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Kyoto University
The duality of Tsuchihashi cusp singularities

Masa-Nori ISHIDA (石田正典)
Mathematical Institute, Faculty of Science
Tohoku University, Sendai 980, Japan
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Introduction

A cusp singularity is known as a normal surface singularity whose exceptional divisor of a suitable resolution is a cycle of nonsingular rational curves. In [N], Nakamura showed that each cusp singularity \((V,p)\) has natural dual cusp singularity \((V^*,p^*)\), and that invariants of these singularities have some dual relations.

One of these relations is the following:

Let \(D_1 + \cdots + D_r\) and \(E_1 + \cdots + E_s\) are the exceptional divisor of the resolution of \(V\) and \(V^*\), respectively. We assume these are cycles of nonsingular rational curves. Then the following equality consisting of selfintersection numbers holds.

\[
D_1^2 + \cdots + D_r^2 + 3r = -(E_1^2 + \cdots + E_s^2 + 3s)
\]

On the other hand, higher dimensional cusp singularities are introduced by Tsuchihashi [T]. We established an equality which is a generalization of the equality for these cusp singularities.

Let \(N\) be a free \(\mathbb{Z}\)-module of rank \(r < \infty\) and \(M\) the dual \(\mathbb{Z}\)-module. We assume that \(r\) is at least 2. We consider a pair \((C, \Gamma)\) of an open convex cone \(C\) in \(N_{\mathbb{R}} := N \otimes \mathbb{Z} \otimes \mathbb{R}\) and a subgroup \(\Gamma\) of \(\text{Aut}(N) \simeq GL(r, \mathbb{Z})\) with the following properties.

1. For the closure \(\overline{C}\) of \(C\), \(\overline{C} \cap (-\overline{C}) = \{0\}\).
2. \(gC = C\) for every \(g \in \Gamma\).
3. The action of \(\Gamma\) on \(C\) is properly discontinuous and free.
(4) The quotient \((C/\mathbb{R}_+)/\Gamma\) is compact.

For such a pair \((C, \Gamma)\), Tsuchihashi \([T]\) constructed a complex analytic isolated singularity \(V(C, \Gamma)\) by using the theory of toric varieties and called it a cusp singularity.

This cusp singularity has a natural dual. Namely, let \(C^*\) be the interior of the cone \(\{x \in M_\mathbb{R} \mid \langle x, a \rangle \geq 0, \forall a \in C\}\) and \(\Gamma^* := \Gamma^t\), where \(M_\mathbb{R} := M \otimes \mathbb{Z} \otimes \mathbb{R}\) and \(\langle, \rangle : M_\mathbb{R} \times N_\mathbb{R} \to \mathbb{R}\) is the natural bilinear map. Then the pair \((C^*, \Gamma^*)\) satisfies similar condition and hence defines a cusp singularity \(V(C^*, \Gamma^*)\). We call \(V(C^*, \Gamma^*)\) the dual cusp singularity of \(V(C, \Gamma)\). Clearly, the dual of \(V(C^*, \Gamma^*)\) is equal to \(V(C, \Gamma)\).

The arithmetic genus defect \(\chi_\infty\) and Ogata's zeta zero \(Z(0)\) are numerical invariants defined for cusp singularities. Here note that our cusp singularities are called "Tsuchihashi singularities" in \([SO]\), and the zeta function is defined by

\[Z(s) = \sum_{u \in (C \cap M)/\Gamma} \phi_C(u)^s,\]

where \(\phi_C(x)\) is the characteristic function of the cone \(C\) \([SO, 4.2]\). As it is mentioned in \([SO, 4.2]\), this zeta function is slightly different from the one defined by the norm function in the case of self-dual homogeneous cones. However, the values at zero of these zeta functions are equal \([SO, 4.2]\). In this note, we denote this value by \(Z(0)(C, \Gamma)\).

On the other hand, \(\chi_\infty(p)\) for a cusp singularity \(p\) of dimension \(r\) is described explicitly as follows: We take a resolution of the singularity such that the exceptional set is a toric divisor \(\bigcup_{i=1}^s D_i\) with simple normal crossing. Then \(\chi_\infty(p)\) is equal to the intersection number

\[\prod_{i=1}^s \frac{D_i}{1 - \exp(-D_i)}\]  

We get the following theorem \([I5]\).

**Theorem**  The rational number \(\chi_\infty(C^*, \Gamma^*)\) is equal to \((-1)^r Z(0)(C, \Gamma)\).

This is a generalization of the equality \((1)\) since it is written as

\[-12Z(0)(C, \Gamma) = -12\chi_\infty(C^*, \Gamma^*)\]

in our new notation.
For the convenience to understand the theorem, we will explain $Z(0)$ for $V(C, \Gamma)$ and $\chi_{\infty}$ for $V(C^{\ast}, \Gamma^{\ast})$.

We introduce here some notations in this note.

Besides some open cones as $C$ and $C^{\ast}$, cones are always closed convex rational polyhedral cones. Namely, a cone $\pi$ in $N_{R}$ is equal to $R_{0}n_{1} + \cdots + R_{0}n_{s}$ for a finite subset $\{n_{1}, \cdots, n_{s}\}$ of the lattice $N$, where $R_{0} := \{c \in R ; c \geq 0\}$. For a cone $\pi$ in $N_{R}$, the linear subspace $\pi + (-\pi)$ of $N_{R}$ is denoted by $H(\pi)$. The interior of $\pi$ as a subset of $H(\pi)$ is called the \textit{relative interior} of $\pi$ and is denoted by $\text{rel. int} \pi$.

We denote $\sigma \prec \pi$ if $\sigma$ is a face of a cone $\pi$. We denote by $F(\pi)$ the set of faces of $\pi$. $\pi$ is said to be strongly convex if $\pi \cap (-\pi) = \{0\}$ or equivalently if the zero cone $0 := \{0\}$ is in $F(\pi)$.

A nonempty collection $\Phi$ of strongly convex cones in $N_{R}$ is said to be a \textit{fan} if (1) $\pi \in \Phi$ and $\sigma \prec \pi$ imply $\sigma \in \Phi$, and (2) if $\sigma, \tau \in \Phi$, then $\sigma \cap \tau$ is a common face of $\sigma$ and $\tau$. For a subset $\Psi$ of a fan $\Phi$ and an element $\rho \in \Phi$, we denote $\Psi(\prec \rho) := \{\sigma \in \Psi ; \sigma \prec \rho\}$ and $\Psi(\rho \prec) := \{\sigma \in \Psi ; \rho \prec \sigma\}$. For an integer $d$ we denote $\Psi(d) := \{\sigma \in \Psi ; \dim \sigma = d\}$.

For a subset $S \subset N_{R}$, we denote $S_{\downarrow} := \{x \in M_{R} ; \langle x, a \rangle = 0, \forall a \in S\}$ and $S^{\vee} := \{x \in M_{R} ; \langle x, a \rangle \geq 0, \forall a \in S\}$. For a (closed convex) cone $\pi \subset N_{R}$, $\pi^{\vee} \subset M_{R}$ is called the dual cone of $\pi$. It is known that the correspondences $\sigma \mapsto \pi^{\vee} \cap \sigma_{\downarrow}$ define a bijection of $F(\pi)$ and $F(\pi^{\vee})$ [O, Prop.A.6].

We use same notations for cones in the other real vector spaces with lattices.

\section{The \textit{T}-complexes}

The notion of \textit{T}-complexes was introduced in [I2] in order to describe the combinatorial structures of toric divisors. We briefly review the definition.

Let $r$ be a positive integer and let $\mathcal{C}_{r}$ be the category of pairs $\alpha = (N(\alpha), c(\alpha))$ of free $\mathbb{Z}$-module $N(\alpha)$ of rank $r$ and a strongly convex rational polyhedral cone $c(\alpha) \subset N(\alpha)_{R}$. For two objects $\alpha, \beta$ of $\mathcal{C}_{r}$, a morphism $u : \alpha \rightarrow \beta$ consists of an isomorphism $u_{\mathbb{Z}} : N(\alpha) \rightarrow N(\beta)$ such that $u_{R}(c(\alpha))$ is a face of $c(\beta)$, where $u_{R} := u_{\mathbb{Z}} \otimes 1_{R}$. For a morphism $u$, we denote by $i(u)$ the source and by $f(u)$ the target, respectively, of $u$.

A subcategory $\Sigma$ of $\mathcal{C}_{r}$ is said to be a \textit{graph of cones} of dimension $r$ if the objects and the morphisms in $\Sigma$ are finite in number. The set of morphisms in $\Sigma$ is denoted
by \text{mor} \Sigma.

Let \rho be an object of a graph of cones \Sigma. We define graphs of cones \Sigma(\prec \rho) and \Sigma(\prec \rho) as follows: \Sigma(\prec \rho) consists of the pairs \beta' = (\beta, v) \in \beta \in \text{mor} \Sigma and v \in \text{mor} \Sigma with i(v) = \rho and f(v) = \beta for which we define \( N(\beta') = N(\beta) \) and \( c(\beta') = c(\beta) \). For \beta' = (\beta, v) and \gamma' = (\gamma, w) in \Sigma(\prec \rho), a morphism \( u' : \beta' \to \gamma' \) consists of \( u : \beta \to \gamma \) with \( u \circ v = w \). Similarly, \Sigma(\prec \rho) consists of pairs \alpha' = (\alpha, v) with v \in \text{mor} \Sigma of the source \alpha and the target \rho.

For each \beta' = (\beta, v) \in \Sigma(\prec \rho), we define \beta'_{\rho} by \( N(\beta'_{\rho}) := N(\beta)[v_{R}(c(\rho))] \) and \( c(\beta'_{\rho}) := c(\beta)[v_{R}(c(\rho))] \), and for each \( u' : \beta' \to \gamma' \in \text{mor} \Sigma(\prec \rho) \), we define \( u'_{\rho} : N(\beta'_{\rho}) \to N(\gamma'_{\rho}) \) to be the isomorphism induced by \( u_{\rho} \). Then we get a graph of cones \Sigma[\rho] of dimension \( r - \dim \rho \) which is equivalent to \Sigma(\prec \rho) as categories.

For a finite fan \( \Delta \) of \( N_{R} \), any subset \( \Sigma \) of \( \Delta \) is regarded as a graph of cones by defining \( N(\alpha) := N \) and \( c(\alpha) := \alpha \) for each \( \alpha \in \Sigma \) and defining that a morphism \( u : \alpha \to \beta \in \text{mor} \Sigma \) if and only if \( u_{\rho} = 1_{N} \).

A free cone \( \alpha = (N(\alpha), c(\alpha)) \) is said to be nonsingular if \( c(\alpha) \) is a nonsingular cone of \( N(\alpha)_{R} \), i.e., \( c(\alpha) = R_{\alpha}x_{1} + \cdots + R_{\alpha}x_{r(\alpha)} \) for a basis \( \{x_{1}, \cdots , x_{r(\alpha)}\} \).

A graph of cones \Sigma of dimension \( r \) is called a \textit{T-complex}, if it satisfies the following conditions.

(1) \( \Sigma \) is nonempty and connected.

(2) The graph of cones \Sigma(\prec \rho) is isomorphic to \( F(\rho) \backslash \{0\} \) for every \( \rho \in \Sigma \), where \( F(\rho) \) is the fan consisting of the faces of \( \rho \).

(3) For each \( \rho \in \Sigma \), the graph of cones \( \Sigma[\rho] \) is isomorphic to a complete fan of \( N(\rho)_{R} \).

A \textit{T-complex} \( \Sigma \) is said to be \textit{nonsingular} if it consists of nonsingular free cones.

We define the support \( |\Sigma| \) of a \textit{T-complex} \( \Sigma \) as the disjoint union

\[
\bigsqcup_{\alpha \in \Sigma} (c(\alpha) \backslash \{0\})
\]

modulo the equivalence relation generated by \( a \sim u_{R}(a) \) for \( u : \alpha \to \beta \in \text{mor} \Sigma \) and \( a \in c(\alpha) \backslash \{0\} \).

A \textit{morphism} \( \varphi : \Sigma' \to \Sigma \) of \textit{T-complexes} consists of a functor \( \varphi : \Sigma' \to \Sigma \) and a collection \( \{\varphi^{\alpha} : \alpha \in \Sigma\} \) of injective \( Z \)-homomorphisms \( \varphi^{\alpha} : N(\alpha) \to N(\varphi(\alpha)) \) such that \( \varphi^{\alpha}_{R}(\text{rel. int } c(\alpha)) \subset \text{rel. int } c(\varphi(\alpha)) \) and the diagram
is commutative for every $u : \alpha \to \beta \in \text{mor} \Sigma$.

A morphism $\varphi : \Sigma' \to \Sigma$ of $T$-complexes induces a map $|\varphi| : |\Sigma'| \to |\Sigma|$ of the supports. The morphism $\varphi$ is said to be a subdivision if the all $\varphi^\alpha$'s are isomorphic and $|\varphi|$ is a bijection.

**Example 1.1** Let $\tilde{\Sigma}$ be a fan of $N_R$ such that $C_1 := |\tilde{\Sigma}| \setminus \{0\}$ is an open cone and $\tilde{\Sigma}$ is locally finite at each point of $C_1$. Assume that a subgroup $\Gamma_1 \subset \text{Aut}(N)$ induces a free action on $\tilde{\Sigma} \setminus \{0\}$ and the quotient is finite. Let $\Sigma$ be the set of representatives of the free quotient. For each $\alpha \in \Sigma$, we set $N(\alpha) := N$ and $c(\alpha) := \alpha$ and we define $\text{mor} \Sigma := \{u : \alpha \to \beta ; u_Z \in \Gamma_1\}$.

Then $\Sigma$ is a $T$-complex. Let $\tilde{\Sigma}'$ be a $\Gamma_1$-equivariant subdivision of $\tilde{\Sigma}$ and $\Sigma'$ the $T$-complex obtained from $\tilde{\Sigma}'$ by the action of $\Gamma_1$. Then $\Sigma'$ is a subdivision of $\Sigma$, since both $|\Sigma'|$ and $|\Sigma|$ are naturally bijective to the quotient $C_1/\Gamma_1$.

Let $(C, \Gamma)$ be the pair which defines a cusp singularity.

We take a $\Gamma$-invariant nonsingular fan $\tilde{\Sigma} \cup \{0\}$ of $N_R$ with the support $C \cup \{0\}$ which is locally finite at each point of $C$. Similarly, we take $\Gamma^*$-invariant nonsingular fan $\tilde{\Delta} \cup \{0\}$ of $M_R$ with the support $C^* \cup \{0\}$ which is locally finite at each point of $C^*$.

Here we assume $0 \notin \tilde{\Sigma}$ and $0 \notin \tilde{\Delta}$ for the convenience of the notations.

Then these are the cases of the above example, and we get $T$-complexes $\Xi = \tilde{\Sigma}/\Gamma$ and $\Delta = \tilde{\Delta}/\Gamma^*$. 
The invariants $Z(0)$ and $\chi_{\infty}$

For a graph of cones $\Phi$, the set of morphisms in $\Phi$ is denoted by $\text{mor}\,\Phi$. This is a finite set by definition. For a covariant functor

$$A : \mathcal{C} \to \text{(Additive groups)},$$

we denote by $A_{\Phi}$ the restriction of $A$ to $\Phi$. In other words, $A_{\Phi}$ is the finite system of the additive groups $(A(\alpha))_{\alpha \in \Phi}$ and the homomorphisms $(A(u) : A(i(u)) \to A(f(u)))_{u \in \text{mor}\,\Phi}$. The inductive limit $\text{ind lim } A_{\Phi}$ of the system $A_{\Phi}$ is described as the cokernel

$$\bigoplus_{u \in \text{mor}\,\Phi} A(i(u)) \longrightarrow \bigoplus_{\alpha \in \Phi} A(\alpha) \longrightarrow \text{ind lim } A_{\Phi},$$

where $p$ consists of the identities $1_{A(i(u))} : A(i(u)) \to A(i(u)) \subset \bigoplus_{\alpha \in \Phi} A(\alpha)$ and $q$ consists of the homomorphisms $A(u) : A(i(u)) \to A(f(u)) \subset \bigoplus_{\alpha \in \Phi} A(\alpha)$.

For a nonsingular free cones $\alpha = (N(\alpha), c(\alpha))$, we set $\text{gen }\alpha := \{x_{1}, \cdots, x_{d(\alpha)}\}$ and $x(\alpha) := \prod_{x \in \text{gen }\alpha} x \in S^{d(\alpha)}(N(\alpha)_{Q})$, where $S^{d}$ means the $d$-th symmetric power over the rational number field $Q$. We denote by $\mathcal{C}^{n.s.}$ the subcategory of $\mathcal{C}$ consisting of nonsingular free cones.

A functor $D^{0} : \mathcal{C}^{n.s.} \to (Q\text{-vector spaces})$ is defined by

$$D^{0}(\alpha) := \{f/x(\alpha) ; f \in S^{d(\alpha)}(N(\alpha)_{Q})\}.$$ 

For $u : \alpha \to \beta$, $D^{0}(u) : D^{0}(\alpha) \to D^{0}(\beta)$ is defined to be the natural injection induced by the isomorphism $u_{Q} : N(\alpha)_{Q} \to N(\beta)_{Q}$. Note that $\text{gen }\alpha$ is mapped into $\text{gen }\beta$ by $u_{\Sigma}$.

Let $Q^{\sim} : \mathcal{C} \to (Q\text{-vector spaces})$ be the constant functor defined by $Q^{\sim}(\alpha) := Q$ and $Q^{\sim}(u) := 1_{Q}$ for all $\alpha \in \mathcal{C}$ and $u \in \text{mor}\,\mathcal{C}$. Since $\Xi$ is connected as a graph of cones, we have $\text{ind lim } Q^{\sim}_{\Xi} = Q$.

Since each $D^{0}(\alpha)$ contains $Q$, there exists a natural morphism of functors $\epsilon_{\Xi} : Q^{\sim}_{\Xi} \to D^{0}_{\Xi}$. By [11, Lem.3.1], the $Q$-linear map

$$Q = \text{ind lim } Q^{\sim}_{\Xi} \to \text{ind lim } D^{0}_{\Xi}$$

is injective. Hence we regard $Q$ as a linear subspace of $\text{ind lim } D^{0}_{\Xi}$.

We recall some notations in [12] with exchanging the roles of $M$ and $N$. 

We denote by $\mathbb{Q}(N)$ the quotient field of the group ring $\mathbb{Q}[N] = \bigoplus_{n \in N} \mathbb{Q}e(n)$. For a nonsingular cone $\sigma$ in $N_{\mathbb{R}}$, the elements $Q_0(\sigma)$ and $Q(\sigma)$ are defined by

$$Q_0(\sigma) = \prod_{y \in \text{gen } \sigma} \frac{e(y)}{1-e(y)} \in \mathbb{Q}(N)$$

and

$$Q(\sigma) = \prod_{y \in \text{gen } \sigma} \frac{1}{1-e(y)} \in \mathbb{Q}(N).$$

Let $\epsilon : M \otimes \mathbb{C} \to M \otimes \mathbb{C}^*$ be the holomorphic map defined by $\epsilon(m \otimes z) := m \otimes \exp(-z)$. For each $y \in N$, $e(y)$ is a regular function on $M \otimes \mathbb{C}^*$, and the pull back $\epsilon^*e(y)$ is equal to $\exp(-y)$. For a nonsingular cone $\sigma$ in $N_{\mathbb{R}}$,

$$x(\sigma)\epsilon^*Q_0(\sigma) = \prod_{y \in \text{gen } \sigma} \frac{y\exp(-y)}{1-\exp(-y)} = \prod_{y \in \text{gen } \sigma} \frac{y}{\exp(y)-1}.$$ 

is an entire function on $M \otimes \mathbb{C}$. We denote by $[\epsilon^*Q_0(\sigma)]_0$ the rational function $f_0$ on $M \otimes \mathbb{C}$, and the pull back $\epsilon^*e(y)$ is equal to $\exp(-y)$. For each $\alpha$ of the $T$-complex $\Xi$, we set

$$\omega(\alpha) := [\epsilon(\alpha)^*Q_0(c(\alpha))]_0 \in D^0(\alpha),$$

where $\epsilon(\alpha) = 1_{M(\alpha)} \otimes \exp(-*) : M(\alpha) \otimes \mathbb{C} \to M(\alpha) \otimes \mathbb{C}^*$. The class of $(\omega(\alpha))_{\alpha \in \Xi}$ in $\text{ind lim } D^0$ is denoted by $\omega(\Xi)$.

The main result of [I1] is the following.

**Theorem 2.1** The class $\omega(\Xi) \in \text{ind lim } D^0$ is in $\mathbb{Q}$, and this rational number is equal to the zeta zero value $Z(0)(C, \Gamma)$ of the cusp $V(C, \Gamma)$.

The value $Z(0)(C, \Gamma)$ can be calculated as follows:

A morphism of functors $\nu : D^{\Xi}_\mathbb{Q} \to \mathbb{Q}^\Xi$ is said to be a retraction if the composition $\nu \cdot \epsilon_\Xi$ is the identity. It was shown that retractions always exist [I1, Lem.3.1]. Then $Z(0)(C, \Gamma)$ is equal to $\sum_{\alpha \in \Xi} \nu(\alpha)(\omega(\alpha))$.

Now, we consider the nonsingular fan $\tilde{\Delta} \cup \{0\}$ of $M_{\mathbb{R}}$. For $\rho \in \tilde{\Delta}$ and an integer $n \geq 0$, we denote by $\text{Index}(\rho, n)$ the set of maps $f : \text{gen } \rho \to \mathbb{Z}_+ := \{c \in \mathbb{Z} ; c > 0\}$ with $\sum_{a \in \text{gen } \rho} f(a) = n$. We use mainly $\text{Index}(\rho, r)$ and denote it simply by $\text{Index}(\rho)$. An element $f$ of $\text{Index}(\rho, n)$ is said to be an index of norm $n$ on $\rho$. 

Let $\sigma$ be a nonsingular cone of maximal dimension in $M_R$. Then $\sigma^\vee$ is a nonsingular cone of dimension $r$ in $N_R$. The bijection $x(\sigma, ) : \text{gen } \sigma \to \text{gen } \sigma^\vee$ is defined so that $(a, x(\sigma, b))$ is 1 if $a = b$ and is zero otherwise for $a, b \in \text{gen } \sigma$. We set $x^*(\sigma) := \prod_{a \in \text{gen } \sigma} x(\sigma, a) = x(\sigma^\vee)$.

For $f \in \text{Index}(\rho, n)$ and $\sigma \in \Delta(\rho \prec)(r)$, we set
\[ I(\sigma, f) := \frac{\prod_{a \in \text{gen } \rho} x(\sigma, a)^{f(a)}}{x^*(\sigma)} \]
and we define
\[ I(\tilde{\Delta}, f) := \sum_{\sigma \in \tilde{\Delta}(\rho \prec)(r)} I(\sigma, f) \].

Then $I(\tilde{\Delta}, f)$ is an integer if $n = r$ (cf. [12, Thm.3.2]).

For each integer $n \geq 0$, we define $b_n := B_n / n!$, where $B_n$‘s are the Bernoulli numbers defined by $1/(1-\exp(-z)) = \sum_{n=0}^{\infty} (B_n / n!) z^{n-1}$. For an index $f$ on a cone $\rho$, we set $b_f := \prod_{a \in \text{gen } \rho} b_{f(a)} \in \mathbb{Q}$.

Let $(\tilde{V}, X)$ be the toroidal desingularization of the cusp singularity $V(C^*, \Gamma^*)$ associated to the fan $\tilde{\Delta} \cup \{0\}$. Then there exists a natural one-to-one correspondence between $\tilde{\Delta}(1)/\Gamma^*$ and the set of irreducible components of $X$. We denote $D(\gamma)$ the prime divisor corresponding to $\gamma \in \tilde{\Delta}(1)$. If we assume that the fan $\tilde{\Delta} \cup \{0\}$ is sufficiently fine, then these prime divisors are nonsingular and $X$ has only normal crossings. Then by expanding the formula for $\chi_{\infty}$ in the introduction, we get an equality
\[ \chi_{\infty}(C^*, \Gamma^*) = \sum_{\rho \in \tilde{\Delta}/\Gamma^*} \sum_{f \in \text{Index}(\rho)} b_f \prod_{a \in \text{gen } \rho} D(\gamma(a))^{f(a)} \]
where $\gamma(a) := R_0 a \in \tilde{\Delta}(1)$ and the products of divisors mean the intersection numbers.

The following theorem is a consequence of Sczech’s equality [S2]
\[ I(\tilde{\Delta}, f) = \prod_{a \in \text{gen } \rho} D(\gamma(a))^{f(a)} \]
which is written in our notation in [12, Thm.3.2].

**Theorem 2.2** The rational number
\[ \sum_{\rho \in \tilde{\Delta}/\Gamma^*} \sum_{f \in \text{Index}(\rho)} b_f I(\tilde{\Delta}, f) \]
is equal to the arithmetic genus defect $\chi_{\infty}(C^*, \Gamma^*)$ of the cusp $V(C^*, \Gamma^*)$. 
Note that we need not assume that $X$ has only simple normal crossings by [I2, Thm.4.9].

By Theorems 2.1 and 2.2, the both invariants $Z(0)(C, \Gamma)$ and $\chi_{\infty}$ are described by the elements of the homogeneous quotient of the polynomial ring $S^*(N_Q)$. This fact makes it possible to compare these two invariants.

For the proof of Theorem, we need some systematic calculation on the $T$-complexes. For the detail, see [I5]. For the historical meaning of this equality, see also [SO].

References


