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A characterization of log terminal normal graded rings in terms of Pinkham-Demazure's construction.

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This is the note for the former half part of Tomari-Watanabe's talk at RIMS symposium in March 1992. Our theme is "finite cyclic cover and rational singularity - characteristic 0 and p ". Our note is the part for characteristic 0. In the later half of RIMS talk, K.-i. Watanabe studied the characteristic positive analogy for log terminal singularity and log canonical singularity by F-regular singularity and F-pure singularity in some sense.

The purpose of this note is to give some criterion for the normal graded singularity to be log terminal singularity which includes the Gorenstein rational singularity as the case of index one. We also discuss the log canonical condition. Our main result (3.2) is a natural continuation of the §4 of [14], where we characterized the normal graded isolated singularity in terms of Pinkham-Demazure's construction. Let$R = \bigoplus_{k\geq 0} R_k$ be a normal graded ring which is a finitely generated algebra over a field k. As a geometric representation for such R , the following theorem due to M. Demazure is fundamental.

THEOREM (DEMAZURE [1]). *Let the situation be as above. Let T be a homogeneous element of degree one of the quotient field $Q(R)$ of R. Then there exists an ample $\mathbb{Q}-$Cartier divisor D on $X = Proj(R)$ which satisfies the relations $R_n = H^0(X, O_X(n.D)).T^n$ for $n \in \mathbb{Z}$. Further this D is uniquely determined by the choice of T.*

For 2-dimensional case, H. Pinkham characterize the rational singularity in terms of (X, D) [9]. Later, Flenner and K.-i. Watanabe gave some criterion for the higher dimensional case [18,3,19] (cf.Theorem (3.2.1) of the present paper.). In their theorems, the condition that Spec$(R) - V(R_+)$ has only rational singularity was a major assumption. In the present paper we study this condition in terms of (X, D) in the same line as in [14], [20] and §5 of [13]. In §1 we first recall the correspondence of the canonical module of the cyclic cover and the base ring after [14]. Then the remaining argument using the log ramification formula would be rather familiar for the specialists. In §2 a formula for Goto-Watanabe's invariant $a(\tilde{R})$ is reviewed. This is a key for the reduction of the proof in §3 to the case of index 1.

For the basic terminologies about "log terminal singularity, rational singularity , canonical singularity, ... etc ", we refer to [10,11,8,7]. We will employ the same notation of [14] about ring theoretic objects. Throughout this note, we assume that all singularities are defined over $\mathbb{C}$, and all rings contains $\mathbb{C}$.

§1. Finite cyclic cover and log terminal (and log canonical) condition.

(1.1) First we recall the description of finite cyclic covers of normal domains from [14]. Let $S = \overset{r-1}{\underset{i=0}{\oplus}} S_i$ be a Noetherian normal $\mathbb{Z}_r$-graded domain. That is, S is the direct sum of subgroups S_i ($i = 0, 1, ..., r-1$) satisfying $S_i \cdot S_j \subset S_k$ with $i + j \equiv k$ (mod r) and $1 \in S_0$. We will denote $R = S_0$. Then by our condition, each S_i is an $R-$module. For simplicity, we assume $S_i \neq 0$ for every i. Let $K = Q(R)$ and we fix $u \in S_1 \otimes K$, $u \neq 0$. Then $\mathbf{a}_i = \{ \ \frac{a_i}{u^i} \in K \mid a_i \in S_i \ \}$ is a fractional divisorial ideal of R and $S_i = \mathbf{a}_i \cdot u^i$. Let $f = u^r \in K$ and $(f) = P_1^{(a_1)} \cap \ ... \ \cap P_s^{(a_s)}$, where $P_1, \ ... \ , P_s$ are prime ideals of height one of R. Then for each i, we have

$$\mathbf{a}_i = \{ \ x \in K \mid v_j(x) \geq -\frac{i \cdot a_j}{r} \ \ (\ j = 1, ..., s \ ) \text{ and } v(x) \geq 0 \text{ for every valuation } v$$

of K such that $v(R) \geq 0$ and $v \neq v_j$, $j = 1, ...s$, $\}$.

We have associated the triple $(R, \sum_{i=1}^{s} \frac{a_i}{r} \cdot V_i, f)$ to the pair (S, u), where the fractional divisor $D = \sum_{i=1}^{s} \frac{a_i}{r} \cdot V_i$ ($V_i = \mathrm{Spec}(R/P_i)$) satisfies the condition

$$r \cdot D = \sum_{i=1}^{s} a_i \cdot V_i = div_R(f). \tag{1.1.1}$$

We will always write

$$D = \sum_{i=1}^{s} \frac{p_i}{q_i} V_i, \qquad \text{where } q_i \text{ and } p_i \text{ are} \tag{1.1.2}$$

relatively prime integers with $q_i > 0$ $(i = 1, ..., s)$.

For this description , we attach the divisor D' as

$$D' = \sum_{i=1}^{s} \frac{q_i - 1}{q_i} V_i. \tag{1.1.3}$$

Conversely, from the triple (R, D, f) which satisfies (1.1.1), we can recover S by $S = \overset{r-1}{\underset{i=0}{\oplus}} \mathbf{a}_i \cdot u^i$, where $x \in \mathbf{a}_i$ if and only if $div_R(x) + i \cdot D \geq 0$ in $Div(R) \otimes_{\mathbb{Z}} \mathbb{Q}$ for $x \in K$. We sometimes denote

$$R(D) = \{x \in K \mid div_R(x) + D \geq 0 \text{ in } Div(R) \otimes_{\mathbb{Z}} \mathbb{Q}\}$$

for $D \in Div(R) \otimes_{\mathbb{Z}} \mathbb{Q}$.

In this terminology we can denote $S(R, D, f) = \overset{r-1}{\underset{i=0}{\oplus}} R(i \cdot D) \cdot u^i$ and call such S a cyclic r-cover of R. If D is an integral divisor of R (that is, if $D \in Div(R)$), we say that S is an integral cyclic r-cover of R.

In this note , we always assume that

(1.1.4) $\qquad r = \min\{i \in \mathbb{Z} \mid i > 0 \text{ and } i \cdot D \text{ is a principal divisor}\}.$

By [14], the cyclic cover S is also a normal d-dimensional normal domain.

We will study the criterion for S to be a Gorenstein rational singularity.

LEMMA (1.2). *Let t be a positive integer. S is a log-terminal (resp. log-canonical) singularity of index t if and only if the following two conditions hold.*

(i) There is an integer a' such that $t(K_R + D') - a'D$ is a principal divisor of R.

(ii) $(SpecR, D')$ is a log-terminal (resp. log-canonical) singularity of $(K_R + D')$-index $t \cdot u$ where u is the torsion index of a' in $\mathbb{Z}/r\mathbb{Z}$.

Proof . First of all, we recall the following correspondence of the pluricanonical modules of S and R.

In §2 of [14], the structure of the homogeneous divisor class group $HCl(S)$ is studied. $HCl(S)$ is a subgroup of $Cl(S)$ and we have the relation

$$HCl(S) = \frac{Div(R, D)}{P(R) \oplus \mathbb{Z}/r\mathbb{Z} \cdot D}$$

where $Div(R, D) = \{E \in Div(R) \otimes \mathbb{Q} \mid E = \sum_{V \in Irr^1(R)} r_V \cdot V \text{ with } r_V q_V \in \mathbb{Z}\}$ and $P(R)$ is the group of principal divisors of R. For the element $m(K_R + D') \in Div(R, D)$, we can control the corresponding element of $Cl(S)$ as : For $m \in \mathbb{N}$

$$K_S^{[m]} = \overset{r-1}{\underset{k=0}{\oplus}} R(m(K_R + D') + kD)T^k.$$

Further $K_S^{[m]} \cong S$ if and only if there exists an integer $\alpha \in \mathbb{Z}$ such that $m(K_R + D') - \alpha D \in P(R)$.

In the below we always assume the equivalent these conditions and $t|m$.

There exists an integer N such that $Nm(K_R+D')$ is a Cartier divisor on R. Hence there exists $\varphi \in R(Nm(K_R + D')) = K_R^{[mN]}(mN(D'))$ where $K_R^{[Nm]}(NmD') = \varphi \cdot R$. We obtain the relation

$$\begin{aligned}\varphi \cdot S &= \bigoplus_{k=0}^{r-1} R(Nm(K_R + D') + kD)T^k \\ &= K_S^{[Nm]}.\end{aligned}$$

Let us take resolution of singularities of Spec(R) and Spec(S) in the following.

$$\begin{array}{ccc} \tilde{W} & \xrightarrow[\eta]{} & \mathrm{Spec}S = W \\ \tilde{\pi} \downarrow & & \pi \downarrow \\ \tilde{V} & \xrightarrow{\xi} & \mathrm{Spec}R = V \end{array}$$

where $\eta : \tilde{W} \longrightarrow W$ is a good resolution and $\xi : \tilde{V} \longrightarrow V$ is a good resolution such that the total transform of Supp$(D - [D])$ is normal crossing. $\tilde{\pi} : \tilde{W} \longrightarrow \tilde{V}$ is a map which is naturally induced and is surjective morphism.

Here we recall the following log-ramification formula.

LEMMA (1.2.1) (IITAKA [5, THEOREM 11.5]). *Let $f : Y \longrightarrow X$ be a generically finite and generically surjective morphism of non-singular algebraic varieties, and let L be a reduced divisor on X with only normal crossings. Assume that f is étale outside of $M =_{def} (f^* L)_{red}$.*

Then the following logarithmic ramification formula obtains :

$$K_Y + M = f^*(K_X + L) + R,$$

where R is an effective divisor on Y whose irreducible components are all mapped to lower dimensional subvarieties by f, i.e., we have $f_* R = 0$.

Now $\xi^*(\varphi)$ is a meromorphic Nm-ple d-form on $\tilde{V}$, and we have the relations

$$\begin{aligned}(*) \qquad \xi^*(K_R^{[Nm]}(NmD')) &= \xi^*(\varphi) \cdot O_{\tilde{V}} \\ &= O_{\tilde{V}}(Nm(K_{\tilde{V}} + \tilde{D}') - div_{\tilde{V}}(\xi^*(\varphi))|_E) \\ &= O_{\tilde{V}}(Nm(K_{\tilde{V}} + E + \tilde{D}') - div_{\tilde{V}}(\xi^*(\varphi))|_E - NmE)\end{aligned}$$

where $E = \xi^{-1}(p)_{red} = \bigcup_{j=0}^{m} E_j$ is the decomposition of the exceptional locus into the irreducible components and $\tilde{D}'$ is the strict transform of D', and $div_{\tilde{V}}(\xi^*(\varphi))|_E$ is the part over E of the divisor $div_{\tilde{V}}(\xi^*(\varphi))$.

By definition (V, D') is log-terminal (resp. log-canonical) when $div(\varphi)|_E + NmE$ is an effective divisor whose support coincides with E (resp. is an effective .

Next $\tilde{\pi}^*(\xi^*(\varphi)) = \eta^*(\pi^*(\varphi))$ is a meromorphic Nm-ple d-form on $\tilde{W}$, and we have the relations.

$$(**) \qquad \begin{aligned} \eta^*(K_S^{[Nm]}) &= \eta^*(\pi^*(\varphi)) \cdot O_{\tilde{W}} \\ &= O_{\tilde{W}}(Nm(K_{\tilde{W}}) - div_{\tilde{W}}(\eta^*(\pi^*(\varphi)))) \\ &= O_{\tilde{W}}(Nm(K_{\tilde{W}} + F) - div_{\tilde{W}}(\eta^*(\pi^*(\varphi))) - NmF) \end{aligned}$$

where $F = \eta^{-1}(\tilde{p})_{red} = \bigcup_{j=0}^{m} F_j$ is the decomposition of the exceptional locus into the irreducible components.

By definition W is log-terminal (resp. log-canonical) when $div(\varphi) + NmF$ is an effective divisor whose support coincides with F (resp. an effective divisor).

By (*) and (**) , we obtain the relation

$$(***) \qquad \begin{aligned} &\tilde{\pi}^*\{Nm(K_{\tilde{V}} + E + \tilde{D}') - div_{\tilde{V}}(\xi^*(\varphi))|_E - NmE\} \\ &= Nm(K_{\tilde{W}} + F) - div_{\tilde{W}}(\eta^*(\pi^*(\varphi))) - NmF \end{aligned}$$

Let $\tilde{D}$ be the support of the strict transform of D'. Since $\tilde{\pi} : \tilde{W} \longrightarrow \tilde{V}$ is étale outside $E \cup \tilde{D}$, we have the relation

$$\begin{aligned} &K_{\tilde{W}} + F + \tilde{\pi}^*(\tilde{D})_{red} \\ &= \tilde{\pi}^*(K_{\tilde{V}} + E + \tilde{D}) + R \end{aligned}$$

where R is an effective divisor whose irreducible components are all mapped to lower dimensional subvarieties by $\tilde{\pi}$, in particular Supp$(R) \subset F$. We have

$$\begin{aligned} &Nm(K_{\tilde{W}} + F + \tilde{\pi}^*(\tilde{D})_{red}) \\ =&\tilde{\pi}^*(Nm(K_{\tilde{V}} + E + \tilde{D})) + NmR. \\ &Nm(K_{\tilde{W}} + F) + Nm\tilde{\pi}^*(\tilde{D})_{red} \\ =&\tilde{\pi}^*(Nm(K_{\tilde{V}} + E + \tilde{D}')) + NmR + \tilde{\pi}^*(Nm(\tilde{D} - \tilde{D}')). \end{aligned}$$

By (***),

$$\tilde{\pi}^*\{div_{\tilde{V}}(\xi^*(\varphi))|_E + NmE\} + NmR$$
$$= div_{\tilde{W}}(\eta^*(\pi^*(\varphi))) + NmF + \{Nm\tilde{\pi}^*(\tilde{D})_{red} - \tilde{\pi}^*(Nm(\tilde{D} - \tilde{D}'))\}.$$

Suppose (V, D') is log-terminal (resp. log-canonical) , then $div_{\tilde{V}}(\xi^*(\varphi))|_E + NmE$ is an effective divisor whose support is E (resp. effective). Since $R \geq 0$, $div_{\tilde{W}}(\eta^*(\pi^*(\varphi))) + NmF$ is an effective divisor whose support is F (resp. effective).

Converse implications are also clear, because $\tilde{\pi}_*(R) = 0$.

Finally we discuss the index of $K_R + D'$ more closely. Let us consider the following two integers.

$H = \min\{\alpha \in \mathbb{Z} \mid \alpha > 0, \text{ and } \alpha \cdot (K_R + D') - 0 \cdot D \in P(R)\}$,

$t = \min\{\alpha \in \mathbb{Z} \mid \alpha > 0, \text{ and there exists } \beta \text{ such that } \alpha \cdot (K_R + D') - \beta \cdot D \in P(R)\}$.

There exists $u \geq 1$ with $H = ut$ and

$$u = \min\{\gamma \in \mathbb{Z} \mid \gamma \cdot a' = 0 \text{ in} \mathbb{Z}/r\mathbb{Z}\}.$$

This completes the proof.

§2. The canonical cover of normal graded rings ([15,16]).

We recall the description of the canonical covers in terms of Demazure's construction.

THEOREM (2.1). *Suppose the canonical module K_R of $R(X, D)$ is $\mathbb{Q}-$Cartier of index r. Let the integer a' satisfy the condition that $r.(K_X + D') - a'.D$ is an integral principal divisor on X ; where $D' = \sum_{V \in Irr^1(X)} \frac{q_V - 1}{q_V}.V$ as in [18]. Then the canonical cover $\tilde{R}$ is isomorphic to the graded ring $R(Y, \tilde{D})$ as follows ;*

(i) the normal projective variety Y is defined by the finite covering

$$\rho : Y = Spec_X(\bigoplus_{l=0}^{s-1} O_X(l\left(\frac{r}{s}(K_X + D') - \frac{a'}{s}D\right))) \to X,$$

where $s = (r, a')$,

(ii) the ample $\mathbb{Q}$−Cartier divisor $\tilde{D}$ on Y is defined by $\tilde{D} = \rho^*\{\alpha(K_X + D') + \beta D\}$, where α and β are integers with $\alpha a' + \beta r = s$.

(iii) Further we obtain the relation $K_{\tilde{R}} = \tilde{R}(\frac{a'}{s})$.

One can find the proof of this in [15,16].

REMARK (2.2). *In the situation of (2.1), we have the following relation*

$$K_Y + \left(\tilde{D}\right)' = \rho^{-1}\left(K_X + D'\right) \quad in \quad Div(Y) \otimes \mathbb{Q}.$$

By this we can conclude that $K_Y + \left(\tilde{D}\right)' - \frac{a'}{s} \cdot \tilde{D}$ is an integral principal divisor on Y. This fact also provides the relation $a(\tilde{R}) = \frac{a'}{s}$ (cf. [18,19], (1.6) of [15]).

Proof. We have

$$\frac{r}{s}(K_X + D') - \frac{a'}{s}D = \frac{r}{s}K_X + \sum_{V \in Irr^1(X)} \frac{\frac{r}{s}(q_V - 1) - \frac{a'}{s}p_V}{q_V} \cdot V$$

and $\frac{r(q_V - 1) - a' p_V}{q_V} \in \mathbb{Z}$. We represent

$$\frac{\frac{r}{s}(q_V - 1) - \frac{a'}{s}p_V}{q_V} = \frac{t_V}{s_V}, \quad t_V \in \mathbb{Z}, s_V \in \mathbb{N} \quad \text{with} \quad (t_V, s_V) = 1$$

where s_V is the ramification index of ρ at $V' \in Irr^1(Y)$ which dominates V. we have

$$\rho^{-1}(V) = s_V \cdot \rho^{-1}(V)_{red}$$

and

$$K_Y = \rho^{-1}(K_X) + \sum_{V \in Irr^1(X)} (s_V - 1) \cdot \rho^{-1}(V)_{red}.$$

Hence

$$\tilde{D} = \rho^{-1}(\alpha \cdot K_X) + \sum_{V \in Irr^1(X)} \frac{s_V\{\alpha(q_V - 1) + \beta \cdot p_V\}}{q_V} \cdot \rho^{-1}(V)_{red}.$$

From the equality

$$1 = \frac{a'}{s}(\alpha(1 - q_V) - \beta \cdot p_V) + \{\frac{a'}{s} \cdot s_V \alpha + \beta(s_V \cdot \frac{r}{s} - t_V)\}\frac{q_V}{s_V},$$

we see that $\frac{q_V}{s_V}$ and $\alpha(q_V - 1) + \beta \cdot p_V$ are relatively prime. Hence

$$\left(\tilde{D}\right)' = \sum_{V \in Irr^1(X)} \frac{\frac{q_V}{s_V} - 1}{\frac{q_V}{s_V}} \cdot \rho^{-1}(V)_{red}.$$

Therefore

$$K_Y + \left(\tilde{D}\right)' = \rho^{-1}(K_X) + \sum_{V \in Irr^1(X)} \frac{(q_V - 1)s_V}{q_V} \cdot \rho^{-1}(V)_{red} = \rho^{-1}(K_X + D').$$

Now $K_Y + \left(\tilde{D}\right)' - \frac{a'}{s} \cdot \tilde{D} = \beta \cdot \rho^{-1}\left(\frac{r}{s}(K_X + D') - \frac{a'}{s} D\right)$ and this is an integral principal divisor on Y.

§3. Log terminal graded singularity and log canonical graded singularity.

As same as Theorem (3.12) of [14] we can show the following.

PROPOSITION (3.1). *Let $R(X, D)$ be a normal d-dimensional graded ring over a field k with char$(k) = 0$. Then $U(X, D) \cong$ Spec$(R(X, D) - V(R_+)$ has log terminal (resp. log canonical) singularity at any point $x \in X$, if and only if the following two conditions hold.*

(i) At any point x , there are integers a'_x and t_x (≥ 1) such that

$$t_x(K_X + D') - a'_x D$$

is a principal divisor at x.

(ii) At any point x, (X, x) has log terminal (resp. log canonical) singularity with respect to $K_{X,x} + D'$.

Proof. Let $V(P) = x \in \mathrm{Spec}(R) \subset X$ be a closed point of X and set $U(X, D)_x = U(R_P, D_P)$ be the fiber over $x \in X$ with respect to $U(X, D) \to X$. Here we denote

$$U(R, D) = \mathrm{Spec}_R(A(R, D))$$

by

$$A(R, D) = \oplus_{k \in \mathbb{Z}} R(kD)T^k \subset Q(R)[T, T^{-1}]$$

(cf. Watanabe [20]). We have the k^*-fiber structure $U(R,D) \to \mathrm{Spec}(R)$. We will discuss the log terminal property (resp. the log canonical property) of $\mathbb{Z}$-graded singularity

$$U(R_P, D_P) = \mathrm{Spec}_R(A(R_P, D_P)).$$

We choose f, r as $div_R(f) = rD$ in $Div(R_P)$ and r is the minimal at R_P. Then $f^{-1}T^r \in R(rD)_P T^r$ is a unit of $A(R_P, D)$. We obtain

$$A(R_P, D)/(f^{-1}T^r - 1)A(R_P, D) \cong S(Rx, D, f) \cong S \cong \oplus_{k=0}^{r-1} R_P(kD)T^k.$$

Following Flenner [3], we define $\alpha : A(R_P, D) \to S[U, U^{-1}]$ with $\alpha(g) = \{g \mod (f^{-1}T^r - 1)A(R_P, D)\} \cdot U^m$ for $g \in R_P(mD)T^m$. Since the characteristic of the base field is zero, α is étale ([3, §2]). Hence S is log terminal (resp. log canonical) of index t if and only if so is $U(R_P, D)$. So the assertion follow from Lemma (1.2).

As an application , we will show the following .

THEOREM (3.2). *Let $R = R(X, D)$ be a normal d-dimensional graded singularity represented by Demazure's construction. Let $t \geq 1$ be an integer. Then $R(X, D)$ is a log terminal singularity of index t if and only if the following two conditions hold.*

(i) There is an integer $a' \in \mathbb{Z}$ with $a' \leq -1$ such that $t(K_X + D') - a'D$ is an integral divisor which is a principal divisor on X. Further $t \in \mathbb{N}$ is the minimal integer such that there exists a' as above.

(ii) At each point x of X , (X, x) has log terminal singularity with respect to $K_X + D'$.

Proof. First we prove the assertion in the case $t = 1$.

We recall the result of Watanabe [18,19, cf. 3]

THEOREM (3.2.1). *Let $R = R(X, D) = \oplus_{k \geq 0} R_k$ be a normal graded ring over the field R_0 with $charR_0 = 0$. Then $R(X, D)$ has canonical singularity of index 1 if and only if the following two conditions hold.*

(3.2.1) There is an integer a with $a \leq -1$, such that $K_X + D' - aD \in P(X)$. Here $P(X)$ is the set of principal divisor of X.

(3.2.2) $U(X, D) = SpecR - V(R_+)$ has only rational singularity.

Now the two conditions of (3.2) with $t = 1$ imply the log terminal condition of $U(X, D)$ by Proposition (3.1). Further, by Theorem (3.2.1), R has only canonical singularity of index 1.

Here recall canonical singularity of index 1 is nothing but Gorenstein rational singularity (Elkik-Flenner [2,3]).

So the sufficiency are proved . The converse implication also followed by (2.3.1) and (2.2). The proof of the case of $t = 1$ is finished.

Assume R is a log terminal singularity of index t . Here we recall $R(X, D)$ has log terminal singularity if and only if the canonical module of $R(X, D)$ is $\mathbb{Q}$-Cartier and the canonical cover of $R(X, D)$ is canonical singularity of index 1.

Then (i) follows from [18, (1.6), (2.8)]. Since $U(X, D)$ is log terminal singularity , (ii) follows from Proposition (3.1).

Next assume the conditions (i) and (ii) hold. By [18, (1.6) and (2.8)], the canonical module K_R of R is $\mathbb{Q}$-Cartier of index t. Let $\tilde{R}$ be the canonical cover of R and $\tilde{R} = R(Y, \tilde{D})$ be the representation by Demazure's construction. By Theorem (2.1), we have $a(R(Y, \tilde{D})) = a'/s < 0$ with $s = (t, a')$. Since $U(Y, \tilde{D}) \to U(X, D)$ is étale in codimension one and $U(X, D)$ has log terminal singularity by (3.1), $U(Y, \tilde{D})$ has also log terminal singularity. (cf. [8,7]) Since the canonical module of $R(Y, \tilde{D})$ has the index 1, $U(Y, \tilde{D})$ has log terminal singularity of index 1, that is Gorenstein rational singularity. Hence , by [3,18,19], $R(Y, \tilde{D})$ has Gorenstein rational singularity. Therefore $R(X, D)$ is a log terminal singularity.

Next we will consider the condition of R to have log canonical singularity. The following follows easily from (3.1).

PROPOSITION (3.3). *Let $R = R(X, D)$ be a normal d-dimensional graded singularity represented by Demazure's construction. Let $t \geq 1$ be an integer and assume that R is a log canonical singularity of index t. Then the following two conditions hold.*

(i) There is an integer $a' \in \mathbb{Z}$ with $a' \leq 0$ such that $t(K_X + D') - a'D$ is an integral divisor which is a principal divisor on X.And $t \in \mathbb{N}$ is the ;minimal integer such that there exists a' as above.

(ii) At each point x of X , (X, x) has log canonical singularity with respect to

$K_X + D'$.

Proof. As in the same way of the proof of Theorem (3.2), we will discuss the log canonical condition for the canonical cover $\tilde{R} = R(Y, \tilde{D})$. Let $C = \text{Proj}\left(\underset{k\geq 0}{\oplus} \tilde{R}|_k \cdot U^k\right)$ $\xrightarrow{\pi}$ Spec$\tilde{R}$ be the filtered blowing up with respect to the grading $\tilde{R}$. C is normal and we have the relation

$$\omega_C \cong O_C(-(a(R)+1)\cdot Y) \cong \underset{k\geq a(R)+1}{\oplus} O_Y(k\tilde{D})T^k.$$

By the log canonical condition for the partial resolution $C \to \text{Spec}\tilde{R}$, we obtain the condition $a(\tilde{R}) = \dfrac{a'}{s} \leq 0$

We will discuss the sufficient condition for $R(X, D)$ to have log canonical condition. We will prove the following.

THEOREM (3.4) (CF. (4.8) OF [12]). *Let $R = R(X, D)$ be a normal d-dimensional graded singularity represented by Demazure's construction. Let $t \geq 1$ be an integer. Suppose the following conditions hold.*

(i) There is an integer $a' \in \mathbb{Z}$ with $a' \leq 0$ such that $t(K_X + D') - a'D$ is an integral divisor which is a principal divisor on X.And $t \in \mathbb{N}$ is the minimal integer such that there exists a' as above.

(ii) At each point x of X , (X, x) has log terminal singularity with respect to $K_X + D'$.

Then $R(X, D)$ is a log canonical singularity of index t .

Proof. (One can find a similar argument in [6]. See also §4 of [12].) We assume $t = 1$. By (3.1), $U(X, D)$ has only log terminal singularity. If $a' < 0$, then we had already seen that R has only log terminal singularity (3.2). Hence we will assume $a' = 0$.

By [3, 18], $C = \text{Proj}\left(\underset{k\geq 0}{\oplus} R|_k \cdot T^k\right) \cong \text{Spec}_X\left(\underset{k\geq 0}{\oplus} O_X(kD)T^k\right)$ has only rational singularity. Let $\varphi : \tilde{C} \to C$ be a morphism induced from resolution of singularity of C and we assume that $\varphi^{-1}(X) \subset \tilde{C}$ is a simple normal crossing divisor. We denote the proper transform of $X \subset C$ as $\tilde{X} \subset \tilde{C}$. Since the canonical module of R is locally principal, we can represent the canonical divisor of $\tilde{C}$ as follows:

$$K_{\tilde{C}} = -E_J + E_I$$

where E_J and E_I are effective divisors on $\tilde{C}$ whose supports have no common irreducible component. Further , since $\mathrm{Spec}(R) - V(R_+)$ has only rational singularity, the support of E_J is contained in $\varphi^{-1}(X)$. Clearly we have $E_J \geq \tilde{X}$.

We will show the relation $E_J = \tilde{X}$ by a contradiction method. Assume $E_J \neq \tilde{X}$. Then $E_J - \tilde{X}$ is an non-zero effective divisor. Hence

$$0 \neq O_{E_J - \tilde{X}} \subset O_{E_J - \tilde{X}}(E_I).$$

We have the natural inclusion relations

$$\begin{array}{ccc} H^0(O_{\tilde{C}}(E_I)) & \xrightarrow{\xi} & H^0(O_{E_J-\tilde{X}}(E_I)) \\ \uparrow & & \uparrow \\ H^0(O_{\tilde{C}}) & \xrightarrow{\tau} & H^0(O_{E_J-\tilde{X}}). \end{array}$$

Since $\tau(1) \neq 0$, ξ is not the zero-map. We have the commutative diagram of exact sequences:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & \omega_{\tilde{C}} & = & \omega_{\tilde{C}} & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \to & \omega_{\tilde{C}}(\tilde{X}) & \to & \omega_{\tilde{C}}(E_J) \cong O_{\tilde{C}}(E_I) & \to & O_{E_J-\tilde{X}}(E_I) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow\cong & & \\ 0 & \to & \omega_{\tilde{X}} & \to & \omega_{E_J} & \to & \omega_{E_J}/\omega_{\tilde{X}} & \to & 0 \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & & & \end{array}$$

and we have

$$\begin{array}{ccccccccc} 0 & \to & H^0(\omega_{\tilde{C}}(\tilde{X})) & \to & H^0(O_{\tilde{C}}(E_I)) & \xrightarrow{\xi} & H^0(O_{E_J-\tilde{X}}(E_I)) & \to & \\ & & \downarrow & & \downarrow & & \downarrow\cong & & \\ 0 & \to & H^0(\omega_{\tilde{X}}) & \xrightarrow{\alpha} & H^0(\omega_{E_J}) & \xrightarrow{\beta} & H^0(\omega_{E_J}/\omega_{\tilde{X}}) & \to & . \end{array}$$

Hence β is not the zero-map and α is not an isomorphism.

In the resolution of singularity $\varphi|_{\tilde{X}} : \tilde{X} \to X$, X has only rational singularity. Hence we have the relation

$$H^0(\omega_{\tilde{X}}) \cong H^0(\omega_X).$$

By the Grauert-Riemenshneider vanishing theorem $H^1(\tilde{C}, \omega_{\tilde{C}}) = 0$, we have the exact sequence

$$0 \to H^0(\tilde{C}, \omega_{\tilde{C}}) \to H^0(\tilde{C}, \omega_{\tilde{C}}(E_J)) \to H^0(\omega_{E_J}) \to 0.$$

We have a natural isomorphism

$$H^0(\tilde{C}, \omega_{\tilde{C}}(E_J)) \cong H^0(\tilde{C} - \varphi^{-1}(X), \omega_{\tilde{C}}).$$

For , since the support of E_J is contained in $\varphi^{-1}(X)$, the relation $\subset$ is obvious. We will show the converse inclusion relation. Since $\mathrm{Spec}(R) - V(R_+)$ has only rational singularity,

$$\begin{aligned} H^0(\tilde{C} - \varphi^{-1}(X), \omega_{\tilde{C}}) &\cong H^0(\mathrm{Spec}(R) - V(R_+), \omega_R) = H^0(\mathrm{Spec}(R), \omega_R) \\ &\cong H^0(\tilde{C}, \omega_{\tilde{C}}(E_J - E_I)) \subset H^0(\tilde{C}, \omega_{\tilde{C}}(E_J)). \end{aligned}$$

Since C has only rational singularity , we obtain

$$H^0(\tilde{C}, \omega_{\tilde{C}}) \cong H^0(C, \omega_C) \quad \text{and} \quad H^0(\tilde{C} - \varphi^{-1}(X), \omega_{\tilde{C}}) \cong H^0(C - X, \omega_C).$$

Hence

$$H^0(\omega_{E_J}) \cong \frac{H^0(C - X, \omega_C)}{H^0(C, \omega_C)} \cong \bigoplus_{k \leq 0} H^0(O_X(K_X + D' + kD)T^k.$$

Since $a(R) = 0$, we have $H^0(\omega_{E_J}) \cong H^0(O_X(K_X))$.

But this contradicts to the fact that α is not an isomorphism.

PROBLEM (3.5). *Do the two conditions of (3.3) imply the log canonical condition of $R(X, D)$?. Obviously the conditions of (3.4) are too strong.*

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