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Kyoto University
On the irregularity of cyclic coverings of the projective plane

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(Preliminary Version)

1 Introduction

The aim of this note is to give a survey on the irregularity of cyclic coverings of the projective plane $P^2$. Let $f(x, y)$ be a polynomial of degree $d$ over $C$. Let us consider the cyclic multiple plane:

$z^n = f(x, y)$.

Decompose $f$ into irreducible components: $f = f_1^{m_1} \cdots f_r^{m_r}$. We assume that the condition: $\text{GCD}(n, m_1, \ldots, m_r) = 1$ is satisfied. This is nothing but the condition that the above surface is irreducible. We pass to the projective model. Let $\tilde{f}(x_0, x_1, x_2)$ be the homogeneous polynomial associated to $f$ so that $\tilde{f}(1, x, y) = f(x, y)$. Let $C$ be the plane curve defined by the equation: $\tilde{f} = 0$. Let $C_\ell$ be the irreducible component $\tilde{f}_\ell = 0$. Let $L$ denote the infinite line: $x_0 = 0$. Define $e$ to be the smallest integer with the condition: $e \geq \gamma / d$. Set $m_0 = ne - d$. Note that $m_0 = 0$ if and only if $n$ divides $d$. Let $W_n$ be the normalization of the following weighted hypersurface in $P(1, 1, 1, e)$:

$x_3^n = x_0^{m_0} \tilde{f}(x_0, x_1, x_2)$.

The covering map $W_n \rightarrow P^2$ ramifies over $C$ in case $m_0 = 0$ or over $C \cup L$ in case $m_0 \neq 0$. Let $\pi : X_n \rightarrow W_n$ be a desingularization. Let $\varphi : X_n \rightarrow P^2$ be the composed map.

Definition. The irregularity $q(X_n)$ of $X_n$ has three equivalent expressions:

$q(X_n) = \dim H^1(X_n, \mathcal{O}) = \dim H^0(X_n, \Omega^1) = \frac{1}{2} \dim H^1(X_n, \mathbb{R})$

There are four classical references on this topics: de Franchis [dF], Comessatti [C], Zariski [Z1], [Z2]. My personal motivation to this question is its application to the analysis of singular plane curves. Cf. [S].

Proposition 1 (Easy Bound).

$2q(X_n) \leq \sum_{i=0}^r d_i (n - n_i) - 2(n - 1)$

where $n_i = \text{GCD}(n, m_i)$, $d_i = \deg f_i$ and $d_0 = 1$. Note that $n_0 = \text{GCD}(n, d)$.

Proof. Let $\Gamma \in X_n$ be the inverse image of a general line on $P^2$. We can easily prove that $H^1(X_n, \mathcal{O})$ injects to $H^1(\Gamma, \mathcal{O})$. The Hurwitz formula gives the genus of $\Gamma$. 
Corollary.

\[ 2q(X_n) \leq \begin{cases} (n-1)(\sum_{i=1}^{r} d_i - 2) & \text{if } n \mid d \\ (n-1)(\sum_{i=1}^{r} d_i - 1) & \text{otherwise} \end{cases} \]

Let us exhibit examples with positive irregularity. Let \( \Gamma_k \rightarrow \mathbb{P}^1 \) be a k-fold cyclic covering. Given a rational map \( \phi : \mathbb{P}^2 \rightarrow \mathbb{P}^1 \). Suppose that \( \Gamma_k \) is given by the equation:

\[ y_2^k = \Pi(b_i y_0 - a_i y_1)^{p} \]

and that the map \( \phi \) is given by \((G(x_0, x_1, x_2), H(x_0, x_1, x_2))\) where both \( G \) and \( H \) are homogeneous polynomials of degree \( \ell \). If \( n \mid \ell \cdot \sum \ell_i \) and \( k \mid n \), then the multiple plane \( X_n \) defined by the equation:

\[ x_3^n = \Pi(a_i G(x) - \zeta \ell_i H(x)) \]

factors through \( \Gamma_k \). In this case, we say that \( X_n \) factors through a pencil. We see that \( X_n \) has positive irregularity if \( \Gamma_k \) has positive genus.

In order to investigate the irregularity of cyclic coverings of \( \mathbb{P}^2 \), there are three approaches:

(i) through the behavior of rational differential forms, cf. Esnault [E], Zuo [Z]

(ii) through the action of the cyclic group \( \mathbb{Z}_n \) on the Albanese variety, cf. Khashin [K], Catanese-Ciliberto [CC]

(iii) through the topology of complements of the branch curves, cf. Libgober [L], Randell [R], Kohno [Ko], Loeser-Vaqüé [LV], Dimca [D].

2 Differential forms

Let \( \psi : S \rightarrow \mathbb{P}^2 \) be a composition of blow-ups so that the inverse image of \( C \cup L \) has normal crossings. Write

\[ \psi^*(\sum_{i=0}^{r} m_i C_i) = \sum \nu_j D_j. \]

Here we set \( C_0 = L \). We understand that if \( j \leq r \), \( D_j \) is the strict transform of \( C_j \) and \( \nu_j = m_j \), and that for \( j > r \), \( D_j \) is exceptional for \( \psi \). Since \( \psi^*(\sum_{i=0}^{r} m_i C_i) \in |n\psi^*\mathcal{O}(\epsilon)| \), one can construct an n-fold covering of \( S \), which ramifies over \( \psi^*(\sum_{i=0}^{r} m_i C_i) \). Let \( W'_n \) denote its normalization. Up to birational equivalence, we have the commutative diagram:

\[
\begin{array}{ccc}
W'_n & \xrightarrow{\phi} & X_n \\
\downarrow & & \downarrow \\
\mathbb{P}^2 & \xrightarrow{} & S
\end{array}
\]

Set \( \z = e^{2\pi i/n} \). The eigenspace decomposition of the structure sheaf \( \mathcal{O}_{X_n} \) has the following consequence:

**Proposition 2** (Esnault [E]). *In this situation, we have*

\[ H^0(X_n, \mathcal{O}_{X_n})^{(i)} \cong H^0(S, \mathcal{L}^{(i)-1}), \]

where \( \mathcal{L}^{(i)} = \psi^*\mathcal{O}(i\epsilon) \otimes \mathcal{O}(-\sum [\nu_j/n] D_j) \).

As for the eigenspace decomposition of the sheaf \( \Omega^1 \), we have
Proposition 3 ([E], [Zu]). One has
\[ H^0(X_n, \Omega^1)^{C_i} \cong H^0(S, \Omega^1(\log D(i)) \otimes \mathcal{L}^{(i)^{-1}}), \]
where \( D(i) = \sum (iv_j - n[iu_j/n]D_j). \)

Remark. Note that \( D_j \not\subset D(i) \) if and only if \( n|iv_j. \)

The Bogomolov type vanishing theorem gives the following criterion for the vanishing of the irregularity.

Theorem 1 ([E], [Zu]). If \( D(i) \) is big for all \( i \), then \( q(X_n) = 0 \).

Proof. If \( H^0(X_n, \Omega^1)^{C_i} \neq 0, \) then one finds that \( \mathcal{L}^{(i)} \rightarrow \Omega^1(\log D(i)) \), which is impossible if \( D(i) \) is big, since \( D(i) \in |(\mathcal{L}^{(i)})^\otimes n| \).

Since \( q(X) = p_g(X) + 1 - \chi(\mathcal{O}) \), one gets the irregularity \( q(X_n) \) if one knows \( p_g(X_n) \) and \( \chi(\mathcal{O}_{X_n}) \).

Proposition 4.
\[ H^0(X_n, \Omega^2)^{C_i} \cong H^0(S, \Omega^2(\log D(i)) \otimes \mathcal{L}^{(i)^{-1}}). \]

On the other hand, one has the following formula for the term \( \chi(\mathcal{O}) \).

Proposition 5.
\[ \chi(\mathcal{O}_{X_n}) = \sum_{i=0}^{n-1} \chi(\mathcal{O}_{p^1(-(ie - \sum[iu_j/n]d_j)})) - \dim R^1\pi_*\mathcal{O}_{X_n}. \]

Proof. Taking the direct image sheaf, we see that
\[ \chi(\mathcal{O}_{X_n}) = \chi((\psi \circ \phi)_*\mathcal{O}_{X_n}) - \dim R^1(\psi \circ \phi)_*\mathcal{O}_{X_n}. \]

We have
\[ (\psi \circ \phi)_*\mathcal{O}_{X_n} \cong \psi_*\mathcal{L}^{(i)^{-1}} \cong \mathcal{O}(-(ie - \sum[iu_j/n]d_j)), \]
and
\[ \dim R^1(\psi \circ \phi)_*\mathcal{O}_{X_n} = \dim R^1\pi_*\mathcal{O}_{X_n}. \]

Problem. Discuss those line arrangements \( C \) such that \( X_n \) has positive irregularity for some \( n \).
3 Albanese map

Let $X_n$ be a non-singular model of a cyclic multiple plane as defined in Introduction. We denote by $G$ the cyclic group $\mathbb{Z}_n$ and let $\sigma$ be its generator. Suppose $q(X_n) > 0$. We have the Albanese map $\alpha : X_n \to \text{Alb}(X_n)$. The group $G$ acts on $X_n$ and naturally on $\text{Alb}(X_n)$.

**Proposition 6.** If the Albanese image $\alpha(X_n)$ is a curve, then $X_n$ factors through a pencil.

**Proof.** Set $\Gamma = \alpha(X_n)$. The group also acts on $\Gamma$. We infer that $\Gamma/G$ is isomorphic to $\mathbb{P}^1$, because there exists a rational map from $\mathbb{P}^2$ onto it.

**Proposition 7.** Suppose that there exist two linearly independent holomorphic one forms $\omega, \omega'$ such that $\sigma^* \omega = \lambda \omega, \sigma^* \omega' = \lambda^{-1} \omega'$ for some $\lambda$. Then the Albanese image $\alpha(X_n)$ is a curve.

**Proof.** By hypothesis, we find that $\sigma^*(\omega \wedge \omega') = \omega \wedge \omega'$. So $\omega \wedge \omega'$ must be a pull-back of a holomorphic 2-form on $\mathbb{P}^2$, hence $\omega \wedge \omega' = 0$. The assertion follows from the Castelnuovo-de Franchis theorem.

**Proposition 8.** Suppose that there exists an $n$-th root of unity $\lambda$ ($\lambda \neq \pm 1$) such that $\sigma^* \omega = \lambda \omega$ for all $\omega \in H^0(X_n, \Omega^1)$. Then $\lambda$ can take one of the values $\pm i, \pm \rho, \pm \rho^2$ where $\rho = e^{2\pi i/3}$. Furthermore, $\text{Alb}(X_n) \cong E^q_\lambda$, where $E_\lambda$ is the elliptic curve $\mathbb{C}/\mathbb{Z} \oplus \mathbb{Z}_\lambda$.

**Proof.** Cf. Comessatti [C].

**Theorem 2 (de Franchis [dF]).** If $q(X_2) > 0$, then $X_2$ factors through a pencil.

**Proof.** In case $n = 2$, one must have $\sigma^* \omega = -\omega$ for all $\omega \in H^0(X_n, \Omega^1)$. So the assertion follows from Propositions 6 and 7.

**Theorem 3 (Comessatti [C]).** If $q(X_3) > 0$ and if the Albanese image of $X_3$ is a surface, then $\text{Alb}(X_3) \cong E^q_\rho$.

**Proof.** This follows from Propositions 7 and 8.

We can prove this type of results for the cases $n = 4, 6$, which were also proved by Catanese and Ciliberto [CC].

**Theorem 4.** If $q(X_4) > 0$, then either $X_4$ factors through a pencil, or $\text{Alb}(X_4) \cong E^q_\rho$.

**Proof.** If $H^0(X_4, \Omega^1)^{(-1)} \neq 0$, then the surface: $x_3^2 = z_0^m z_0^{-1}$ factors through a pencil, so does $X_4$. In case $H^0(X_4, \Omega^1)^{(-1)} = 0$, by Propositions 7 and 8, we see that either $X_4$ factors through a pencil or $\text{Alb}(X_4) \cong E^q_\rho$.

**Example.** $z^4 = (y^2 - 2x^3)x^2(x^2 + 1)^2(y + 2x)$. In this case, $X_4 \cong E^2_1$. 

4 Alexander polynomials

Set
\[ U = C^2 \backslash \{ f = 0 \} = \mathbb{P}^2 \backslash C \cup L. \]

Write \( U_n = \varphi^{-1}(U) \subset X_n \). We see that \( \varphi : U_n \to U \) is an unramified covering of degree \( n \). We have a commutative diagram:

\[
\begin{array}{ccc}
U_n & \xrightarrow{\varphi} & U \\
\downarrow & & \downarrow f \\
C^* \ni z & \rightarrow & z^n \in C^*
\end{array}
\]

The idea of the topological approach is to calculate the first Betti number of \( X_n \) through that of \( U_n \). Namely, we write:

\[ b_1(X_n) = b_1(U_n) - B.C. \]

The term \( B.C. \) (the boundary contribution) is given by the following:

**Proposition 9.** We have

\[ B.C. = \# \{ \text{the irreducible components of } \varphi^{-1}(C \cup L) \} - 1. \]

This follows from the following:

**Proposition 10.** Let \( S \) be a smooth projective surface and let \( D = D_1 \cup \ldots \cup D_n \) be a divisor having simple normal crossings. Then

\[ b_1(S) = b_1(S \backslash D) - (n - \rho(D)), \]

where \( \rho(D) = \dim \{ \sum \mathbb{R}[D_i] \} \) in \( NS(S) \otimes \mathbb{R} \).

**Proof (Esnault [E]), cf. [He]).** One can deduce this from the Residue sequence:

\[ 0 \to H^0(S, \Omega^1) \to H^0(S, \Omega^1(\log D)) \to H^0(\hat{D}, \mathcal{O}) \to H^1(S, \Omega^1) \]

**Corollary.** \( B.C. \geq r \).

**Example.** If \( f \) is reduced and if \( L \) meets \( C \) transversely, then \( B.C. = r \). Cf. [L].

One can construct an infinite cyclic covering \( \tilde{U} \) of \( U \) as follows.

\[
\begin{array}{ccc}
\tilde{U} & \xrightarrow{\Phi} & U \\
\downarrow f_\infty & & \downarrow f \\
C \ni \tau & \rightarrow & e^{2\pi i \tau} \in C^*
\end{array}
\]

It is well known that \( H_1(U, \mathbb{Z}) = \mathbb{Z}^r \), which is generated by the meridian loops \( \gamma_i \) around \( C_i \). The map \( f_* : \pi_1(U) \to \pi_1(C^*) = \mathbb{Z} \) factors through \( H_1(U, \mathbb{Z}) \) and it sends
It turns out that $\tilde{U}$ is nothing but the quotient of the universal covering of $U$ by the kernel of the above homomorphism.

Let $T$ be the deck transformation on $\tilde{U}$ corresponding to the above infinite cyclic covering. The transformation $T$ induces a linear transformation $T_* : H_1(\tilde{U}) \to H_1(\tilde{U})$. We have the exact sequences ([M2]):

$$H_1(\tilde{U}) \xrightarrow{T_* - I} H_1(\tilde{U}) \to H_1(U) \to$$

Since $H_1(U, \mathbb{Z}) = \mathbb{Z}'$, we infer that the sequence:

$$H_1(\tilde{U}, \mathbb{Z})_0 \xrightarrow{T_* - I} H_1(\tilde{U}, \mathbb{Z})_0 \to \mathbb{Z}' - 1 \to 0 \quad (1)$$

is exact, where $H_1(\tilde{U}, \mathbb{Z})_0 = H_1(\tilde{U}, \mathbb{Z})/\text{Tor}$.

**Definition.** Under the assumption that $H_1(\tilde{U}, C)$ is finite dimensional, the Alexander polynomial of $f$ is defined as follows (cf. [L]):

$$\Delta_f(t) = \det(tI - T_*).$$

Since $T_*$ is defined on $H_1(\tilde{U}, \mathbb{Z})_0$, we infer that $\Delta_f(t) \in \mathbb{Z}[t]$. It follows from (1) that $\Delta_f(t) = (t - 1)^{r-1} \cdot G(t)$ but $G(1) \neq 0$.

**Example.** Suppose that $f(x, y)$ is weighted homogeneous. Let $(a, b)$ be the weights of $(x, y)$ and let $N$ be the degree of $f$ as a weighted homogeneous polynomial. Then $U \to \mathbb{C}^*$ is a fibre bundle, of which fibre is $F = \{(x, y)|f(x, y) = 1\}$. Set $\xi = e^{2\pi i/N}$. Let $h : F \ni (x, y) \to (\xi^a x, \xi^b y) \in F$ be the monodromy map and we denote by $h_*$ the induced linear map on $H_1(F, C)$. In this case, $H_1(\tilde{U}) \cong H_1(F)$ and $\Delta_f(t) = \det(tI - h_*)$. Clearly, the origin $p$ is the only singularity of the affine curve $f = 0$ and $\det(tI - h_*)$ is known to be the local Alexander polynomial $\Delta_p(t)$ of $p$ [M1].

**Definition.** In case $N = \dim H_1(\tilde{U}, C) < \infty$, let $e_j(t)$, $j = 1, \ldots, N$, be the elementary divisors of $tI - T_*$. Set

$$N(n, T_*) = \sum \#\{\text{distinct n-th roots of unity which are roots of } e_j(t)\}.$$
Corollary. If $T_*$ is of finite order, then

$$2q(X_n) = 1 + \#\{n\text{-th roots of unity which are roots of } \Delta_f(t)\} - B.C.$$ 

Definition. We say that $f$ is primitive if the general fibre $f^{-1}(a)$ is irreducible. It is well known that if $f$ is not primitive, then there are polynomials $u$ and $g$ such that $f(x, y) = u(g(x, y))$. Cf. [Su].

Remark. Suppose that $r \geq 2$. If $f$ is not primitive, then (i) $X_n$ factors through a pencil, (ii) the infinite line $L$ does not meet $C$ transversely.

Proposition 11. The vector space $H_1(\tilde{U}, C)$ is finite dimensional if and only if either (i) $r = 1$, or (ii) $r \geq 2$, $f$ is primitive.

Proof. Suppose that $f$ is primitive. The general fibre of the fibration $f_{\infty} : \tilde{U} \rightarrow C$ is irreducible. By Lemma 7 in [Su], we see that $\dim H_1(\tilde{U}, C) \leq \dim H_1(a \text{ general fibre, } C) < \infty$. Note that $f_{\infty}^{-1}(r) = f^{-1}(e^{2\pi i r})$. Assume now that $f$ is not primitive. Writing $f = u(g)$ as above, we set $u^{-1}(0) = \{a_1, \ldots, a_s\}$. Define $V = C \setminus \{a_1, \ldots, a_s\}$. We have the diagram:

$$
\begin{array}{ccc}
\tilde{U} & \rightarrow & U \\
\downarrow & & \downarrow g \\
\tilde{V} & \rightarrow & V \\
\downarrow & & \downarrow u \\
C & \rightarrow & C^*
\end{array}
$$

If $s \geq 2$, it is easy to prove that $\dim H_1(\tilde{V}, C) = \infty$. It follows that $\dim H_1(\tilde{U}, C) = \infty$. If $s = 1$, then $\tilde{V} = C$ and so $\dim H_1(\tilde{U}, C) < \infty$.

Remark. In case $r = 1$, this fact was pointed out in [L].

Now we come to Zariski’s result.

Theorem 6 (Zariski [Z1]). Suppose $r = 1$. If $n = p^a$ ($p$ is a prime number), then $q(X_n) = 0$.

Proof. Since $r = 1$, we infer from (1) that $\Delta_f(1) = \det (I - \tilde{h}_*) = \pm 1$. If a primitive $p^i$-th root of unity ($1 \leq i \leq a$) is a root of the integral polynomial $\Delta_f(t)$, then $\Delta_f(t)$ must be divided by the cyclotomic polynomial $\Phi_{p^i}(t)$. Since $\Phi_{p^i}(1) = p$, this is impossible.

We can generalize this result to the case in which $C$ is reducible.

Theorem 7. Suppose $r \geq 2$. Assume that $f$ is primitive or that $n|d$. If $n = p^a$ ($p$ is a prime number), then

$$2q(X_n) \leq (n - 1)(r - 1).$$

Proof. Assume first that $f$ is primitive. By Proposition 11, $N = \dim H_1(\tilde{U}, C) < \infty$. Let $d_j(t)$ (resp. $d_j$) be the GCD of all $j$-minors of the matrix $tI - T_*$ (resp. $I - T_*$). By the
exact sequence (1), we see that the elementary divisors of $I - T_*$ are $1, \ldots, 1, 0. \ldots 0$. We infer that $d_j = 1$ for $j \leq N - (r - 1)$ and $d_j = 0$ for $j > N - (r - 1)$. Since $d_j(1)|d_j$, we find that $d_j(1) = \pm 1$ for $j \leq N - (r - 1)$ and $d_j(1) = 0$ for $j > N - (r - 1)$. As in the proof of Theorem 6, any primitive $p'$-th root of unity other than 1 cannot be a root of $d_j(t)$ for $j \leq N - (r - 1)$. Let $e_1(t), \ldots, e_N(t)$ be the elementary divisors of $tI - T_*$. We know that $d_j(t) = b_j e_1(t) \cdots e_j(t), b_j \in \mathbb{Q}$. Thus any primitive $p'$-th root of unity other than 1 cannot be a root of $e_j(t)$ for $j \leq N - (r - 1)$. It follows that $N(n, T_*) \leq n(r - 1)$.

Since $B.C. \geq r$, we conclude that $b_1(X_n) \leq (n - 1)(r - 1)$.

In case $n|d$, since the infinite line $L$ does not appear in the branch locus of $X_n \to \mathbb{P}^2$, by taking a suitable line as the infinite line, we may assume that $f$ is primitive.

**Corollary.** If $n = 2$, $r = 2$ and $d$ is even, then $q(X_2) = 0$.

**Definition.** Set $\tilde{F} = \{(x_0, x_1, x_2) \in \mathbb{C}^3|\tilde{f}(x_0, x_1, x_2) = 1\}$. Since $\tilde{f}$ is homogeneous, $\tilde{f}: \mathbb{C}^3\setminus\{\tilde{f} = 0\} \to \mathbb{C}^*$ is a fibre bundle. The typical fibre is $\tilde{F}$. Letting $\eta = e^{2\pi i/d}$, we have the monodromy transformation $\tilde{h}: \tilde{F} \ni (x_0, x_1, x_2) \mapsto (\eta x_0, \eta x_1, \eta x_2) \in \tilde{F}$. It induces a linear transformation $\hat{h}_*: H_1(\tilde{F}, \mathbb{Z}) \to H_1(\tilde{F}, \mathbb{Z})$. Define

$$\Delta_C(t) = \det (tI - \hat{h}_*) \in \mathbb{Z}[t],$$

which is called the Alexander polynomial of the plane curve $C$. Cf. [R], [D].

**Proposition 12.** Under the assumption that the infinite line $L$ is in a general position, we have the equality: $\Delta_f(t) = \Delta_C(t)$.

**Proof.** Cf. [R], [D]. We see that $U \cong (\mathbb{C}^3\setminus\{\tilde{f} = 0\}) \cap \{x_0 = 1\}$. The affine version of the Lefschetz theorem ([H]) asserts that $\pi_1(\mathbb{C}^3\setminus\{\tilde{f} = 0\}) \to \pi_1(U)$ is an isomorphism. It follows that $H_1(U, \mathbb{Z}) \cong H_1(\tilde{F}, \mathbb{Z})$. Furthermore, the transformation $T_*$ corresponds to $\hat{h}_*$. Q.E.D.

**Theorem 8.** Assume that $L$ is in a general position. We have

$$2q(X_n) = 1 + \#\{n\text{-th roots of unity which are roots of } \Delta_C(t)\} - B.C.$$  

**Corollary.** Under the same hypothesis, if $\text{GCD}(n, d) = 1$, then $q(X_n) = 0$.

**Proof.** By hypothesis, we find that $b_1(U_n) = r - 1$ and $B.C. = r$.

We quote two divisibility theorems of the Alexander polynomials. See also [Ko], [LV].

**Theorem 9 (Libgober [L]).** Suppose $f$ is irreducible. Then

$$\Delta_f(t) \mid \prod \Delta_{\tilde{p}}(t),$$

where $\tilde{p}$ moves all local branches of Sing($C \cup L$).
Theorem 10 (Dimca [D]). Suppose $f$ is reduced. Then

$$\Delta_C(t) \mid \prod_{p \in \text{Sing}(C)} \hat{\Delta}_p(t),$$

where $\hat{\Delta}_p(t)$ is the reduced local Alexander polynomial of $p$.

Corollary (Zariski [Z2]). Suppose $L$ is in a general position. If $C$ has only nodes and ordinary cusps as its singularities, then $q(X_n) = 0$ unless $6|n$ and $6|d$.

Proof. We know that $\Delta_p(t) = t - 1$ if $p$ is a node, $= t^2 - t + 1$ if $p$ is an ordinary cusp. Thus $\Delta_C(t) = (t - 1)^{(r-1)}(t^2 - t + 1)^{\ell}$ for some $\ell$. In view of Theorem 8, the assertion follows from this.

Remark. The assumption that $L$ is in a general position is necessary in the above result. Let us consider the case: $f = (x+y)(x+y+1)$. In this case, we find that $q(X_3) = 1$.

References


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