_{ij}, 0)$ with $|g_{ij}| = S_{ij}^{--}$ k,l. Moreover, $\sigma_{kl} = \varphi_{kl,ikl}\varphi_{kli,I}\sigma_I$, and

$|\sigma_{kl}| = (S_i^- \cup S_j^-) - (S_{ikl}^{---} \cup S_{jkl}^{---})$ i,j,

Also $\psi_{i,j}\psi_{j,i}$ does not vanish on X_{kl}.

Thus the determinant of the above two vector $= x_{kl}x(\text{unit})$.

FPV(Aug/92) For type 4a, we restrict $(g(I),g(il))$ to X^2_{jk}:

$$(g(I),g(il))|_{jk} = (g_{jk}\psi_{j,k}\psi_{k,j}\sigma_{jk},\ (g_i\mu_i)|_{ijk} \dot{+} (g_l\mu_l)|_{jkl})$$

where $g_{jk} = (g_{jk,1}, g_{jk,2})$ with $|g_{jk,1}| = S^{--}_{jk}$ l *and* $|g_{jk,2}| = S^{++}_{jk}$ i.Moroever, $g_{i|ijk} = (g_i, 0)$ with $|g_i| = S^-_i$ and $g_{l|jkl} = {}^t(0, g_l)$ with $|g_l| = S^+_l$ i.

Also $(\psi_{j,k}\psi_{k,j}\sigma_{jk})$ and μ_l do not vanish.

Using these we check that the the determinant of the above two vector $\neq 0$.

Finally we consider the case of type 3c. In this case, the situation is subtle and seems to be a new factor which arises at codimension four:First note

the first and second components of g_i(resp.g_l) contain z_{kl} and z_j(resp.z_{ij} and z_k),and those of g_j(resp.g_k) contains z_l and z_{jk}(resp. z_i and z_{jl}) .

Moroevr, $\sigma_\alpha \equiv 0$ (X_I). Thus the degree of $g(I:1)$ and $g(I:2) \geqq 2$. Thus, lettting $(g(I:1)\wedge g(I:2))^{(I)}$ denote the term

$$(x_I)^{-1}(g(I:1)\wedge g(I:2)),$$

we have:

$$\textit{the term} = \Sigma_{\alpha\beta}\ g^{(\alpha\beta)}(I:1)\wedge g^{(\gamma\delta)}(I:2),\ \text{with } (\gamma\delta) = I - (\alpha\beta),$$

where $g^{(\alpha\beta)}(I:1)$ denotes $\{(\partial^2/\partial x_\alpha x_\beta)g(I:1)\}_{|I}$.

First we have:

(*) The summand = 0, unless $(\alpha\beta) = (il)$ or (jk).

Actually assume that $(\alpha\beta) = (ij)$ or (ik)(and so $(\gamma\delta) = (kl)$ or (jl)). Then $g^{(\alpha\beta)}(I:1) = (\partial^2/\partial x_\alpha x_\beta)(g(I:1)_{|\gamma\delta})_{|I} = (g_{\gamma\delta}\psi_{\gamma,\delta}\psi_{\delta,\gamma}\sigma_{\gamma\delta}(1))^{(\alpha\beta)}$ and similarly for $g^{(\gamma\delta)}$. If $(\alpha\beta) = (ij)$ then

$g_{kl} = {}^t(*,0)$ and $g_{ij} = {}^t(*,0)$, i.e., the second term = 0. Thus $g^{(\alpha\beta)}(I:1)\wedge g^{(\gamma\delta)}(I:2)$ obviously vanishes. If $(\alpha\beta) = (ik)$, then the vectors g_{ik} and g_{jl} appear, and the first term of them = 0. Thus the terms should be considered are as follows:

$$g^{(il)}(I:1)\wedge g^{(jk)}(I:2) + g^{(jk)}(I:1)\wedge g^{(il)}(I:2)$$
$$=(g_{jk}\psi_{j,k}\psi_{k,j}\sigma_{jk}(1))^{(il)}\wedge(g_{il}\psi_{i,l}\psi_{l,i}\sigma_{il}(2))^{(jk)}$$
$$+ (g_{il}\psi_{i,l}\psi_{l,i}\sigma_{il}(1))^{(jk)}\wedge(g_{jk}\psi_{j,k}\psi_{k,j}\sigma_{jk}(2))^{(il)}$$

Also note that $\psi_{j,k} \equiv 0(x_l)$ and $\psi_{k,j} \equiv 0(x_i)$ and similar fact. Thus the above quantity is written as:

$$(g_{jk}\wedge g_{jl})(\psi^{(l)}_{j,k}\psi^{(l)}_{k,j}\psi^{(k)}_{i,l}\psi^{(j)}_{l,i})\{\sigma_{jk}(1)\sigma_{il}(2) - \sigma_{il}(1)\sigma_{jk}(2)\} .$$

Nopte that $g_{jk} = {}^t(g_{jk},0)$ with $|g_{jk}| = S^{--}_{jk}$ and $g_{il} = (0,g_{il})$ with $|g_{il}| = S^{++}_{il}$> Thus $(g_{jk}\wedge g_{il})$ does not vanish. On the otherhand,

$\sigma_{jk}(1) = \varphi_{jk,I}\sigma_I(1)$, $\sigma_{il}(2) = \varphi_{il,I}\sigma_I(2)$, while

$\sigma_{jk}(2) = 0$ and $\sigma_{il}(1) = 0$. Thus we have:

where $\varphi_{jk,I} = \varphi_{jk,jki}\varphi_{jki,I} = \varphi_{jk,jkl}\varphi_{jkl,I}$, and similarly for $\varphi_{il,I}$.

$$(g(I:1)\wedge g(I:2))^{(I)} = (g_{jk}\wedge g_{jl})(\psi^{(l)}_{j,k}\psi^{(i)}_{k,j}\psi^{(k)}_{i,l}\psi^{(j)}_{l,k})(\varphi_{jk,I}\sigma_{il,I})\sigma_I(1)\sigma_I(2).$$

and we finish the proof.

(In order to check that, for Type 4b, $\mathfrak{E}$ is not locally free, we should check all 2-vectors. We omit here. The chief objection for the locally freeness is that, for pairs $(\psi_{i,l},\psi_{l,i})$, $(\psi_{k,l},\psi_{l,k})$,..., $Z(\psi_{i,l})\cap Z(\psi_{l,i}) \equiv 0(X_I)$,.....)

Remark: *If* $p = 5$, then $I \subset F_5$ with $\# I = 4$ is of the form $I = (i,i+1,i+2,1+3)$ and is always of type 3c. Thus $\mathfrak{E}$ is locally free and is actually the Horrocks-Mumford bundle. Our hope was to construct reflexive sheaves $\mathfrak{E}$ which are locally free over $(X-X^5)$. Bur if $p > 5$, then type 4b or its dual always appears. However, the content of this paper is limitted to the quadratic residue.

Is it possible to construct reflexive sheaves from some arithmetic (e.g.higher residue) of F_p ? .

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