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Citation
数理解析研究所講究録 (1992), 808: 91-146

Issue Date
1992-09

URL
http://hdl.handle.net/2433/82973

Type
Departmental Bulletin Paper
Local monodromy on the fundamental groups of algebraic curves along a degenerate stable curve

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Introduction.

The purpose of this paper is to prove some result on the local monodromy representation on the fundamental groups for a universal degenerating family of punctured algebraic curves.

Let us explain a typical case in a more precise way, i.e. the case of no puncture. We start with a most degenerate stable curve $C_0$ of genus $g \geq 2$. For such a curve, we can associate the dual graph $Y$ whose vertices correspond to the irreducible components of $C_0$ and edges to double points. Consider a local universal deformation $f : C \to \mathcal{D}$ of $C_0$ in the category of stable curves. Let $\mathcal{D}^o$ be the open subset of $\mathcal{D}$ on which the fibers of $f$ are smooth. Let $t$ be a point on $\mathcal{D}^o$. Then we obtain the monodromy map on the fundamental group $\pi_1(C_t, \ast)$

$$\rho_{C_0} : \pi_1(\mathcal{D}^o, t) \to Out \pi_1(C_t, b).$$

Here $b$ is a base point in $C_t$.

We can consider the weight filtration on the fundamental group of curves, which is preserved by the monodromy homomorphism. The main target of this paper is to describe the relation between the monodromy homomorphism and the weight filtration for the local universal deformation of a most degenerate stable curve. The weight filtration coincides with the lower central series for the fundamental group of a complete curve.

Here is a description of the main result: Let $I_Y$ be the image of the injective homomorphism $\rho_{C_0}$ which is a free abelian group of rank $3g - 3$, and let $\{I_Y^{(m)}\}_{m=0,1,2,\ldots}$ be the induced filtration on $I_Y$ derived from the lower central filtration on $\pi_1(C_t, b)$. Put

$$r_m(Y) = rank_Z I_Y^{(m)}/I_Y^{(m+1)}$$

for all $m \geq 3$, $r_2(Y) = s_2(Y)$, $r_1(Y) = s_1(Y)$, and $r_0(Y) = 3g - 3 - s_1(Y) - s_2(Y)$.

Then the main result tells

$$r_m(Y) = 0, \text{ if } m \geq 3, \quad r_2(Y) = s_2(Y), \quad r_1(Y) = s_1(Y), \quad \text{and } r_0(Y) = 3g - 3 - s_1(Y) - s_2(Y).$$

Here $s_2(Y)$ is the number of bridges in the graph $Y$, and $s_1(Y)$ is also another geometric invariant of $Y$ related with the connectivity (cf. Subsection 1.4 for a precise
We also note here the equality \( r_0(Y) = 3g - 3 - s_1(Y) - s_2(Y) \) is due to Brylinski [Br].

The first motivation was to generalize the transcendental part of the previous paper [O] by one of the authors, in which we discussed a similar problem when the base \( D \) is one-dimensional, and the graph of \( C_0 \) is a tree. Similarly to that paper, we expect that these results have some applications to \( l \)-adic setting.

Now let us explain the outline of the contents of this paper. In Section 1, we recall some basic notions on stable curves and stable \( n \)-pointed curves, and their associated graphs. Defining some combinatorial invariants for graphs, we formulate the main result of this paper. In Section 2, we recall basic facts on the graph of group by Bass and Serre [S]. We define the notion of edge twists, which is used to describe Dehn twists in an algebraic language. Section 3 is the corner stone of this paper. In this section, we translate the problem of the local monodromy on the fundamental group into a completely algebraic and combinatorial language of the graph of groups. We start with a special case of the Seifert-van Kampen theorem. The key proposition here is the non-abelian Picard-Lefschetz formula (Theorem (3.2)).

In Section 4, we discuss the algorithm to compute Dehn twists for the monodromy explicitly. Some examples are discussed for the low genus cases. These examples also serve as the initial step of the inductive proof of the main result in Sections 5 and 6.

In Sections 5 and 6, we give an inductive proof of the main result. In the first place, we discuss the case of no puncture which is simpler compared with the general case. After that the general case is reduced to this former special case by a simple idea.

Though we do not discuss, our results have purely topological interpretation in terms of Dehn twists associated to pants decomposition of punctured Riemann surfaces.

By the results of J. Morgan and R. Hain, we can equip the Malcev Lie algebras of the fundamental groups of algebraic varieties with mixed Hodge structures. It seems an interesting problem to push forward our result toward this direction.

We thank H. Nakamura for pointing out a clue for proving our main results. We also thank Y. Ihara for valuable and stimulating discussion on the theme of this paper, and S. Morita for informing us of the literature on low dimensional topology.
1. Formulation of the main result.

1.1 Stable $n$-pointed curves and their graph.

Let us recall the definition of stable $n$-pointed curves [Kn, §1].

**Definition 1.1** A stable $n$-pointed curve $(C, S)$ of genus $g$ is a pair $(C, S)$ of a proper connected curve $C$ over the complex number field $\mathbb{C}$ and a subset of $n$-distinct smooth points on $C$ satisfying the following conditions:

(i) $C$ has only ordinary double points as singularities. $C_{\text{sing}}$ denotes the locus of singularities. Let $p : C^{*} \rightarrow C$ be the normalization of $C$. Then we set $C^{*}_{\text{sing}} = p^{-1}(C_{\text{sing}})$ and identify $p^{-1}(S)$ with $S$ via $p$.

(ii) (stability) On the normalization $D^{*}$ of each irreducible component $D$ of $C$ which is isomorphic to $\mathbb{P}^{1}$, the sum of numbers of $D^{*} \cap C_{\text{sing}}$ and $D^{*} \cap S$ is at least 3.

When $n = 0$, the above definition gives the notion of stable curves [DM].

A graph of a stable $n$-pointed curve

For each stable $n$-pointed curve $(C, S)$, we can associate the (dual) graph $Y$ in the following manner [DM], [N].

**Definition 1.2**

(1) Each vertex $P$ of the graph $Y$ corresponds uniquely to an irreducible component $C_{P}$ of $C$. Or equivalently, each vertex $P$ corresponds uniquely to a connected component of the normalization $C^{*}$ of $C$.

(2) A pair $\{y, \overline{y}\}$ of mutually inverse (oriented) edges of $Y$ corresponds uniquely to a singular point $q_{\{y, \overline{y}\}}$ of $C$. If necessary, we refer to the pair $\{y, \overline{y}\} = |y|$ as a geometric edge associated with $y$ or with $\overline{y}$. We also denote $q_{\{y, \overline{y}\}}$ by $q_{y}, q_{\overline{y}}$, or $q_{|y|}$. The set of geometric edges is denoted by $\text{Edge}(Y)_{\text{geom}}$.

(3) For each edge $y$, its two extremities are given by the vertices $P_{1}, P_{2}$ so that

$$q_{y} = C_{P_{1}} \cap C_{P_{2}} \quad (\text{if } P_{1} \neq P_{2}),$$

$$q_{y} = C_{P_{1}} \cap C_{\text{sing}} \quad (\text{if } P_{1} = P_{2}).$$

(4) There is a function

$$v : \text{Vert}(Y) \rightarrow \mathbb{Z} \times \mathbb{Z}$$

from the set of vertices $\text{Vert}(Y)$ of $Y$ to the product of the set of non-negative integers defined by $v(P) = (g_{P}, n_{P})$. Here $g_{P}$ is the genus of the normalization $C^{*}_{P}$ of $C_{P}$, and $n_{P}$ is the cardinality of the set $S \cap C_{P}$.

For each edge $y$, we denote by $o(y)$ the origin and by $t(y)$ the terminus of $y$, respectively. Choose one edge from each geometric edge $|y| = \{y, \overline{y}\}$, and form a subset $\text{Edge}(Y)_{+}$. Then $\#(\text{Edge}(Y)_{+}) = \#(\text{Edge}(Y)_{\text{geom}}) = \frac{1}{2} \#(\text{Edge}(Y))$. We have the following condition for the above connected graph $Y$ with function $v$. 
Proposition (1.1).

\[ g = \sum_{P \in Vert Y} g_P + h^1(Y), \]

where \( h^1(Y) = \#(Edge(Y)_{geom}) - \#(Vert(Y)) + 1 \). Moreover

\[ n = \sum_{P \in Vert(Y)} n_P. \]

If \( g_P = 0 \), then

\[ n_P + \#\{ y \in Edge(Y) \mid o(y) = P \} \geq 3. \]

Remark 1.1 The graph \((Y, v)\) determines the homotopy type of \( C - S \).

The following is easy to prove.

Lemma (1.2). Let \((Y, v)\) be the graph of a stable \( n \)-pointed curve of genus \( g \). Then

\[ #(Vert(Y)) \leq 2(g - 1) + n; \quad #(Edge(Y)_{geom}) = \frac{1}{2} #Edge(Y) \leq 3(g - 1) + n. \]

The most degenerate case

Definition 1.3 A stable \( n \)-pointed curve \((C, S)\) is called most degenerate, if it has no deformation with the same homotopy type.

In this case, any irreducible component of \( C \) is of genus 0. Moreover the graph \((Y, v)\) of \((C, S)\) satisfies the following conditions.

Lemma (1.3). If \((Y, v)\) is the graph of a most degenerate stable \( n \)-pointed curve \((C, S)\) of genus \( g \). Then

\[ #(Vert(Y)) = 2(g - 1) + n; \quad #(Edge(Y)_{geom}) = \frac{1}{2} #Edge(Y) = 3(g - 1) + n. \]

In particular,

\[ h^1(Y) = #(Edge(Y)_{geom}) - #(Vert(Y)) + 1 = g. \]

For each \( P \in Vert Y \),

\[ n_P = 3 - \#\{ y \in Edge(Y) \mid t(y) = P \}. \]

Since \( Y \) is connected, \( n_P = 0, 1, \) or \( 2 \). When \( n_P = 2 \), \( P \) is a terminal point of \( Y \).

This is very easy to prove and more or less well-known. We omit a proof.

Example. For \( g = 2 \) and \( n = 1 \), there are three types of the graphs of most degenerate stable \( n \)-pointed curves. The pictures are the following

(a) \hspace{2cm} (b) \hspace{2cm} (c)

Here the assigned number denotes \( n_P \) for each \( P \) if \( n_P > 0 \).
1.2 Weight filtration on the fundamental groups and induced filtration on the automorphism groups.

A group isomorphic to the fundamental group of a compact Riemann surface of genus $g$ is called a surface group of genus $g$. The fundamental group of an $n$-punctured Riemann surface is a free group, if $n > 0$. On these groups, we can define the weight filtration in the following way.

1.2.1 The weight filtration.

Let $\pi_1$ be the surface group of genus $g$. Then we can introduce the weight filtration $\{W_{-m}(\pi_1)\}_{m \geq 1}$ on it, by the lower central series

$$W_{-m}(\pi_1) = \Gamma_m \pi_1$$

for each $m \geq 1$.

Here the higher commutators $\Gamma_m \pi_1$ are defined inductively by

$$\Gamma_1 \pi_1 = \pi_1, \text{ and } \Gamma_{m+1} \pi_1 = [\Gamma_m \pi_1, \pi_1]$$

for each $m \geq 1$.

The case of the fundamental group of a punctured Riemann surface is slightly more complicated (cf. Kaneko [K]).

Let $C$ be a compact Riemann surface of genus $g$, and $S$ be a finite subset of $C$ with cardinality $n$. Choose a base point * in $C - S$. When $n$ is arbitrary, the weight filtration on $\pi_1 = \pi_1(C - S, *)$ is defined as follows. Let $N$ be the kernel of the canonical surjection

$$\pi_1(C - S, *) \to \pi_1(C, *)$$

which is a normal subgroup of $\pi_1$ generated by the homotopy classes which correspond to the puncture.

We set $W_{-1}(\pi_1) = \pi_1$ the whole group, and $W_{-2}(\pi_1) = [\pi_1, \pi_1]N$. Then the weight filtration $\{W_{-n}(\pi_1)\}_{n \geq 1}$ is defined as the fastest decreasing central filtration.

Note that the quotient group $\pi_1(C - S, *)/W_{-1}(\pi_1)$ is isomorphic to the 1-st homology group $H_1(C, \mathbb{Z})$.

1.2.2 The induced filtration.

Now we consider the induced filtration on the outer automorphism group of $\pi_1$ and its subgroup.

Let $Aut_S \pi_1$ be the subgroup of the automorphism group $Aut \pi_1(C - S, *)$ consisting elements which preserve the normal subgroup $N$. Also by $Aut_{S}^{+} \pi_1$ the subgroup of $Aut_S \pi_1$ given as the kernel of the composition of the canonical homomorphisms

$$Aut_S \pi_1(C - S, *) \to Aut \pi_1(C) \to Aut H_2(\pi_1(C), \mathbb{Z}).$$

When $g = 0$, $Aut_{S}^{+} \pi_1 = Aut_S \pi_1$, and when $g > 1$, $Aut_{S}^{+} \pi_1$ is an index 2 subgroup of $Aut_S \pi_1$. 
**Notation 1.1** We denote by $\tilde{\Gamma}_{g,n}$ the group $Aut^+_S\pi_1$, and by $\Gamma_{g,n}$ the group $Out^+_S\pi_1$.

**Remark 1.2**
By a classical theorem of Nielsen, $\Gamma_{g,n}$ is isomorphic to a mapping class group or a Teichmüller group (cf. [ZVC], §§5.7).

The weight filtration on $\pi_1(C - S, \ast)$ canonically induces a filtration on $\tilde{\Gamma}_{g,n}$ by

$$\tilde{\Gamma}_{g,n}[k] = \{ \sigma \in \tilde{\Gamma}_{g,n} \mid \text{any } l \geq 1, \text{and any } x \in W_{-l}(\pi_1), \sigma(x)x^{-1} \in W_{-k-l}(\pi_1) \}.$$

Passing to the quotient $\Gamma_{g,n} = \tilde{\Gamma}_{g,n}/Inn(\pi_1(C - S, \ast))$, we can define the induced filtration on $\Gamma_{g,n}$, by the image of the canonical homomorphism:

$$\Gamma_{g,n}[k] = Image(\tilde{\Gamma}_{g,n}[k] \to \Gamma_{g,n})$$

for each $k$. Then we have the following

**Proposition (1.4).**

1. $\Gamma_{g,n}[0] = \Gamma_{g,n}$, and

$$[\Gamma_{g,n}[k], \Gamma_{g,n}[l]] \subset \Gamma_{g,n}[k + l] \text{ for any } k, l \geq 0;$$

2. The quotient $\Gamma_{g,n}/\Gamma_{g,n}[1]$ is isomorphic to the Siegel modular group $Sp(g; \mathbb{Z})$

3. For any $m$ ($m \geq 1$), the quotient group $\Gamma_{g,n}[m]/\Gamma_{g,n}[m + 1]$ is a free abelian group of finite rank.

**Proof.** The statements (1) and (2) are well-known. When $n = 0$, (3) is proved by Asada [A]. In the case of $n > 0$, a pro-$l$ analogy is proved by Kaneko [K]. Although the discrete case can be treated almost in the same way, we shall give a proof for the sake of completeness. Also the case of $m = 2$ is not explicitly stated in [K].

For simplicity, we write $\tilde{\Gamma}$ and $\Gamma$ instead of $\tilde{\Gamma}_{g,n}$ and $\Gamma_{g,n}$, respectively. And for each $m \geq 0$, we write $\tilde{\Gamma}[m]$ and $\Gamma[m]$ for $\tilde{\Gamma}_{g,n}[m]$ and $\Gamma_{g,n}[m]$, respectively. We write $gr_m(\pi_1) = W_{-m}(\pi_1)/W_{-m-1}(\pi_1)$ for each $m \geq 1$.

First, we define a group homomorphism

$$\tilde{h}_m : \tilde{\Gamma}[m]/\tilde{\Gamma}[m + 1] \to gr_{m+1}(\pi_1)^{\oplus 2g} \times gr_m(\pi_1)^{\oplus(n-1)}$$

as follows. For $\sigma \in \tilde{\Gamma}$, put $s_i(\sigma) = \sigma(\alpha_i)\alpha_i^{-1}$, $s_{g+i}(\sigma) = \sigma(\beta_i)\beta_i^{-1}$ ($1 \leq i \leq g$), and let $t_j$ be an element of $\pi_1$ such that $\sigma(\gamma_j) = t_j\gamma_jt_j^{-1}$ ($1 \leq j \leq n - 1$). Since $\pi_1$ is a free group of rank $2g + n - 1 > 1$, the centralizer of $\gamma_j$ is an infinite cyclic group generated by $\gamma_j$. Hence, if $m \neq 2$, $t_j$ is uniquely determined. If $m = 2$, we normalize $t_j$ as follows. Since $gr_2(\pi_1)$ is a free $\mathbb{Z}$-module with a basis

$$[\alpha_i, \alpha_j], [\beta_i, \beta_j] \quad (1 \leq i < j \leq g);$$

$$[\alpha_i, \beta_j] \quad (1 \leq i, j \leq g, (i, j) \neq (g, g)); \quad \gamma_j \quad (1 \leq j \leq n - 1),$$
we can normalize $t_j$ uniquely in such a way that the coefficients of $\gamma_j$ is 0 when $\{t_j \mod W_{-3}(\pi_1)\}$ is expressed as a $\mathbb{Z}$-linear combination of this basis. Now, for $\sigma \in \tilde{\Gamma}[m]$, we define

$$\tilde{h}_m(\bar{\sigma}) = (s_i(\sigma) \mod W_{-m-2}(\pi_1))_{1 \leq i \leq 2g} \times (t_j(\sigma) \mod W_{-m-1}(\pi_1))_{1 \leq j \leq n-1}$$

($\bar{\sigma}$ denotes the class of $\sigma$). The fact that $\tilde{\Gamma}[m]$ acts trivially on $gr_{m+1}(\pi_1)$ and the formula

$$s_i(\sigma \tau) = \tau(s_i(\sigma)) s_i(\tau) \quad \sigma, \tau \in \tilde{\Gamma}$$

implies that $\tilde{h}_m$ is a homomorphism. Obviously, $\tilde{h}_m$ is injective.

For each positive integer $m$, set

$$Int_{\pi_1}(W_{-m}(\pi_1)) = \{ \sigma \in Int(\pi_1) \mid \sigma = Int(g) \text{ with } g \in W_{-m}(\pi_1) \}.$$ 

Here $Int(g)$ is the inner automorphism of $\pi_1$ induced from the transform by $g : Int(g)(x) = gxg^{-1}$ ($x \in \pi_1$). Let us consider the following two homomorphisms:

$$\iota : gr_m(\pi_1) \to Int_{\pi_1}(W_{-m}(\pi_1))/Int_{\pi_1}(W_{-m-1}(\pi_1));$$

$$\overline{t} \to \text{ the class of } Int(t)$$

$$h : gr_m(\pi_1) \to (gr_{m+1}(\pi_1))^\oplus 2g \times (gr_m(\pi_1))^\oplus (n-1).$$

$$\overline{t} \to ([t_i x_i])_{1 \leq i \leq 2g} \times (\overline{t}_j)_{1 \leq j \leq n-1}$$

Then, since the Lie algebra $gr^W(\pi_1) = \bigoplus_{m=1}^{\infty} gr_m(\pi_1)$ has trivial center, it follows that $\iota$ is an isomorphism, $h$ is injective, and

$$\tilde{\Gamma}[m] \cap Int(\pi_1) = Int_{\pi_1}(W_{-m}(\pi_1)) \quad \text{for all } m \geq 1$$

[A, Lemma 4]. Hence we have the following commutative diagram:

$$\begin{array}{ccc}
0 & \to & Int_{\pi_1}(W_{-m}(\pi_1))/Int_{\pi_1}(W_{-m-1}(\pi_1)) \to \tilde{\Gamma}[m]/\Gamma[m+1] \to \Gamma[m]/\Gamma[m+1] \to 0 \text{(exact)}
\end{array}$$

Since $\tilde{h}_m$ is injective, to prove Proposition, it suffices to show that the cokernel of $h$ is a free $\mathbb{Z}$-module of finite rank. Now, $gr_m(\pi_1)$ and $gr_{m+1}(\pi_1)$ are both free $\mathbb{Z}$-module of finite rank, $h$ is injective, and $h \otimes_{\mathbb{Z}} F_p$ is also injective for all prime number $p$ since $gr^W(\pi_1) \otimes_{\mathbb{Z}} F_p$ has trivial center. Therefore, by Lemma 4 in [A], the cokernel of $h$ is a free $\mathbb{Z}$-module of finite rank. (q.e.d)
1.3 The non-abelian monodromy homomorphism.

Let \((C_0, S_0) = (C, S)\) be a most degenerate stable \(n\)-pointed curve of genus \(g\) with the graph \(Y\). Consider the local universal deformation of \((C_0, S_0)\). For each geometric edge \(e = |y| (y \in \text{Edge}(Y))\), let

\[
u_e v_e = 0 \quad (\text{in } (u_e, v_e) \in C^2)
\]

be the local defining equation of the singularity associated with \(e\). Let

\[
u_e v_e = t_e \quad (\text{in } (u_e, v_e, t_e) \in C^3)
\]

be the local universal deformation of the above singularity [DM, §1] [Kn, §2]. For each \(e\), we can associate a small complex disk \(D_e = \{ t_e \in C \mid |t_e| < \epsilon \}\). Then over the polydisk \(D = \prod_{e \in \text{Edge}(Y)_{\text{geom}}} D_e\), we have a local universal family

\[
f : C \rightarrow D, \quad S : \{1, \ldots, n\} \times D \rightarrow C.
\]

If \(t = (t_e)_{e \in \text{Edge}(Y)_{\text{geom}}}\) satisfies \(t_e \neq 0\) for any \(e \in \text{Edge}(Y)_{\text{geom}}\), the fiber \(f^{-1}(t) = C_t\) is a smooth proper curve of genus \(g\), and \(S(t) = S_t\) is a set of \(n\) distinct points on \(C_t\). Let \(D^0\) be the open subset of \(D\) consisting of such points. Choose such a point \(t_0\) in \(D^0\). Let

\[
\pi_1(C_{t_0} - S_{t_0}, *)
\]

be the fundamental group of the \(n\)-punctured Riemann surface \(C_{t_0} - S_{t_0}\) with a base point \(*\). Then we have the non-abelian monodromy homomorphism

\[
\rho_{(C_0, S_0)} : \pi_1(D^0, t_0) \cong \mathbb{Z}^{3g - 3 + n} \rightarrow \text{Out} \pi_1(C_{t_0} - S_{t_0}, *)
\]

By using a transcendental result, we can assure that the monodromy homomorphism \(\rho_{(C_0, S_0)}\) is injective [BLM].

Now we want to see the fact that this monodromy homomorphism is compatible with the weight filtration. In fact,

**Proposition (1.5).** The monodromy homomorphism \(\rho_{(C_0, S_0)}\) preserves the weight filtration on \(\pi_1(C_{t_0} - S_{t_0}, *)\). In particular, for any \(\sigma\) of \(\pi_1(D^0, t_0)\), we have \(\sigma(N) = N\), where \(N\) is the kernel of the canonical surjective homomorphism

\[
\pi_1(C_{t_0} - S_{t_0}, *) \rightarrow \pi_1(C_{t_0}, *)
\]

The proof of above proposition is given in Subsection 3.4.
1.4 Bridges, cut systems, and invariants $s_1$, $s_2$ in a graph.

Definition 1.4

(1) An edge $y$ is called a bridge, if the subgraph $Y - \{|y|\}$ is not connected.
(2) A pair $\{|y_1|, |y_2|\}$ of geometric edges is called a cut pair, if neither $|y_1|$ nor $|y_2|$ is a bridge, and the subgraph $Y - \{|y_1| \cup |y_2|\}$ is not connected.

The following is easy to prove.

Lemma (1.6). Let $\{|y_1|, |y_2|\}$ be a cut pair, and $\{|y_2|, |y_3|\}$ ($|y_3| \neq |y_1|$) be another cut pair. Then $\{|y_1|, |y_3|\}$ is also a cut pair.

Definition 1.5. We call a set $E$ of geometric edges a maximal cut system, if

(1) it contains at least two distinct geometric edges;
(2) any pair of two distinct geometric edges $|y|$, $|y'|$ in $E$ is a cut pair;
(3) and no edge $y''$ outside $E$ makes a cut pair with an edge in $E$.

Now we define two invariants of a graph $Y$ which is used to describe the main result of this paper.

Definition 1.6

(1) Let $s_2(Y)$ be the number of bridges in the graph $Y$.
(2) Put $s_1(Y) = \sum_E \{|E| - 1\}$, where $E$ runs over the maximal cut systems in $Y$. 
### 1.5 Main results.

Let \((Y, v)\) be a graph of a most degenerate \(n\)-pointed stable curve of genus \(g\). Recall the monodromy homomorphism \(\rho_{(C_{0}, S_{0})}\) in Subsection 1.3.

**Definition 1.7** We denote by \(I_{Y}\) the image of the monodromy homomorphism

\[
\rho_{(C_{0}, S_{0})} : \pi_{1}(D^{0}, t_{0}) \cong \mathbb{Z}^{3g-3+n} \rightarrow \text{Out} \pi_{1}(C_{t_{0}} - S_{t_{0}}, *)
\]

in \(\Gamma_{g,n} = \text{Out}_{\mathcal{S}}^{+}(\pi_{1})\).

Since \(\rho_{(C_{0}, S_{0})}\) is injective, \(I_{Y}\) is a free abelian subgroup of rank \(3g - 3 + n\). Let \(I^{(m)}_{Y} = I_{Y} \cap \Gamma_{g,n}[m]\) for each \(m \geq 1\), and define the numbers \(\{r_{m}(Y)\}_{m \geq 0}\) by

\[
r_{m}(Y) = \text{rank}_{\mathbb{Z}} I^{(m)}_{Y}/I^{(m+1)}_{Y}
\]

for each \(m \geq 0\).

Note here each \(I^{(m)}_{Y}/I^{(m+1)}_{Y} \subset \Gamma_{g,n}[m]/\Gamma_{g,n}[m + 1]\) is a free abelian group of finite rank by Proposition 1.4, if \(m \geq 1\). We will see later that \(I^{(0)}_{Y}/I^{(1)}_{Y}\) is also a free \(\mathbb{Z}\)-module (Subsection *.*).

Here is the main result of this paper.

**Theorem (1.7).** Let \(Y\) be an associated graph with a most degenerate stable pointed curve of type \((g, n)\). Then

1. \(r_{0}(Y) = 3g - 3 + n - s_{1}(Y) - s_{2}(Y)\);
2. \(r_{1}(Y) = s_{1}(Y)\);
3. \(r_{2}(Y) = s_{2}(Y)\);
4. \(I^{(3)}_{Y} = \{0\}\).

**Remark 1.3** The first statement (1) is due to Brylinski [Br, Prop. 5].

**Corollary (1.8).**

1. When \(n = 0\), the naturally induced homomorphism

\[
\rho_{(C_{0}, S_{0})}(\text{mod } 3) : \pi_{1}(D^{0}, t_{0}) \cong \mathbb{Z}^{3g-3+n} \rightarrow \text{Out}(\pi_{1}(C_{t_{0}} - S_{t_{0}}, *)/W_{-4}\pi_{1})
\]

is injective;

2. When \(n > 0\), the homomorphism

\[
\rho_{(C_{0}, S_{0})}(\text{mod } 4) : \pi_{1}(D^{0}, t_{0}) \cong \mathbb{Z}^{3g-3+n} \rightarrow \text{Out}(\pi_{1}(C_{t_{0}} - S_{t_{0}}, *)/W_{-5}\pi_{1})
\]

is injective.
2. Graph of groups and edge twists.

In this section, we give a preparatory result for a combinatorial description of Dehn twist. In the next section, we specialize the results of this section to the graph of surface groups, and apply them to describe the local monodromy for the fundamental group associated with a given degenerate stable n-pointed curve. We recall the basic contents of Serre’s book [S] in Subsection 2.1. The notion of edge twist does not seem to be found in the literature.

As in the previous section, $Y$ denotes a connected non-empty graph, with oriented edges. For each $y \in \text{Edge}(Y)$, $\overline{y} \in \text{Edge}(Y)$ is the inverse edge of $y$, $o(y)$ and $t(y)$ are the origin and the terminus of $y$, respectively.

2.1 The fundamental groupoid of a graph of groups.

**Definition 2.1** (graph of groups)

A graph of groups $(G, Y)$ is

1. a group $G_P$ assigned for each vertex $P \in \text{Vert}(Y)$.
2. a group $G_y$ assigned for each edge $y \in \text{Edge}(Y)$, with a monomorphism

$$G_y \rightarrow G_{t(y)}, \quad \text{denoted by } a \rightarrow a^y.$$ 

We impose $G_y = G_{\overline{y}}$ for any $y \in \text{Edge}(Y)$.

Let $(G, Y)$ be a graph of groups. Then Serre [S,§5] defines an auxiliary group $F(G, Y)$. Let us recall its definition. Let $F_Y$ be the free group generated over $\text{Edge}(Y)$. Then $F(G, Y)$ is the quotient group of the free product

$$F_Y \ast \left( \prod_{P \in \text{Vert}(Y)} G_P \right)$$

by the subgroup normally generated by the relations:

$$y\overline{y} = 1 \quad (y \in Y); \quad ya^y\overline{y} = a^y, \quad \text{for } y \in \text{Edge}(Y), \ a \in G_y.$$ 

Here $*$ is the product symbol for free product.

**Words of $F(G, Y)$.**

Let $c$ be a path in $Y$ whose origin is a vertex $P_0$. We let $y_1, \ldots, y_n$ denote the edges of $c$, where $n = l(c)$ is the length of $c$, and put

$$P_i = o(y_{i+1}) = t(y_i).$$

**Definition 2.2** A word of type $c$ in $F(G, Y)$ is a pair $(c, \mu)$ where $\mu = (r_0, \ldots, r_n)$ is a sequence of elements $r_i \in G_{P_i}$. The element

$$|c, \mu| = r_0 y_1 r_1 y_2 \ldots y_n r_n \quad \text{of } F(G, Y)$$

is said to be associated with the word $(c, \mu)$. When $n = 0$, we have $|c, \mu| = r_0$. An element of $F_Y \ast \left( \prod_{P \in \text{Vert}(Y)} G_P \right)$ is admissible if it has the form of $|c, \mu|$ for some $c, \mu$.

One says that $(c, \mu)$ is reduced if it satisfies the following condition: if $n = 0$ then one has $r_0 \neq 1$; if $n \geq 1$ then one has $r_i \notin G_{y_i}^{y_i}$ for each index $i$ such that $y_{i+1} = \overline{y_i}$, where $G_{y_i}^{y_i}$ denotes the image of the monomorphism $G_{y_i} \rightarrow G_{t(y_i)}$. 

Fundamental groupoid.

Let us consider composable paths $c_1, c_2$, i.e. $t(c_1) = o(c_2)$. Let $c_1 * c_2$ be the concatenation of $c_1$ and $c_2$. Two words $(c_1, \mu_1), (c_2, \mu_2)$ are said composable, if $c_1, c_2$ are composable. We define the concatenation $(c_1, \mu_1) * (c_2, \mu_2)$ by

$$(c_1 * c_2, \mu_1 * \mu_2),$$

where $l(c_1)$-th element in $\mu_1 * \mu_2$ is given by $r_n^{(1)} r_0^{(2)} \in G_{t(c_1)}$.

We write $\pi_1(G, Y; P_0, P_1)$ for the set of elements of $F(G, Y)$ of the form $|c, \mu|$ with $o(c) = P_0$, $t(c) = P_1$. The sets $\{ \pi_1(G, Y; P_0, P_1) | P_0, P_1 \in Vert(Y) \}$ form a groupoid. In particular,

$$\pi_1(G, Y; P_0, P_0) = \pi_1(G, Y; P_0)$$

is the fundamental group of the graph of groups $(G, Y)$ with the base point $P_0$.

Another realization of the fundamental group

Let us recall another realization of the fundamental group of a graph of groups, i.e. realization as a quotient group of the ambient group $F(G, Y)$.

Let us choose a spanning (or maximal) tree $T$ in $Y$. Then we define the group $\pi_1(G, Y, T)$ as the quotient group of $F(G, Y)$ by the subgroup normally generated by the elements

$$y \ (y \in \text{Edge}(T)).$$

It is shown in Serre[S] (Chap. I, §5, Prop. 20) that this group is isomorphic to the fundamental group $\pi_1(G, Y, P_0)$ by the composition of the canonical homomorphisms

$$\pi_1(G, Y, P_0) \rightarrow F(G, Y) \rightarrow \pi_1(G, Y, T).$$

2.2 Edge Twist.

We choose an edge $y \in \text{Edge}(Y)$, and an element $d$ in the center $Z(G_y)$ of the group $G_y$. Let $D_{y,d}$ be the endomorphism of $F_Y * (\star_{P \in Vert(Y)} G_P)$ defined by

$$D_{y,d}(y) = yy^d, \quad D_{y,d}(\overline{y}) = \overline{y}(d^\overline{y})^{-1},$$

$$D_{y,d}(y') = y' \quad \text{for other edges } y' \not\in \{y, \overline{y}\}$$

and

$$D_{y,d}(x) = x \quad \text{for any element } x \in \star_{P \in Vert(Y)} G_P.$$

Then, since $D_{y,d} D_{\overline{y},d} = 1$, $D_{y,d}$ is an automorphism of $F_Y * (\star_{P \in Vert(Y)} G_P)$. 

Lemma (2.2). $D_{y,d}$ induces an automorphism of $F(G,Y)$.

Proof. We have to check that the defining relation is preserved under the map $D_{y,d}$. In fact, the relation $y\bar{y}=1$ is mapped to

$$yd^y(y^d)^{-1} = (yd^y)(d^y)^{-1} = d^y (d^y)^{-1} = 1.$$ 

Also $ya^y\bar{y} = a^y$ is mapped to

$$(yd^y)a^y(yd^y)^{-1} = (da)^y(a^y)^{-1} = (da)^y(d^y)^{-1} = (dad^{-1})^y,$$

and since $d$ belongs to the center of $G_y$, $dad^{-1} = a$. Here we use the assumption that $d$ belongs to the center of $G_y$. Since the other relators are preserved trivially by $D_{y,d}$, this settles the proof of our proposition.

Definition 2.3 By an abuse of notation, we denote by the same symbol $D_{y,d}$ the automorphism of $F(G,Y)$ induced from $D_{y,d} \in Aut(F_{Y}(\ast \ast G_{P}))_{P \in Vert(Y)}$, and call it the edge twist associated with $(y,d)$.

The following is immediate from the above lemma.

Proposition (2.3).

1. The automorphism $D_{y,d}(w)$ induces a bijection

$$\pi_1(G,Y;P_0,P_1) \simarrow \pi_1(G,Y;P_0,P_1)$$

for each $P_0$ and $P_1$, compatible with composition of groupoid. In other words, $D_{y,d}$ defines an automorphism of the fundamental groupoid of $(G,Y)$. In particular, $D_{y,d}$ defines an automorphism of the fundamental group $\pi_1(G,Y;P_0)$.

2. For any pair $d \in Z(G_y)$ and $d' \in Z(G_{y'})$, the twists $D_{y,d}$ and $D_{y',d'}$ commute.

Thus we can define a homomorphism

$$\prod_{y \in Edge(Y)} Z(G_y) \rightarrow Aut \pi_1(G,Y;P_0).$$

By applying the above construction to a graph of surface groups, we can obtain an algebraic description of Dehn twists in the next section.

3.1 Graph of surface groups.

For each graph of a stable $n$-pointed curve of genus $g$, we can assign a graph of groups naturally, and recover the fundamental group of an $n$-punctured Riemann surface of genus $g$ as the fundamental group of the graph of groups.

Let $(Y, v)$ be the graph of a stable $n$-pointed curve of genus $g$. For such a graph, we consider the following more specialized version of graph of groups.

**Definition 3.1** (graph of surface groups)

1. For each vertex $P, G_P$ is the fundamental group of $C_P - C_P \cap (C_{sing} \cup S)$.
2. For each edge $y, G_y$ is an infinite cyclic group with an assigned generator $\iota_y$.

We put $\iota_y = \iota_y^{-1}$. The monomorphism

$$G_y \to G_{t(y)}$$

is defined by mapping $\iota_y$ to $x$ in $G_{t(y)}$ which is free-homotopically equivalent to a closed path encircling the deleted point $q_y$ in counter-clockwise.

Choose one vertex $P$ of $Y$. If $v(P) = (g_P, n_P)$ and let $Y_P$ be the subset of edges $y$ in $Y$ such that $t(y) = P$. We fix some order on the set $Y_P$. Then the group $G_P$ has a presentation:

$$< \alpha_1, \beta_1, \ldots, \alpha_{g_P}, \beta_{g_P}, \gamma_1, \ldots, \gamma_{n_P}, \gamma_y (y \in Y_P) |$$

$$[\alpha_1, \beta_1] \cdots [\alpha_{g_P}, \beta_{g_P}] \gamma_1 \cdots \gamma_{n_P} \prod_{y \in Y_P} \gamma_y = 1 > .$$

For $y \in Y_P$, the image of the generator $\iota_y$ of $G_y$ is an element $\gamma_y'$ which is conjugate to $\gamma_y$ in $G_{t(y)} = G_P$.

**Remark 3.1** In the above definition of graph of surface groups, the choice of $\iota_y \mapsto x \in G_{t(y)}$ has ambiguity, since only the conjugacy class of $x$ is specified. However, this ambiguity does not affect the definition in the following sense.

Let $(G, Y)$ be a graph of groups. Let $(G, Y')$ be a graph of groups obtained from $(G, Y)$ by “changing the choice of $x$ in the same conjugacy class in $G_{t(y)}$”. Then, there is an isomorphism between $F(G, Y)$ and $F(G, Y')$ compatible with edge twists.

To be precise, let us fix $s_y \in G_{t(y)}$ for each $y \in \text{Edge}(Y)$. Let $(G, Y')$ be the graph of groups defined as follows. The graph $Y'$ is isomorphic to $Y$, with $\text{Vert}(Y) = \text{Vert}(Y')$ and $\text{Edge}(Y) \cong \text{Edge}(Y')$ by $y \mapsto y'$. The groups $G_P$ on $P \in \text{Vert}(Y')$ are identical with the ones in $(G, Y)$, and $G_{y'} = G_y$. We define the monomorphisms $G_{y'} \to G_{t(y')}$ by

$$a \mapsto a^{y'} := s_y a^y s_y^{-1}.$$
The isomorphism $F(G, Y) \rightarrow F(G, Y')$ is defined on generators by $g \mapsto g$ for $g \in G_P$ and

$$y \mapsto s_y^{-1}y's_y$$

for $y \in \text{Edge}(Y)$. Then relators are mapped as

$$y\bar{y} = 1 \mapsto s_y^{-1}y's_y \bar{y}'s_{\bar{y}} = 1,$$

and

$$ya^y\bar{y} \mapsto s_y^{-1}y'a^y s_y \bar{y}'s_{\bar{y}} = s_y^{-1}a^\bar{y}'s_{\bar{y}} = a^\bar{y}.$$
$C_p^*$ are compatible. Then we obtain a Riemann surface $R$ with no boundary of genus $g$ and $n$ punctures.

We have to show $\pi_1(R, \ast) \cong \pi_1(G, Y, P)$ which is nothing but a variant of van Kampen theorem. Since we could not find a good reference, we discuss how to reduce our claim to a simpler well-known case.

Choose a maximal tree $T$ in $Y$, and consider the surface $R_T$ which is the image of $(\bigcup_{P \in \text{Vert}(Y)} C_p^*) \cup (\bigcup_{|y| \in \text{Edge}(T)_{geom}} A_y)$, in $R$ with respect to the natural map. Let $G_{|T}$ be the restriction of $G$ to $T$. Then the usual van Kampen theorem implies $G_T = \lim_{\to}(G_{|T}, T)$ is isomorphic to $\pi_1(R_T, \ast)$.

Let $Y' = Y/T$ be the graph obtained from $Y$ by contracting every edges in $T$ to a point. Then $Y'$ is a graph with a unique vertex $P'$. Define a function $v'$ on $\text{Vert}(Y')$ by $v'(P') = (g, n)$. Then setting $G_{P'} = G_T \cong \pi_1(R_T, \ast)$, we obtain a graph of surface groups $(G', Y')$.

By the definition of the fundamental group of a graph of groups, it is easy to check that there is a canonical isomorphism $\pi_1(G, Y, P) \cong \pi_1(G', Y', P')$. The surface $R$ is obtained from $R_T$ by attaching $g$ handles $A_y$ ($|y| \in \text{Edge}(Y)_{geom} - \text{Edge}(T)_{geom}$). Meanwhile $\pi_1(G', Y', P')$ is $g$ times iterated HNN-extension of $G_{P'}$. It is well known that the isomorphism $G_{P'} \cong \pi_1(R_T, \ast)$ implies $\pi_1(G', Y', P') \cong \pi_1(R, \ast)$. (q.e.d)
3.3 Non-abelian Picard-Lefschetz formula.

Let $Y$ be a graph of a stable $n$-pointed curve $(C_0, S_0)$ of genus $g$. Then we consider the graph of surface groups $G$, naturally associated to $Y$: a free group of rank 2 with a set of three assigned generators for each vertex, and an infinite cyclic group for each edge.

There are $n$-generators corresponding to the $n$-assigned points in $S$. The fundamental group of $(G, Y)$ is isomorphic to the fundamental group of an $n$-punctured Riemann surface of genus $g$. Then $n$-generators of assigned points are free-homotopically equivalent to the simple curves which bound small disks centered at $n$ punctures, respectively.

Let $P_0$ be a vertex of $Y$, and $\pi_1(G, Y, P_0)$ be the fundamental group of the graph of groups with base point $P_0$. Then for each edge $y$ of $Y$, we can associate the edge twist $D_{y, t_y}$, where $t_y$ is a canonical generator of the free cyclic group $G_y$.

**Remark 3.3** Let $\bar{y}$ be the inverse edge of $y$. Then we put $t_{\bar{y}} = t_y^{-1}$ with respect to the identification $G_y = G_{\bar{y}}$. Then we have $D_{y, t_y} = D_{\bar{y}, t_{\bar{y}}}$. Hence we may consider $D_{y, t_y}$ depends only on geometric edge $|y|$. Thus we denote it by $D_{|y|}$, and call it the edge twist associated to $|y|$. We also denote by the same symbol $D_{|y|}$ the induced element in $Out \pi_1(G, Y, P_0)$.

Let us consider the local universal deformation of $(C_0, S_0)$ in the category of stable $n$-pointed curves $f : C \to D$, where the base space $D$ is a $3g - 3 + n$ dimensional polydisk with coordinates $\{(t_i)\}_{1 \leq i \leq 3g-3+n}$. Moreover for the parameters $t_i$, we may assume that the first $\#(Edge(Y)_{geom})$-parameters are the parameters of the local universal deformation of the singularities on $C_0$.

Let $D_e$ be the complex disk associated to a geometric edge $e$ of $Y$ with coordinates $t_e$. Put $D_Y = \prod_{e \in Edge(Y)_{geom}} D_e$. Then $D$ has a product decomposition $D_Y \times D'$ (non-canonical). Here $D'$ is a polydisk of dimension $3g - 3 + n - \#(Edge(Y)_{geom})$. For each punctured disk $D^0_e = \{ t \in C \mid |t| < \epsilon, t \neq 0 \}$, we denote by $\gamma_e$ the associated generator of $\pi_1(D_{e_0}^0, t_{e_0}) (t_{e_0} \neq 0)$, which encircle the origin in counter-clockwise. Then for $D^0_Y = \prod_{e \in Edge(Y)_{geom}} D_e$, $\pi_1(D^0_Y, t^0_e)$ is generated by $\{\gamma_e \mid e \in Edge(Y)_{geom}\}$.

Let $D^0$ be the open subset of $D$ consisting of points whose fibers are smooth. Then $D^0 = D^0_Y \times D'$ and $\pi_1(D^0, t_0) \cong \oplus_{e \in Edge(Y)_{geom}} Z$.

The following result plays a crucial role to reduce the proof of the main result to a combinatorial problem for graph of groups.

**Theorem (3.2).** (non-abelian Picard-Lefschetz formula) We have a commutative diagram

$$\pi_1(D^0, t_0) \xrightarrow{\rho} Out \pi_1(G_{t_0} - S_{t_0}, *)$$

$$\Pi_{e \in Edge(Y_{geom})} ZD_e \longrightarrow Out \pi_1(G, Y, P_0)$$
Here the left vertical arrow is defined by mapping each \( \gamma_e \) to the corresponding edge twist \( D_e \), and the right vertical arrow is induced from \( \pi_1(C_t - S_{t_0}, \ast) \cong \pi_1(G, Y, P_0) \) obtained in the previous theorem, which is unique up to inner automorphisms.

**Proof.** Assume that \( n = 0 \), i.e. \( S_0 \) is empty. Then the proof is a generalization of Main Lemma (1.7) of the transcendental part of the previous paper [O].

Let \( \pi : \tilde{C}_0 \to C_0 \) be the normalization of \( C_0 \). Then

\[
\tilde{C}_0 = \bigcup_{P \in \text{Vert}(Y)} C_P \quad \text{(disjoint)}
\]

and we can number the singularities of \( C_0 \) by \( \{p_e\}_{e \in \text{Edge}(Y)_+^+} \).

Let \( f : \mathcal{C} \to \mathcal{D} \) be the local universal deformation of \( C_0 \). Let \( \{P, Q\} \) be two vertices of an edge \( e \). Then using the parameter of deformation \( t_e \) of each double point \( p_e \) of \( C_0 \), the local defining equation of the smooth analytic space \( C \) at \( p_e \) is written as

\[
u_{P,e}u_{Q,e} = t_e \quad \text{in} \quad (t_e, u_{P,e}, u_{Q,e}) \in C^3
\]

with certain local coordinates \( u_{P,e} \) and \( u_{Q,e} \). Moreover at \( t_e = 0 \), we may assume that \( u_{P,e} = 0 \) is the local defining equation of the component \( C_P \) at \( p,e \), and \( u_{Q,e} = 0 \) the local defining equation of \( C_Q \) at \( p_e \).

For each edge \( e \), choose a sufficiently small positive real number \( \varepsilon_e \). For any \( \varepsilon \in (0, \varepsilon_e) \) we define a chart

\[
U_{\varepsilon}(\varepsilon) = \{(u_{P,e}, u_{Q,e}) \in C_\varepsilon; |u_{P,e}| < \varepsilon, |u_{Q,e}| < \varepsilon\}
\]

of a neighbourhood of \( p_e \) in \( C \), which is identified with that neighbourhood in \( \mathcal{C} \).

Let \( t = (t_e)_{e \in \text{Edge}(Y)_+} \) be a point in \( \mathcal{D}^0 \). Set \( \varepsilon = \varepsilon_e/2 \) and put \( A_{e,t} = U_{\varepsilon_e}(\varepsilon) \cap f^{-1}(t) \) for each \( e \in \text{Edge}(Y)_+ \). Then each \( A_{e,t} \) is an annulus in the Riemann surface \( C_t = f^{-1}(t) \), and the complement \( B_t = C_t - \bigcup_{e \in \text{Edge}(Y)_+} A_{e,t} \) consists of \( \#(\text{Vert}(Y)) \) connected components, each of them corresponding to a unique vertex \( P \) of \( Y \), and a deformation retract of \( C^0_P = C_P - \{\text{double point}\} \). We denote this component by \( B_{P,t} \) for each vertex \( P \in V(Y) \).

Put

\[
B_{P,t}^* = B_{P,t} \cup \bigcup_{e \in \text{St}(P)} \{(u_{P,e}, u_{Q,e}) \in U_e(\varepsilon_e/2) \cap C_t; |u_{P,e}| \geq \eta\}
\]

for a sufficiently small positive real number \( \eta \), smaller than \( |t_e|^{1/2} \) for each \( e \). Here \( \text{St}(P) \) is the set of edges with vertex \( P \).

Then \( B_{P,t}^* \) has \( \#(\text{St}(P)) \) boundary components. The curve \( C_t \) is written as a union

\[
C_t = \bigcup_{P \in \text{Vert}(Y)} B_{P,t}^* \bigcup_{e \in \text{Edge}(Y)_{\text{geom}}} A_{e,t}.
\]
Each $B_{P,t}^{\ast}$ is a deformation retract of $B_{P,t}$, which is homotopically equivalent to $C_{P}^{0}$. Therefore, the $C^{\infty}$-fibration $\cup_{t \in D^{0}} B_{P,t}^{\ast} \rightarrow D^{0}$ is homotopically equivalent to a product $pr_{2} : C_{P}^{0} \times D^{0} \rightarrow D^{0}$. Thus in order to describe the Deck transformation with respect to $\gamma_{t}$, it suffices to see its action on $A_{e,t}$’s and the change of the patching condition with $B_{P,t}^{\ast}$.

Choose a point $t_{0}$. For each $P$, we choose a base point $b_{P}$ in $B_{P,t_{0}}^{\ast}$, and for each tube $A_{e,t_{0}}$, we fix a base point $b_{e}$. When the vertex $P$ is on the edge $e$, we connect the base points $b_{P}$ and $b_{e}$ by an oriented arc $c_{P,e}$ emanating from $b_{P}$. If we consider the graph with vertices $b_{P}$’s and $b_{e}$’s and with edges $c_{P,e}$, this is canonically identified with the barycentric subdivision of the geometric graph $Y_{geom}$.

For each oriented edge $y$ with $o(y) = P$ and $t(y) = Q$, we associate an oriented arc $c_{y} = c_{P,|y|}c_{Q,|y|}^{-1}$ starting from $b_{P}$ and ending at $b_{Q}$.

Let us choose a vertex $P_{0}$ of $Y$ and a base point $b_{P}$ in $B_{P_{0},t_{0}}^{\ast}$. Then we can regard $b_{P}$ as a point on $C_{t_{0}}$. If we fix the arcs $c_{P,e}$ once for all, then we have a canonical isomorphism

$$\pi_{1}(C_{t_{0}}, b_{P}) \cong \pi_{1}(G, Y, P_{0}).$$

Via the above isomorphism of the fundamental groups, any element of $\pi_{1}(C_{t_{0}}, b_{P})$ is written as a product

$$u_{0}c_{y_{1}}u_{1}c_{y_{2}} \cdots u_{n-1}c_{y_{n}}u_{n}.$$ 

Here $y_{1}, \ldots, y_{n}$ is a loop of the graph $Y$, such that

$$o(y_{1}) = t(y_{1}) = P_{0}; \quad t(y_{i}) = o(y_{i+1}) \quad \text{for each} \ (1 \leq i \leq n-1).$$

For each $i \ (0 \leq i \leq n)$, $u_{i}$ is an element of $\pi_{1}(B_{P_{i},t_{0}}^{\ast}, b_{P_{i}})$, with $P_{i} = t(y_{i})$ for $1 \leq i \leq n$.

Let $t_{e} = r_{e}e^{2\pi i \theta_{e}}$ be the polar coordinates of $t_{e}$ for each geometric edge $e$ of $Y$. We may assume that $t_{0} = (r_{e})_{e \in \text{Edge}(Y)_{geom}}$. Then by the relation $u_{P,e}u_{Q,e} = r_{e}e^{2\pi i \theta_{e}}$, $B_{P,t}^{\ast}$ and $B_{Q,t}^{\ast}$ are patched along the two annuli

$$\{u_{P,e} \in C|\eta \leq |u_{P,e}| \leq \frac{r_{e}}{\eta}\} \text{ and } \{u_{Q,e} \in C|\eta \leq |u_{Q,e}| \leq \frac{r_{e}}{\eta}\}$$

in $A_{e,t}$. The increase of $\theta_{e}$ from 0 to 1 rotates the patching condition of two annuli. Hence the arc $c_{y} = c_{P,e}c_{Q,e}^{-1}$ is transformed to $dc_{y}$, where $d$ is an element of $\pi_{1}(B_{P_{0},t_{0}}^{\ast}, b_{P})$ which is free-homotopically equivalent to the generator of $\pi_{1}(A_{e,t_{0}}, *)$.

Thus the proof is completed for the case $n = 0$.

Now let us discuss the general case. Let $f : C \rightarrow D$ and $s : \{1, 2, \ldots, n\} \times D \rightarrow C$ be the local universal deformation of $(C_{0}, S_{0})$. Similarly to the case $n = 0$, we can define $A_{e,t}$ and $B_{P,t}^{\ast}$ for each edge $e$ and vertex $P$. Define a subset $S_{t} = \cup_{i=1}^{n} s(i, t)$ in $C_{t}$ for each point $t$. Then $B_{P,t}^{\ast} - S_{t}$ is homotopically equivalent to $C_{P} - C_{P} \cap S_{0}$. The rest of the proof proceeds completely similarly as the case $n = 0$. (q.e.d)
3.4 Proof of Proposition (1.5).

Since the weight filtration \( \{W_{-m}(\pi_{1})\}_{m\geq} \) is determined by \( N \) and the characteristic subgroups \( \Gamma_{m}\pi_{1} \), it suffices to show that \( \sigma(N) = N \) for any \( \sigma \in \text{Im}\, \rho(C_{0},S_{0}) = I_{Y} \).

Let \( \gamma \) be an element in \( \pi_{1}(C_{t}-S_{t}, *) \), free-homotopically equivalent to a small circle around a point \( s \in S_{t} \). Then via the isomorphism \( \pi_{1}(C_{t}-S_{t}, *) \cong \pi_{1}(G, Y, P_{0}) \) of the previous subsection, \( \gamma \) is represented by an element which is conjugate to the image of some element \( \gamma'' \) in \( G_{P_{1}} \) corresponding to a puncture in the graph of groups \((G,Y)\).

Therefore, there exists some path \( c \) from \( P_{0} \) to \( P_{1} \) such that \( \gamma \) is identified with \( w\gamma''w^{-1} \) for some element \( w = [c,\mu] \in \pi_{1}(G,Y;P_{0},P_{1}) \). Then for any edge \( e \), the twist \( D_{e} \) maps \( \gamma \) to its conjugation \( D_{e}(w\gamma''w^{-1}) = D_{e}(w)\gamma''D_{e}(w)^{-1} \). This completes the proof of proposition (1.5), because \( N \) is normally generated by the elements of the form \( \gamma \) for various \( s \in S_{t} \).
4. An algorithm to compute Dehn twists and examples for the case of low genus.

The purpose of this section is twofold: one is to describe an algorithm to compute Dehn twists explicitly using the theorems of the previous section; another is to calculate some examples for the case when genus is 2 or 3, which also gives the starter of the inductive proof of the main result.

4.1 Description of the algorithm.

For simplicity, we consider the case when \( n = 0 \), and the curve \( C_0 \) is most degenerate. Then the graph \( Y \) is tri-valent. When \( Y \) is most degenerate, \( G_P \) is isomorphic to a free group of rank 2 for any \( P \in Vert Y \). Let \( y_1, y_2, y_3 \) be the three edges such that \( t(y_i) = P \). Corresponding to each \( y_i \), we can consider the images \( x_{P,y_i} = y_i \in G_P \).

Changing \( x_{P,y_i} \) by its conjugate if necessary, we may assume that \( x_{P,y_i} \) satisfy the relation

\[
x_{P,y_1} x_{P,y_2} x_{P,y_3} = 1.
\]

**Step 1 Search of canonical generators.**

We choose a maximal tree \( T \), and want to find a system of canonical generators in the surface group \( \pi_1(G, Y, T) \) of genus \( g \). We restrict the graph of surface groups \( G \) to \( T \), and investigate the inductive limit \( G_T = \lim_{arrow}(G|_{T}, T) \) in the first place. We want to show that \( G_T \) is isomorphic to a free group of rank \( 2g - 1 \).

In order to prove the above fact by induction, we reformulate it for subtrees \( T' \) of \( T \). Put \( G_{T'} = \lim_{arrow}(G|_{T'}, T') \).

**Holes.**

For each vertex \( P \in Vert(T') \), we can consider \( \{ y \in Edge(Y) \mid t(y) = P \} \). We call the pair \((P,y)\) a hole. The set of total holes of the graph \( Y \) is given by

\[
\{ (P, y) \mid P = t(y), P \in Vert(Y), y \in Edge(Y) \}
\]

When \( P \in Vert(T') \) and \( y \notin Edge(T') \), then we call \((P,y)\) is an open hole for \( T' \). We denote by \( h(T') \) the total number of open holes for \( T' \). Then \( h(T') = 3\#(Vert(T')) - \#(Edge(T')) = \#(Vert(T')) + 2 \).

**Lemma (4.1).** \( G_{T'} \) is isomorphic to a free group of rank \( h(T') - 1 \). The generators are given by

\[
H(T') = \{ x'_{P,y} \mid (P,y) \text{ open hole for } T' \}
\]

with a relation

\[
\prod_{(P,y)\in H(T')} x'_{P,y} = 1,
\]

where the order of the product is considered appropriately. Here \( x'_{P,y} \) are the images of \( x_{P,y} \) via \( G_P \to G_{T'} \).

**The order of generators in the relation.**
If one wants to specify the order of the product of (4.1.1), we can do it as follows. For each open hole \((P,y)\), we can associate a dummy vertex \(Q(P,y)\) and an edge connecting \(P\) and \(Q(P,y)\). Let \(\tilde{T}'\) be the extended tree. Then we can embed \(\tilde{T}'\) in an oriented plane \(\Pi\), so that the orientation of \(\Pi\) is compatible with the order \((P,y_1), (P,y_2), (P,y_3)\) of three holes of \(P\). Namely, the direction of the edges \(y_1, y_2, y_3\) changes in a counter-clockwise for the orientation on \(\Pi\) for each \(P\).

**Tree-traversal search.**

Let us start from a vertex \(P_0\), and choose an edge \(y\) with \(o(y) = P_0\).

(Case 1) If \((P_0,y)\) is not an open hole, we move to the adjacent vertex \(P_1 = t(y)\). Write \(y' = y\).

(Case 2) If \((P_0,y)\) is an open hole of \(T'\), we write \(x_{P_0,y}\) first in the product (4.1.1). Rotate the vector \(o(y)t(y)\) counter-clockwise with \(o(y)\) fixed until to meet another edge \(y_1\) with \(o(y_1) = P_0\).

(Case 2-1) If \((P_0,y_1)\) is also an open hole, then write \(x_{P_0,y_1}\) after \(x_{P_0,y}\) in the product (4.1.1). In this case, \(P_0\) is a terminal vertex of \(T'\), and for the last edge \(y_2\) with \(o(y_2) = P_0\), the hole \((P_0,y_2)\) is not open, unless \(T'\) consists of one vertex \(P_0\), the trivial case. We move to the adjacent vertex \(P_1\) such that \(t(y_2) = P_1\). Write \(y' = y_2\).

(Case 2-2) If \((P_0,y_1)\) is not an open hole, we set \(P_1 = t(y_1)\), and write \(y' = y_1\).

At \(P_1\), we start scanning an adjacent edge \(y''\) lying to the left of \(y'\), i.e. \(y''\) is the first edge with \(o(y'') = P_1\) which is meet if we rotate small vector in counter-clockwise starting from \(o(y')t(y') = P_1P_0\).

**Remark 4.1** The order of three generators \(x_{P,y_1}, x_{P,y_2}, x_{P,y_3}\) for each edge is not essential. Even if we are given a relation of different order

\[
x_{P,y_1}x_{P,y_2}x_{P,y_3} = 1,
\]

we can rewrite it as

\[
x_{P,y_1}x_{P,y_2}(x_{P,y_2}^{-1}x_{P,y_3}x_{P,y_2}) = 1,
\]

and replace the generator \(x_{P,y_3}\) by its conjugate \(x_{P,y_2}^{-1}x_{P,y_3}x_{P,y_2}\). Thus in the above determination of the order of elements in the relator (4.1.1) of Lemma (4.1), the embedding of \(\tilde{T}'\) into an oriented plane \(\Pi\) is not essential.

**Proof of Lemma.**

We prove Lemma by induction on \(\#(Vert(T'))\). If \(\#(Vert(T')) = 1\), it is trivial.

Choose a terminal vertex \(P_0\) of \(T'\), and let \(\{y_0, \tilde{y}_0\}\) be the edges with \(t(y_0) = P_0\), and \(o(\tilde{y}_0) = P_0\). Let \(T''\) be a tree

\[
Vert(T'') = Vert(T') - \{P_0\};
\]

\[
Edge(T'') = Edge(T') - \{y_0, \tilde{y}_0\}.
\]
Then
\[ G_{T'} = G_{T''} * G_{y_0} * G_{P_0}. \]

Put \( P_1 = o(y_0). \) Then \( (P_1, \bar{y}_0) \) is an open hole for \( T''. \) Rearranging the position of \( x''_{P_1,y} \) in the product (4.1.1) by a cyclic rotation if necessary, we may assume that \( x''_{P_1,y_0} \) is the last element in the product (4.1.1). We take generators \( x_{P_0,y_0}, x_{P_0,y_1}, x_{P_0,y_2} \) satisfying
\[ x_{P_0,y_0} x_{P_0,y_1} x_{P_0,y_2} = 1. \]
Then
\[ x'_{P_1,y_0} x'_{P_0,y_0} = 1 \quad \text{in} \quad G_{T'}. \]
Thus the presentation of \( G_{T'} \) is given by
\[ \langle x'_{P,y} | (P,y) \text{ open hole for } T; \]
\[ \prod_{(P,y) \text{ open hole for } T'',(P,y) \neq (P_1,\bar{y}_0)} x'_{P,y} x'_{P_0,y_1} x'_{P_0,y_2} = 1 \rangle. \]
The group \( G_{T'} \) is a free group of rank \( \text{rank}(G_{T''}) + 1. \)

Construction of canonical generators.
We compute the quotient realization \( \pi_1(G,Y,T) \) of the fundamental group of a graph of groups \((G,Y)\) with respect to a maximal or spanning tree \( T \) in \( Y. \)
Since \( h(T) = \#(\text{Vert } T) + 2 = \#(\text{Vert } Y) + 2 = 2g, \) \( G_T \) is a free group of rank \( 2g - 1 \) with generators \( \{x'_{P,y} | (P,y) \in H(T)\}. \) From now on we delete the "" in the symbol \( x'_{P,y} \) to simplify notation.
Consider the contracted graph \( Y' = Y/T, \) which has a unique vertex \( T/T \) and \( g \) geometric edges. Let \( y_1, \ldots, y_g \) be \( g \) oriented edges which represent all \( g \) geometric edges (i.e. \( |y_i| \neq |y_j|, \) if \( i \neq j \)). Then for each edge \( y_i, \) two open holes \((o(y_i), y_i)\) and \((t(y_i), y_i)\) are associated. Now the ambient group \( F(G,Y) \) is generated by \( G_T \) and \( y_1, \ldots, y_g \) with relations
\[ y_i t(y_i), y_i^{-1} = x_{o(y_i),\bar{y}_i}^{-1}. \]
Decompose the word \( \prod_{(P,y) \in H(T)} x_{P,y} \) into segments. Then it has a form
\[ w f x_{o(y_1),\bar{y}_1} w x_{t(y_1),y_1} w t, \]
or
\[ w f x_{t(y_1),y_1} w x_{o(y_1),\bar{y}_1} w t. \]
Reversing the orientation of the edge \( y_1 \) for the second case, we may discuss only the first case. Then we put \( \alpha_1 = x_{o(y_1),\bar{y}_1} \) and \( \beta_1 = y_1^{-1} = \bar{y}_1. \) The original word is written as
\[ w f [\alpha_1, \beta_1] x_{t(y_1),y_1}^{-1} w x_{t(y_1),y_1} w t, \]
and changing the order of words cyclically, we may assume that the relator is of the form

$$[\alpha_1, \beta_1]x_{t(y_1), y_1}^{-1}wx_{t(y_1), y_1}w_1w_f.$$

Now for each $i$ ($2 \leq i \leq y$), we want to rewrite the generators $x_{o(y_i), \bar{y}_i}$, $x_{t(y_i), y_i}$, and $y_i$ as follows.

(i) If both $x_{o(y_i), \bar{y}_i}$ and $x_{t(y_i), y_i}$ are contained in the segment $w_1w_f$, then we keep them and $y_i$ the same.

(ii) If both $x_{o(y_i), \bar{y}_i}$ and $x_{t(y_i), y_i}$ are contained in the segment $w$, then we replace them and $y_i$ by their transforms with respect to $x_{t(y_1), y_1}^{-1}$. In this case, the relation

$$y_i x_{t(y_i), y_i}^{-1} y_i^{-1} = x_{o(y_i), \bar{y}_i}^{-1}$$

is still valid.

(iii) If one of $x_{o(y_i), \bar{y}_i}$ and $x_{t(y_i), y_i}$ is contained in $w$ and another in $w_1w_f$, then reversing the orientation of the edge $y_i$, we may assume that $x_{t(y_i), y_i}$ is contained in $w_1w_f$. Then we transform $x_{o(y_i), \bar{y}_i}$ by $x_{t(y_1), y_1}^{-1}$, and replace $y_i$ by $x_{t(y_1), y_1}^{-1} y_i$. Then the relation

$$y_i x_{t(y_i), y_i}^{-1} y_i^{-1} = x_{o(y_i), \bar{y}_i}^{-1}$$

is still valid.

Thus the segment after $[\alpha_1, \beta_1]$ is a product of new $x_{o(y_i), \bar{y}_i}$ and $x_{o(y_i), \bar{y}_i}$ ($2 \leq i \leq g$). We can apply the above process for this shorter word of length $2g - 2$. Iterating this process, we can reach the canonical relation

$$[\alpha_1, \beta_1] \ldots [\alpha_g, \beta_g] = 1.$$

**Step 2**

The algorithm to pass from the quotient realization $\pi_1(G, Y, T)$ to a subgroup realization $\pi_1(G, Y, P)$ is described in the book of Serre [S] (§5, Prop. 20). Under this subgroup realization, apply the definition of edge twists in Section 2.
4.2 Examples in the case of genus 2.

Proposition (4.2). The main theorem (1.5) is true when \( g = 2 \) and \( n = 0 \).

Proof. There are two graphs corresponding to the most degenerate stable curves of genus 2.

One of the two graphs consists of two vertices \( P_1, P_2 \) with three edges \( y_i \) (\( i = 1, 2, 3 \)) so that \( t(y_i) = P_2 \) and \( o(y_i) = P_1 \) for any \( i \) (\( i = 1, 2, 3 \)). Other vertices are given by \( \{\bar{y}_i \ (i = 1, 2, 3)\} \). We denote this graph by \( Y_A \).

In this case, \( s_1(Y_A) = s_2(Y_A) = 0 \). Therefore the part (1) of the main theorem for \( n = 0 \) is a result of [Br] implies that the homomorphism

\[
I_{Y_A} \to \text{Aut} \pi_1(C_t, *) \to \text{Out} \pi_1(C_t, *) / W_{-2} \pi_1
\]

is injective. This means \( I_{Y_A}^{(1)} = \{0\} \). Hence \( I_{Y_A}^{(3)} = \{0\} \) and \( r_i(Y_A) = 0 \) for any \( i \geq 1 \).

Thus we can confirm the main theorem for the graph \( Y_A \).

The other graph consists of two vertices \( P_1, P_2 \) with three edges \( y_i \) (\( i = 1, 2, 3 \)) such that \( o(y_2) = t(y_2) = P_1 \), \( o(y_3) = t(y_3) = P_2 \), and \( t(y_1) = P_1 \) and \( o(y_1) = P_1 \). Other edges are given by \( \{\bar{y}_i \ (i = 1, 2, 3)\} \). We denote this graph by \( Y_B \).

In order to compute Dehn twists, from now on, we use the following abridged convention to denote the elements in \( F(G, Y) \). In place to write \( x_{P_1, y_j} \), we simply write \( x_{ij} \), when \( t(y_j) = P_1 \). Similarly for \( x_{P_i, y_j} \), we write \( x_{ij} \).

4.2.1 Computation of the edge twists of the graph \( Y_B \).

Let us start with 9 generators:

\[
x_{11}, x_{12}, x_{12}, x_{21}, x_{23}, x_{23}, y_i \ (i = 1, 2, 3)
\]

with 5 relations:

\[
x_{11}x_{12}x_{12} = 1; \quad x_{21}x_{23}x_{23} = 1;
\]

\[
y_2x_{12}y_2^{-1} = x_{12}^{-1}; \quad y_3x_{23}y_3^{-1} = x_{23}^{-1}; \quad y_1x_{21}y_1^{-1} = x_{11}^{-1}.
\]

If we choose a tree \( T = \{[y_1]\} \), then \( y_1 = 1 \), \( x_{21} = x_{11}^{-1} \), \( x_{11}x_{21} = 1 \). Hence

\[
x_{12}x_{12}x_{23}x_{23} = 1 \text{ with relations:}
\]

\[
x_{12} = \bar{y}_2x_{12}^{-1}\bar{y}_2^{-1}; \quad x_{23} = \bar{y}_3x_{23}^{-1}\bar{y}_3^{-1},
\]

which implies the canonical relation

\[
[x_{12}, \bar{y}_2][x_{23}, \bar{y}_3] = 1.
\]

Thus we should set

\[
\alpha_1 = x_{12}; \quad \beta_1 = \bar{y}_2; \quad \alpha_2 = x_{23}; \quad \beta_2 = \bar{y}_3
\]

in the group \( \pi_1(G, Y_B, T) \).

Choose \( P_1 \) as a base point. Then, we have

\[
\alpha_1 = x_{12}; \quad \beta_1 = \bar{y}_2; \quad \alpha_2 = y_1x_{23}^{-1}y_1^{-1}; \quad \beta_2 = y_1\bar{y}_3y_1^{-1}
\]

in \( \pi_1(G, Y_B, P_1) \). We note here that \( x_{21} = (x_{23}x_{23})^{-1} = (\alpha_2 \beta_2)^{-1} \).

Here is the computation of the Dehn twists \( D_{y_i} \ (i = 1, 2, 3) \).
Computation (4.1). We write $D_i$ for $D_{y_i}$.

1. $D_1$ keeps $\alpha_1$ and $\beta_1$ invariant. $D_1(\alpha_2) = x_{21}\alpha_2 x_{21}^{-1} = [\alpha_2, \beta_2]^{-1}\alpha_2 [\alpha_2, \beta_2]$, and $D_1(\beta_2) = x_{21}\beta_2 x_{21}^{-1} = [\alpha_2, \beta_2]^{-1}\beta_2 [\alpha_2, \beta_2]$.

2. $D_2$ keeps the canonical generators invariant except for $\beta_1$, and $D_2(\beta_1) = \beta_1 \alpha_1$.

3. $D_3$ keeps the canonical generators invariant except for $\beta_2$, and $D_3(\beta_2) = \beta_2 \alpha_2$.

It is clear that $D_2$ and $D_3$ act on $\pi_1(C_t, \ast)^{ab}$ as mutually independent transvections.

Lemma (4.3). $D_1 \not\in I_{Y_B}^{(3)}$.

Proof. The proof is completely the same as that of [O, Lemma (1.12)]. We omit it. Hence we have $r_0(Y_B) = 2$, $r_1(Y_B) = 0$, and $r_2(Y_B) = 1$. Meanwhile, we find $s_1(Y_B) = 0$ and $s_2(Y_B) = 2$ by drawing the picture of $Y_B$. Thus we have confirmed the main theorem for $Y_B$. (q.e.d)

4.3 One example of genus 3.

In order to complete the inductive proof in Section 5, we have to discuss the case of graph $Y_C$ given as follows. It consists of four vertices $P_i$ ($i = 1, 2, 3, 4$), and six unoriented edges. The oriented edges $y_i$ ($i = 1, \ldots, 6$) are defined by

\[
o(y_1) = t(y_1) = P_1, \quad o(y_2) = t(y_2) = P_2, \quad o(y_3) = t(y_3) = P_3, \quad o(y_4) = t(y_4) = P_4, \quad o(y_5) = t(y_5) = P_2, \quad o(y_6) = t(y_6) = P_3.
\]

The generators of the ambient group are

\[x_{12}, x_{11}, x_{11}, \quad x_{43}, x_{44}, x_{44}, \quad x_{22}, x_{26}, x_{25}, \quad x_{33}, x_{35}, x_{36}\]

and $y_i$ ($i = 1, \ldots, 6$) with relations:

\[x_{12}x_{11}x_{11} = 1, \quad x_{43}x_{44}x_{44} = 1, \quad x_{22}x_{26}x_{25} = 1, \quad x_{33}x_{35}x_{36} = 1\]

and

\[y_1 x_{11} y_1^{-1} = x_{11}^{-1}, \quad y_2 x_{22} y_2^{-1} = x_{22}^{-1}, \quad y_3 x_{33} y_3^{-1} = x_{43}^{-1}, \quad y_4 x_{44} y_4^{-1} = x_{44}^{-1}, \quad y_5 x_{35} y_5^{-1} = x_{25}^{-1}, \quad y_6 x_{26} y_6^{-1} = x_{36}^{-1}.
\]

Choose $T = \{y_i, y_i (i = 2, 3, 5)\}$ as a spanning tree. Then in the group $\pi_1(G,Y,T)$, we have

\[x_{12} = x_{22}^{-1}, \quad x_{35} = x_{25}^{-1}, \quad x_{33} = x_{43}^{-1}.
\]

Eliminating the above 6 $x_{ij}$ from the 4 relations between $x_{ij}$, we have the relation

\[x_{11} x_{11} x_{26} x_{38} x_{44} x_{44} = 1,
\]
which in turn implies the canonical relation:

\[
[x_{11}, y_1][x_{26}, y_6][x_{44}, y_4] = 1.
\]

Naturally we should set

\[
\alpha_1 = x_{11}, \quad \beta_1 = y_1, \quad \alpha_2 = x_{26}, \quad \beta_2 = y_6, \quad \alpha_3 = x_{44}, \quad \beta_3 = y_4.
\]

Rewrite these in the fundamental group \(\pi_1(G, Y, P_2)\) with base point \(P_2\). Then we have

\[
\alpha_1 = y_2^{-1}x_{11}y_2, \quad \beta_1 = y_2^{-1}y_1y_2^{-1}, \quad \alpha_2 = x_{26}, \quad \beta_2 = y_5y_6^{-1}, \quad \alpha_3 = y_5y_3^{-1}x_{44}y_5y_3^{-1}, \quad \beta_3 = y_5y_3y_4y_3^{-1}y_4y_3^{-1}.
\]

We write only the result of the computation of the Dehn twists, which is easy to check.

**Computation (4.2).** We write \(D_i\) for \(D_{y_i}\). Then \(D_i (i = 1, \ldots, 6)\) are given as follows.

1. \(D_1\) keeps canonical generators invariant except for \(\beta_1\). \(D_1(\beta_1) = \beta_1 \alpha_1\).
2. \(D_2\) keeps canonical generators invariant except for \(\alpha_1, \beta_1\).
   \[D_2(\alpha_1) = [\alpha_1, \beta_1]^{-1} \alpha_1[\alpha_1, \beta_1], \quad D_2(\beta_1) = [\alpha_1, \beta_1]^{-1} \beta_1[\alpha_1, \beta_1].\]
3. \(D_3\) keeps canonical generators invariant except for \(\alpha_3, \beta_3\).
   \[D_3(\alpha_3) = [\alpha_3, \beta_3]^{-1} \alpha_3[\alpha_3, \beta_3], \quad D_3(\beta_3) = [\alpha_3, \beta_3]^{-1} \beta_3[\alpha_3, \beta_3].\]
4. \(D_4\) keeps canonical generators invariant except for \(\beta_3\). \(D_4(\beta_3) = \beta_3 \alpha_3\).
5. \(D_5\) keeps \(\alpha_1, \beta_1, \) and \(\alpha_2\) invariant.
   \[D_5(\beta_2) = [\alpha_3, \beta_3]^{-1} \beta_2\alpha_2, \quad D_5(\alpha_3) = c_3^{-1}d_2\alpha_3d_2^{-1}c_3, \quad D_5(\beta_3) = c_3^{-1}d_2\beta_3d_2^{-1}c_3,\]
   where \(c_3 = [\alpha_3, \beta_3]\) and \(d_2 = \beta_2\alpha_2\beta_2^{-1}\).
6. \(D_6\) keeps canonical generators invariant except for \(\beta_2\). \(D_6(\beta_2) = \beta_2 \alpha_2\).

Obviously, we have \(D_5 \equiv D_6\) modulo \(I_{Y_C}^{(1)}\).

**Lemma (4.4).** \(D_5D_6^{-1} \notin I_Y^{(2)}\).

**Proof.** Let us compute \(\delta = D_5D_6^{-1} \in I_{Y_C}^{(1)}\). Then

\[
\delta(\alpha_1)\alpha_1^{-1} = 1, \quad \delta(\beta_1)\beta_1^{-1} = 1, \quad \delta(\alpha_2)\alpha_2^{-1} = 1, \quad \delta(\beta_2)\beta_2^{-1} = [\alpha_3, \beta_3]^{-1}, \quad \delta(\alpha_3)\alpha_3^{-1} \equiv [\alpha_2, \alpha_3]\text{ modulo } W_3(\pi_1), \quad \delta(\beta_3)\beta_3^{-1} \equiv [\alpha_2, \beta_3]\text{ modulo } W_3(\pi_1).
\]

Since there exists no element of weight \(-2\) in \(\pi_1\) such that the associated inner automorphism is equal to \(\delta\) modulo \(\Gamma_{g,n}[2]\), \(\delta\) represents a non-zero element in \(I_{Y_C}^{(1)}/I_{Y_C}^{(2)}\).

(q.e.d)
5. Proof of Main Result.

5.1 Restating Main Theorem.

(5.1.1) Let \((G, Y)\) be a graph of surface groups associated with a most degenerate stable \(n\)-pointed curve of genus \(g\) (see Definition 1.3 and Definition 3.1 if necessary). By Lemma 1.3, the number of edges in \(Y\) is \(3g - 3 + n\). From now on, we simply write \(D_y\) for \(D_{y,y}\) and the terms \(D_{y,y}\) imply \(D_{y,y}\) edges.

For the \(i\)-th puncture of \((G, Y), (i = 1, \ldots, n)\), we denote by \(Q_i\) the vertex on which the puncture lies, and denote by \(z_i\) the corresponding element of \(G_{Q_i}\). It may happen that \(Q_i = Q_j\) for distinct \(i, j\).

We denote by \(\pi_{g,n} := \pi_1(G, Y, P)\) the fundamental group with base point \(P \in \text{Vert}(Y)\). This group is uniquely determined by \(g\) and \(n\) up to isomorphism, that is,

\[
\pi_{g,n} \cong \langle \alpha_1, \beta_1, \ldots, \alpha_g, \beta_g, \gamma_1, \ldots, \gamma_n | [\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g] \gamma_1 \cdots \gamma_n = 1 \rangle.
\]

We shall omit subscripts \(g, n\) in \(\pi_{g,n}\) if they are clear.

Let us fix a spanning tree \(T\) in the graph \(Y\) as in Subsection 2.1. For each \(i\), \((i = 1, \ldots, n)\), there exists a unique path from \(P\) to \(Q_i\) in \(T\) and let us denote it by \(q_i\). Then, \(q_i z_i q_i^{-1}\) is an element of \(\pi_{g,n} := \pi_1(G, Y, P)\), corresponding to one of \(\gamma_j\) up to conjugacy. We denote \(q_i z_i q_i^{-1}\) by \(c_i\).

We equip \(\pi_{g,n}\) with the central filtration \(\pi_{g,n} = \pi_{g,n}(1), \pi_{g,n}(2), \ldots\) that decreases fastest with condition that \(c_1, \ldots, c_n \in \pi_{g,n}(2)\). In other words, we define

\[
\pi(1) := \pi \\
\pi(2) := \langle [\pi, \pi], c_1, \ldots, c_n \rangle \\
\pi(3) := \langle [\pi(1), \pi(2)] \rangle \\
\pi(4) := \langle [\pi(1), \pi(3)], [\pi(2), \pi(2)] \rangle \\
\cdots
\]

where \(\langle \cdots \rangle\) denotes the subgroup normally generated by the elements inside. It is easy to see that this filtration coincides with the one provided in 1.2.1; i.e., we have \(\pi(m) = W_{-m}\pi\). We say as usual that \(\gamma \in \pi\) has weight \(-m\) if and only if \(\gamma \in \pi(m) - \pi(m + 1)\). It is known that \(\cap_{m=1}^{\infty} \pi(m) = \{1\}\) holds, and we define the weight of \(1\) as \(-\infty\)(for a proof, see [K], in which the pro-\(l\) case is proved, and the above follows immediately from the fact that \(\pi\) can be embedded into its pro-\(l\) completion preserving the weight.)

Let us recall the definition of the induced filtration on \(\tilde{\Gamma}_{g,n}\) defined in 1.2.2.

**Definition 5.1** We define a subgroup \(\tilde{\Gamma}_{g,n}\) of \(\text{Aut}(\pi_{g,n})\) by

\[
\tilde{\Gamma}_{g,n} := \{\sigma \in \text{Aut}(\pi_{g,n}) | \sigma : \text{orientation preserving}, \sigma(c_i) \sim c_i \text{ for } i = 1, \ldots, n\},
\]
where \( \sim \) denotes conjugacy (see 1.2.2 for the meaning of orientation preserving). We equip \( \tilde{\Gamma}_{g,n} \) with a filtration \( \tilde{\Gamma}_{g,n}[m] \) by

\[
\tilde{\Gamma}_{g,n}[m] := \{ \sigma \in \tilde{\Gamma}_{g,n} | \sigma(\eta)\eta^{-1} \in \pi_{g,n}(m+k) \text{ for any } k \geq 1 \text{ and any } \eta \in \pi_{g,n}(k) \}\]

We define \( \Gamma_{g,n}, \Gamma_{g,n}[m] \) to be the image of \( \tilde{\Gamma}_{g,n}, \tilde{\Gamma}_{g,n}[m] \) in \( \text{Out}(\pi_{g,n}) \) respectively. It is not difficult to see that this definition does not change if we restrict \( \eta \) to be chosen from a fixed generating set of \( \pi_{g,n} \).

Let \( I_{Y} \) denote the subgroup of \( \text{Out}(\pi_{g,n}) \) generated by edge twists. It is known that \( I_{Y} \) is in fact a subgroup of \( \Gamma_{g,n} \) isomorphic to \( \mathbb{Z}^{\oplus 3g-3+n} \) [BLM](§3).

In Definition 1.7, we equipped \( I_{Y} \) with a filtration by

\[
I_{Y}^{(m)} := I_{Y} \cap \Gamma_{g,n}[m]
\]

for \( m = 0, 1, \ldots \).

Let \( H \) denote the set of bridges in \( Y \). We denote by BRG the subset

\[
\{D_{y}|y \in H\}
\]

of \( I_{Y} \), and denote by MCS the subset

\[
\{D_{y_i}D_{y_i}^{-1}|i = 1 \ldots l, y_{i} \in S_{i} - \{y_{i}\}\}
\]

of \( I_{Y} \), where \( S_{1}, \ldots, S_{l} \) are the maximal cut systems in \( Y \) and each \( y_{i} \) is an arbitrarily chosen element from \( S_{i} \). Observe that \( \#\text{BRG} = s_{2}(Y) \) and \( \#\text{MCS} = s_{1}(Y) \) hold (see Definition 1.6 for \( s_{1} \) and \( s_{2} \)).

In this formulation, we shall prove the next theorem from which Main Theorem 1.7 immediately follows by applying Theorem 3.2.

**Theorem 5.1.** Let \( s_{2} \) denote the number of bridges in \( Y \) and let \( s_{1} \) denote the summation \( \sum \{ \#(S) - 1 \} \) over all the maximal cut systems \( S_{1}, S_{2}, \ldots, S_{l} \). Then we have

1. \( \text{rank}_{\mathbb{Z}}(I_{Y}/I_{Y}^{(1)}) = 3g - 3 + n - s_{1} - s_{2}, \)
2. BRG is a base of \( I_{Y}^{(2)}/I_{Y}^{(3)} \),
3. MCS is a base of \( I_{Y}^{(1)}/I_{Y}^{(2)} \),
4. \( I_{Y}^{(3)} = 0. \)

We shall prove this theorem in the following manner.

Step 1. Prove that \( \text{BRG} \subset I_{Y}^{(2)} \) and that \( \text{MCS} \subset I_{Y}^{(1)} \).

Step 2. Prove (1) of Theorem.
Step 3. Prove that BRG is linearly independent modulo $I_Y^{(3)}$ and that MCS is linearly independent modulo $I_Y^{(2)}$.

When the above steps are completed, we have an inequality

$$3g - 3 + n = \text{rank}_Z(I_Y)$$

$$\geq \text{rank}_Z(I_Y/I_Y^{(1)}) + \text{rank}_Z(I_Y^{(1)}/I_Y^{(2)}) + \text{rank}_Z(I_Y^{(2)}/I_Y^{(3)})$$

$$\geq 3g - 3 + n - s_1 - s_2 + \#(\text{BRG}) + \#(\text{MCS})$$

$$= 3g - 3 + n,$$

hence equality must hold. This implies that, when tensored with $Q$, BRG, MCS are respectively bases of $I_Y^{(1)}/I_Y^{(2)}$, $I_Y^{(2)}/I_Y^{(3)}$ and that $I_Y^{(3)} = 0$. Since each $I_Y^{(m)}/I_Y^{(m+1)}$ is a free $Z$-module, we have

$$I_Y = I_Y/I_Y^{(1)} \oplus I_Y^{(1)}/I_Y^{(2)} \oplus I_Y^{(2)}/I_Y^{(3)} \oplus I_Y^{(3)}.$$ 

It is obvious that BRG $\cup$ MCS can be extended to a base of $I_Y$, hence their quotient is torsion free. It follows that (2), (3), and (4) hold.

The hardest part is Step 3. We shall treat this step in Section 6.

From now on, we shall use the following notation. For $y \in \text{Edge}(Y)$, $t_y$ denotes the element $t_y^y$ (see Definition 3.1). By definitions, we have the following

**Lemma 5.2.** For any edge $y \in \text{Edge}(Y)$, We have

$$yt_y \overline{y} t_{\overline{y}} = 1.$$ 

The edge twist $D_y$ maps

$$y \mapsto yt_y, \quad \overline{y} \mapsto \overline{y} t_{\overline{y}}$$

and leaves the other generators unchanged.

**5.2 Step 1-A. BRG $\subset I_Y^{(2)}$.**

**Proposition 5.3.** Let $y$ be a bridge of the graph $Y$. Then the Dehn twist $D_y$ associated with the edge $y$ belongs to $I_Y^{(2)}$. In particular, $D_y$ acts trivially on the group $\pi/\pi(2)$.

**Proof.** It is enough to show that $D_y \in \tilde{\Gamma}_{g,n}[2]$.

Put $Y - |y| = Y_1 \cup Y_2$, where $Y_i$ ($i=1,2$) are both connected. Let us fix the orientation of $y$ by $t(y) \in \text{Vert} Y_2$ and $o(y) \in \text{Vert} Y_1$. Choose $P_i \in \text{Vert} Y_i$ ($i = 1, 2$), and form

$$G_i = \pi_1(G|Y_i, Y_i, P_i) \quad (i = 1, 2).$$
Let $Y' = Y/(Y_1 \cup Y_2)$ be the graph obtained from $Y$ by contracting both $Y_i$ (i = 1, 2) to a point $Q_i$ (i = 1, 2), respectively. Then $Y'$ is a graph with two vertices $Q_1$, $Q_2$ and unique geometric edge $\{y, \overline{y}\}$.

There are canonically induced monomorphisms

$$G_y \rightarrow G_2, \text{ and } G_{\overline{y}} \rightarrow G_1.$$ 

Then $G' = \{ G_i \ (i = 1, 2), \ G_y \cong G_{\overline{y}}, \text{ and the above monomorphisms } \}$ is a graph of groups over $Y'$. Then we have a canonical isomorphism

$$\pi_1(G, Y, P_1) \cong \pi_1(G', Y', Q_1).$$

By Theorem 3.1, each $G_i$ (i = 1, 2) has the following presentation:

$$G_1 = \langle \alpha_1, \beta_1, \ldots, \alpha_i, \beta_i, \gamma_0, \gamma_1, \ldots, \gamma_j \mid (\prod_{k=1}^{i} [\alpha_k, \beta_k])\gamma_0\gamma_1 \cdots \gamma_j = 1 \rangle;$$

$$G_2 = \langle \alpha_{i+1}, \beta_{i+1}, \ldots, \alpha_g, \beta_g, \gamma_{j+1}, \ldots, \gamma_{n+1} \mid (\prod_{k=i+1}^{g} [\alpha_k, \beta_k])\gamma_{j+1} \cdots \gamma_{n+1} = 1 \rangle;$$

with $\gamma_0 = t_{\overline{y}}$ and $\gamma_{n+1} = t_y$, and each of the other $\gamma_i$ corresponds to a puncture up to conjugacy.

Consider the presentation of $\pi_1(G', Y', Q_1)$ with base point $Q_1$. This group is generated by $\xi \in G_1$ and $\xi = y\eta y^{-1}$ for $\eta \in G_2$. It is enough to show that $D_y(\xi)\xi^{-1} \in \pi(2 + l)$ for these $\xi$ with weight $-l$.

Dehn twist $D_y$ has the following description.

$$D_y(\xi) = \xi, \text{ if } \xi \in G_1;$$

$$D_y(y\eta y^{-1}) = yt_\overline{y}\eta t_y = t_\overline{y}^{-1}y\eta y^{-1}t_y \text{ for } \eta \in G_2.$$ 

Since $t_\overline{y}^{-1} = \gamma_0^{-1} = \gamma_1 \cdots \gamma_j(\prod_{k=1}^{i} [\alpha_k, \beta_k]) \in \pi(2)$, we have

$$D_y(\xi)\xi^{-1} = \begin{cases} 1, & \text{if } \xi \in G_1; \\ [t_\overline{y}^{-1}, \xi] \in \pi(2 + l), & \text{if } \xi \in yG_2y^{-1}. \end{cases}$$

(q.e.d.)

5.3 Step 1-B. MCS $\subset I_Y^{(1)}$. 
**Proposition 5.4.** If $\{|y_1|, |y_2|\}$ is a cut pair of edges, then the actions of Dehn twists $D_{y_1}$ and $D_{y_2}$ on the group $\pi/\pi(2)$ coincide.

**Proof.** Let $Y - |y_1| \cup |y_2|$ have two connected components $Y_1$ and $Y_2$. Contract both $Y_1$ and $Y_2$ to points. Denote these points by $P_1$ and $P_2$, respectively. Then, by changing the orientation of edge if necessary, the quotient graph $Y' = Y/(Y_1, Y_2)$ is given by

$$Y' = \{\}.$$ 

The vertex groups are given by

$$G_{P_1} = < \alpha_1, \beta_1, \ldots, \alpha_i, \beta_i, \gamma_1, \ldots, \gamma_j, t_{y_1}, t_{y_2} | (\prod_{k=1}^i [\alpha_k, \beta_k]) \gamma_1 \cdots \gamma_j t_{y_2} t_{y_1} = 1 >;$$

$$G_{P_2} = < \alpha_{i+2}, \beta_{i+2}, \ldots, \alpha_g, \beta_g, \gamma_{j+1}, \ldots, \gamma_n, t_{y_1}, t_{y_2} | (\prod_{k=i+2}^g [\alpha_k, \beta_k]) \gamma_{j+1} \cdots \gamma_n t_{y_1} t_{y_2} = 1 >$$

with each $\gamma_l$ corresponding to a puncture up to conjugacy for $l = 1, \ldots, n$. We also have

$$y_1 t_{y_1} y_1^{-1} = t_{y_1}^{-1}; \quad y_2 t_{y_2} y_2^{-1} = t_{y_2}^{-1}.$$ 

Firstly, we choose $y_1$ as a spanning tree and consider $\pi := \pi_1(G, Y', T)$. Then, since $y_1 = 1$, we have

$$t_{y_1} = t_{y_1}^{-1} \text{ and } t_{y_2} t_{y_2} = y_2 [t_{y_2}^{-1}, y_2^{-1}] y_2^{-1}.$$ 

Thus, the defining equation of $\pi_1(G, Y', T)$ is

$$(\prod_{k=1}^i [\alpha_k, \beta_k]) \gamma_1 \cdots \gamma_j y_2 [t_{y_2}^{-1}, y_2^{-1}] y_2^{-1} (\prod_{k=i+2}^g [\alpha_k, \beta_k]) \gamma_{j+1} \cdots \gamma_n = 1,$$ 

and we have

$$\pi/\pi(2) \cong \mathbb{Z} \alpha_1 \oplus \cdots \mathbb{Z} \alpha_g \oplus \mathbb{Z} \beta_1 \oplus \cdots \mathbb{Z} \beta_g$$

with $\alpha_{i+1} = t_{y_2}^{-1}$ and $\beta_{i+1} = y_2^{-1}$.

Now we choose $P_1$ as a base point, and see the presentation of $\pi_1(G, Y, P_1)$. This group is generated by elements of the following five types:

1. $\xi \in G_{P_1}$,
2. $\xi = y_1 \eta \bar{y}_1, \eta \in G_{P_2}$,
3. $\xi = y_2 \eta \bar{y}_2, \eta \in G_{P_2}$,
4. $\xi = y_1 \eta \bar{y}_2, \eta \in G_{P_2}$,
5. $\xi = y_2 \eta \bar{y}_1, \eta \in G_{P_2}$. 


By symmetry it is enough to prove for the cases (1), (2), and (4) that

$$D_{y_{1}}D_{y_{2}}^{-1}(\xi)\xi^{-1} \in \pi(1 + l),$$

where $-l$ is the weight of $\xi$.

The case (1) is trivial. In the case of (2), $D_{y_{2}}$ acts trivially, and

$$D_{y_{1}}(\xi)\xi^{-1} = y_{1}t_{y_{1}}\eta t_{y_{1}}^{-1}\overline{y}_{1}\xi^{-1} = [y_{1}t_{y_{1}}\overline{y}_{1}, \xi] \in \pi(1 + l).$$

In the case of (4), observe that the weight of $\xi$ is $-1$ because in $\pi/\pi(2)$ we have

$$\xi = \eta + \overline{y}_{2} = \eta + \beta_{i+1},$$

and $\eta$ does not contain $\beta_{i+1}$-component under the canonical basis $\alpha_{1}, \beta_{1}, \cdots, \alpha_{g}, \beta_{g}$ of an abelian group $\pi/\pi(2)$, hence $\xi$ is not zero in $\pi/\pi(2)$. Since

$$D_{y_{1}}D_{y_{2}}^{-1}(\xi)\xi^{-1} = t_{\overline{y}_{1}}^{-1}y_{1}\eta\overline{y}_{2}t_{\overline{y}_{2}}^{-1}\xi^{-1} = t_{\overline{y}_{1}}^{-1}t_{\overline{y}_{2}}^{-1}[t_{\overline{y}_{2}}, \xi]$$

and

$$t_{\overline{y}_{1}}^{-1}t_{\overline{y}_{2}}^{-1} = \prod_{k=1}^{i}[\alpha_{k}, \beta_{k}]\gamma_{1}\cdots\gamma_{j})^{-1},$$

the right hand side is contained in $\pi(2) = \pi(1 + l)$. (q.e.d.)

5.4 Step 2. $\text{rank}_{\mathbb{Z}}(I_{Y}/I_{Y}^{(1)}) = 3g - 3 + n - s_{1} - s_{2}$.

(5.4.1) The compact case of this equality was proved by Brylinski and Baclawski (see Proposition 5 in [Br]).

**Theorem 5.5.** (Brylinski and Baclawski.)

Suppose that $n = 0$. Then the group $I_{Y}$ is a free abelian group of rank $3g - 3$ and

$$\text{rank}_{\mathbb{Z}}(I_{Y}/I_{Y}^{(1)}) = 3g - 3 - s_{1} - s_{2}$$

holds, where $s_{2}$ denotes the number of bridges of $Y$ and $s_{1}$ denotes the summation $\sum\{#(S) - 1\}$ over all the maximal cut systems $S_{1}, S_{2}, \ldots, S_{l}$.

The non-compact cases are reduced to the compact cases as follows.

**Theorem 5.6.** The group $I_{Y}$ is a free abelian group of rank $3g - 3 + n$ and

$$\text{rank}_{\mathbb{Z}}(I_{Y}/I_{Y}^{(1)}) = 3g - 3 + n - s_{1} - s_{2}$$
holds.

In the rest of this subsection, we shall prove this theorem.

(5.4.2) Let \((G^*, Y^*)\) denote the graph of surface groups obtained by "compactification" of \((G, Y)\) as follows.

Let \(Q_1\) be a vertex with at least one puncture (see subsection 5.1). If \(n_{Q_1} = 1\), then we remove the vertex \(Q_1\) and replace the two incident edges with an edge connecting the other ends of the two edges. If \(n_{Q_1} = 2\), then we remove the vertex \(Q_1\) and the unique incident edge, and increase \(n_P\) by one, where \(P\) is the other end of the removed edge.

We denote by \((G', Y')\) the obtained graph. We also define a surjective partial map

\[ * : \text{Edge}(Y) \rightarrow \text{Edge}(Y'), y \mapsto y^* \]

as follows. If \(n_{Q_1} = 1\), then the both two edges in \(Y\) incident to \(Q_1\) are mapped to the new added edge. If \(n_{Q_1} = 2\), then the removed edge is removed from the defining domain of \(*\). Note that this definition does not change the genus. We iterate this process as far as \(Y\) has at least two vertices, and denote by \((G^*, Y^*)\) the obtained graph of surface groups. Let \(* : \text{Edge}(Y) \rightarrow \text{Edge}(Y^*)\) be the composition of all the partial maps defined in each stage. Let \(s_1^*, s_2^*\) be the numbers \(s_1(Y^*), s_2(Y^*)\), respectively. Let \(n^*\) denote the number of punctures in \((G^*, Y^*)\).

There are three cases:

1. \((G^*, Y^*)\) is compact; i.e., \(n^* = 0\).
2. \((G^*, Y^*)\) has \(g = 0\), \(n^* = 3\); i.e., consists of a unique vertex with three punctures and no edge.
3. \((G^*, Y^*)\) has \(g = 1\), \(n^* = 1\); i.e., consists of a unique vertex with one puncture and one loop.

The proof will be completed by showing three identities in

\[ \text{rank}_Z(I_Y^{(0)}/I_Y^{(1)}) = \text{rank}_Z(I_{Y^*}^{(0)}/I_{Y^*}^{(1)}) = 3g - 3 + n^* - s_1^* - s_2^* = 3g - 3 + n - s_1 - s_2. \]

(5.4.3) First we shall treat the second identity. In the compact case (1), this is nothing else but Theorem 5.5. In the case of (2), both sides are trivially zero. In the case of (3), we have one edge twist. It is easy to compute that \(\pi_1(G^*, Y^*)\) is a free group with generators \(\alpha, \beta\) and the weight filtration coincides with the lower central filtration. The edge twist acts on \(\pi/[\pi, \pi]\) by \(\alpha \mapsto \alpha, \beta \mapsto \alpha + \beta\). This action has infinite order, hence both sides of the second identity are one.

(5.4.4) Now we shall treat the first identity. Let \(P\) be a vertex of \(Y^*\). We denote by the same \(P\) the inverse image of \(P\) in \(Y\).

**Definition 5.2** We say that a homomorphism \(G \rightarrow G'\) between filtered groups preserves the filtration if the image of \(G(m)\) is contained in \(G'(m)\) for every \(m\).
Lemma 5.7. There is a filtration preserving surjective homomorphism

\[ \varphi : \pi_1(G, Y) \to \pi_1(G^*, Y^*) \]

inducing

\[ \tilde{\Gamma}_{g,n} \to \tilde{\Gamma}_{g,0} \]

By passing to quotient, we have a filtration preserving homomorphism

\[ \Gamma_{g,n} \to \Gamma_{g,0} \]

and by restricting to \( I_Y \) we get

\[ I_Y \to I_{Y^*} \]

\[ D_y \mapsto D_{y^*}, \]

where \( D_{y^*} \) is defined to be identity if \( y \) is not contained in the defining domain of \( * \).

The kernel of \( \varphi \) is generated by \( << c_1, \ldots, c_j >> \) as a normal subgroup, where each \( c_i = q_i z_i q_i^{-1} \) corresponds to the removed puncture \( z_i \) (see (5.1.1)) for \( i = 1, \ldots, j \).

(Note that \( j = n, n - 3, n - 1 \) according to the cases (1), (2), (3) above, respectively.)

Proof. It is enough to prove in each stage of compactification; i.e., for \( Y^* = Y' \). We shall define a homomorphism \( F(G, Y) \to F(G', Y') \). Note that we can take

\[ \{y, t_y \mid y \in \text{Edge}(Y)\} \cup \{z_1, \ldots, z_n\} \]

as generators of \( F(G, Y) \).

Case 1. \( n_{Q_1} = 1 \).

Let \( y_1, y_2 \) be the two incident edges to \( Q_1 \) with \( t(y_1) = Q_1 = o(y_2) \). Let \( y \) be the newly added edge in \( Y' \) with \( o(y) = o(y_1), t(y) = t(y_2) \).

On generators, we define

\[
\begin{align*}
y_1 & \mapsto y \\
\bar{y}_1 & \mapsto \bar{y} \\
y_2, \bar{y}_2 & \mapsto 1 \\
t_{y_1}, t_{y_2} & \mapsto t_y \\
t_{\bar{y}_1} & \mapsto t_{\bar{y}} \\
t_{\bar{y}_2} & \mapsto t_{\bar{y}}^{-1} \\
z_1 & \mapsto 1,
\end{align*}
\]

and the other generators are left unchanged.

Case 2. \( n_{Q_1} = 2 \).
Let $y_1$ be the unique edge with $t(y_1) = Q_1$, and let $P$ be $o(y_1)$. We may assume $z_1, z_2 \in G_{Q_1}$ with $t(y_1)z_1z_2 = 1$. We map

\[
\begin{align*}
y_1 & \mapsto 1 \\
z_1 & \mapsto 1 \\
z_2 & \mapsto t_{\overline{y}_1}.
\end{align*}
\]

(Note that $G_P$ in $Y'$ is the same group with the one in $Y$, hence $t_{\overline{y}_1} \in G_P$ in $Y$ makes sense.)

It is a tedious and simple task to prove that this map is a group homomorphism, inducing a homomorphism even restricted to $\pi_1$, that the kernel is generated by $q_1z_1q_1^{-1}$ (and $q_1z_2q_1^{-1}$ in the second case), that filtration is preserved, and that this map is compatible with

\[
I_Y \to I_{Y^*} \\
D_y \mapsto D_{y^*}.
\]

The detailed proof goes in parallel with the proofs of Lemmas 6.7–6.12 in Subsection 6.5. Since this case is much simpler than these lemmas, we omit the proof. (q.e.d.)

The map $I_Y \to I_{Y^*}$ defined above is obviously surjective. For the first identity, it is enough to prove that the kernel of

\[
I_Y \to I_{Y^*}/I_Y^{(1)}
\]

coincides with $I_Y^{(1)}$. Trivially $I_Y^{(1)}$ is contained in this kernel. Conversely, if $\sigma \in I_Y$ is mapped to an element in $I_Y^{(1)}$, then

\[
\sigma(\eta)\eta^{-1} \in \langle< \pi_{g,n}(k + 1), c_1, c_2, \ldots, c_j >\rangle
\]

holds for any $\eta \in \pi_{g,n}(k)$. We want to prove that $\sigma \in I_Y^{(1)}$. Since there is a generating set of $\pi_{g,n}$ consisting of elements in $(\pi_{g,n}(1) - \pi_{g,n}(2)) \cup \{c_1, \ldots, c_n\}$, it is enough to check the condition for $\eta$ in this set.

If $\eta \notin \pi_{g,n}(2)$, then we have $\sigma(\eta)\eta^{-1} \in \pi_{g,n}(2)$ mod $< c_1, \ldots, c_j >$, and since $c_k \in \pi_{g,n}(2)$ there is no problem. Otherwise $\eta = c_i$ for some $1 \leq i \leq n$. We have $\sigma(c_i) = sc_is^{-1}$ for some $s \in \pi_{g,n}$ by definition of $\tilde{\Gamma}$, and $\sigma(c_i)c_i^{-1} = [s, c_i] \in \pi_{g,n}(3)$, proving that $\sigma \in \Gamma_{g,n}[1]$.

(5.4.5) The third equality is easily obtained by induction on $n$. Note that filling up one puncture decreases $\#\text{BRG}$ by one if the vertex containing the puncture is incident to a bridge, and decreases $\#\text{MCS}$ by one otherwise.
6. Linear independence of BRG and MCS.
First we shall establish the linear independence for compact cases, and next reduce the non-compact cases to the compact one. In compact cases, the proof is induction on \#BRG and on \#MCS. The induction depends on the graph reduction defined below.

The merit to restrict to compact cases is that the weight filtration of \( \pi_1 \) coincides with the usual lower central series and hence any homomorphism between two fundamental groups is always filtration preserving.

6.1 Graph Reduction.
In this subsection we assume \( n = 0 \). It is true that all results in this section are valid for the cases \( n > 0 \), but we don’t use it here.

**Definition 6.1** (Graph Reduction.)
Let \((G, Y)\) be a graph of surface groups associated with a most degenerate stable curve. Suppose that two distinct edges \( e_1, e_2 \) in \( Y \) satisfy the following two conditions.

1. The vertices \( t(e_1) \) and \( o(e_2) \) belong to the same connected component of \( Y - |e_1| - |e_2| \).
2. Let \( Y_d \) denote the above connected component, and let \( Y_r \) denote \( Y - |e_1| - |e_2| - Y_d \) (\( d \) for deletion and \( r \) for residue). Then, \( Y_r \) contains both \( o(e_1) \) and \( t(e_2) \).

Figure 1. Graph Reduction.

Then, we construct a new graph of surface groups \((G', Y')\) called the reduced graph of groups along the pair \((e_1, e_2)\) as follows. The graph \( Y' \) is obtained from \( Y_r \) by adjoining one edge \( e \) from \( o(e_1) \) to \( t(e_2) \). The new groups \( G' \) is the same one with \( G \) restricted to \( Y_r \), equipped with a free group of one generator \( G_e \) on \( e \) and injections \( G_e \rightarrow G_{o(e)}, G_{t(e)} \) induced from \( G_{e_1} \rightarrow G_{o(e_1)} \) and \( G_{e_2} \rightarrow G_{t(e_2)} \).

Note that the graph reduction along \((e_1, e_2)\) is different from the one along \((e_2, e_1)\).

**Proposition 6.1.** (Graph Reduction.)
Let $G, Y, G', Y', Y_d, e_1, e_2$ be as above. Choose a simple path $\{z_1, \ldots, z_k\}$ in $Y_d$ from $o(e_1)$ to $t(e_2)$ (thus we have $z_1 = e_1$ and $z_k = e_2$). Let $P$ be a vertex in $Y_r$. We can define a group homomorphism

$$\phi : \pi_1(G, Y, P) \to \pi_1(G', Y', P)$$

using $z_i$ (see (6.5.1) for the precise definition).

The kernel of $\phi$ is stable under the action of $I_Y$, and consequently, $\phi$ induces a unique map

$$\varphi : I_Y \to \text{Out}(\pi_1(G', Y', P))$$

satisfying the condition

$$\varphi(D_y) \cdot \phi(x) = \phi(D_y \cdot x).$$

The image of $\varphi$ is contained in $I_{Y'}$; more precisely, $\varphi$ maps the generator of $I_Y$ as follows:

\[
\begin{align*}
D_y &\mapsto D_y \quad \text{if } y \in \text{Edge}_{Y_r} \\
D_y &\mapsto \text{id} \quad \text{if } y \in \text{Edge}_{Y_d} - \{|z_1|, \ldots, |z_k|\} \\
D_{z_i} &\mapsto D_{e_i}.
\end{align*}
\]

Also, $\varphi$ preserves the filtration, and induces a homomorphism

$$\varphi : I_Y^{(m)}/I_Y^{(m+1)} \to I_{Y'}^{(m)}/I_{Y'}^{(m+1)}.$$

This proposition will be proved in Subsection 6.5.

6.2 Independence of BRG in compact cases.

In this subsection we assume $n = 0$.

**Proposition 6.2.** BRG is linearly independent in $I_Y^{(2)}/I_Y^{(3)}$.

**Proof.** We proceed by induction on the cardinality of the set of the bridges $H$. Suppose that $H = \{|y_1|, |y_2|, \ldots, |y_k|\}$. Let $A_1, A_2, \ldots, A_{k+1}$ be the connected components of $Y - H$. It is clear that the graph obtained from $Y$ by contracting every $A_i$ to a vertex $P_i$ is a tree. We call this tree the skeleton of $Y$.

The case $\#(H) = 1$. We have $H = \{|y|\}$ and there exist two connected components $A_1, A_2$ of $Y - |y|$.

It is enough to prove that $D_y^n$ is nonzero in $I_Y^{(2)}/I_Y^{(3)}$ for $n > 0$. The skeleton of $Y$ is shown in Figure 2. Since $Y$ is tri-valent, there are two other edges $e_1, e_2$ incident to $t(y)$ other than $y$, with direction $o(e_1) = t(e_2) = t(y)$. If $e_1 = e_2$, then $A_1$ consists of a loop. Otherwise, using the graph reduction along $(e_1, e_2)$, we have a new graph $Y'$ with $A_1'$ consisting of a loop, and a homomorphism

$$\varphi : I_Y^{(2)}/I_Y^{(3)} \to I_{Y'}^{(2)}/I_{Y'}^{(3)}$$

$$D_y \mapsto D_y.$$
Applying the same operation on $A_2$ again, we may assume that $Y'$ is the graph with two loops and one bridge; that is, both $A'_1$ and $A'_2$ are loops. For this graph, $g = 2$ holds, and from Proposition 4.2 it follows that $D^*_y$ is trivial in $I_{Y'}^{(2)}/I_{Y'}^{(3)}$ if and only if $n = 0$. Thus, we have $n = 0$ if $D^n_y = 0$ in the original graph, by passing to $Y'$. The case $\#(H) > 1$. Let $A_1$ be a leaf of $Y$; in other words, there exists only one $|y_1| \in H$ incident to $A_1$. We may assume $t(y_1) \in A_1$. There are two other edges $e_1, e_2$ incident to $o(y_1)$ with direction $t(e_1) = o(e_2) = o(y_1)$. Since $\#(H) > 1$, we have $e_1 \neq e_2$ (Figure 3).

Case 1. $|e_1|$ is not contained in $H$. In this case, it is clear that $|e_2|$ is not contained in $H$ and that both $o(e_1)$ and $t(e_2)$ belong to the same connected component of $Y - |e_1| - |e_2|$. We apply the graph reduction along $(e_1, e_2)$. Suppose that $\prod_{i=1}^k D^n_{y_i} = 0$ in $I_{Y'}^{(2)}/I_{Y'}^{(3)}$. Then, by passage to $I_{Y'}^{(2)}/I_{Y'}^{(3)}$, we have $\prod_{i=2}^k D^n_{y_i} = 0$ in $I_{Y'}^{(2)}/I_{Y'}^{(3)}$. By induction hypothesis, $n_i = 0$ holds for $i > 1$. Thus, we have $D^n_{y_1} = 0$ in $I_{Y'}^{(2)}/I_{Y'}^{(3)}$. Again applying the graph reduction along $(e_2, e_1)$ (i.e., $Y_r$ corresponds to $A_1 \cup \{o(y_1)\}$), we can reduce to the case $\#(H) = 1$. 

Figure 2.

Figure 3.
Case 2. $|e_1| \in H$. In this case, it is obvious that $|e_2| \in H$. We may assume $y_2 = e_1$ and $y_3 = e_2$. Suppose again that $\prod_{i=1}^{k} D_{y_i}^{n_i} = 0$ in $I_{Y}^{(2)}/I_{Y}^{(3)}$. Then, by the graph reduction along $(y_2, y_3)$, we have $\prod_{i=1}^{k} D_{y_i}^{n_i} = 0$ in $I_{Y}^{(2)}/I_{Y}^{(3)}$, that is, $D_{e}^{n_1+n_2} \prod_{i=4}^{k} D_{y_i}^{n_i} = 0$ in $I_{Y}^{(2)}/I_{Y}^{(3)}$, where $e$ is the added edge to $Y_r$ (see Proposition 6.1). From induction hypothesis, $n_2 + n_3 = 0$ and $n_i = 0$ for $i > 3$ follow. Similarly, by the graph reduction along $(y_2, y_1)$ and $(\overline{y}_1, y_3)$, we have $n_1 + n_2 = 0$ and $n_1 + n_3 = 0$ respectively. (Observe that $D_{\overline{y}} = D_{y}$.) Solving these three equations, we have $n_i = 0$ for $i = 1, 2, 3$. (q.e.d.)

6.3 Independence of MCS in compact cases.

In this subsection we assume $n = 0$. From now on, a pair means a cut pair and a system means a maximal cut system.

**Lemma 6.3.** Let $(e_1, e_2), (e_3, e_4)$ be two disjoint cut pairs of $Y$. Then, at least one of the following holds.

1. $(e_1, e_2, e_3, e_4)$ belong to one system.
2. Both $e_3$ and $e_4$ belong to one connected component of $Y - |e_1| - |e_2|$.

**Proof.** Let $Y_1, Y_2$ denote the connected components of $Y - |e_1| - |e_2|$. Suppose that neither (1) nor (2) holds. We may assume that $e_3 \in Y_1$ and $e_4 \in Y_2$. Then, since (1) does not hold, both $Y_1 - |e_3|$ and $Y_2 - |e_4|$ are connected. Consequently, $Y - |e_3| - |e_4|$ is connected and this is a contradiction.

**Lemma 6.4.** Let $S$ be a system of $Y$. Then, the graph obtained from $Y$ by contracting every connected component $Y_i$, $i = 1, 2, \ldots, n$ of $Y - S$ is a cycle (see Figure 4).

**Figure 4.**

**Proof.** Clear.

By Lemma 6.3, for any system $S' \neq S$, there exists a $Y_i$ which contains $S'$. 
Lemma 6.5. There exists a system $S$ such that a connected component $Y_1$ of $Y - S$ contains all of the other systems.

Proof. Take an arbitrary $S$, and decompose $Y - S$ as in Lemma 6.4. Suppose that a connected component $Y_i$ other than $Y_1$ yet contains a system $S'$. Then, decompose $Y - S'$, and let $Y_1'$ be the connected component of $Y - S'$ containing $Y_1$. By iterating this process, the size of $Y_1^{(k)}$ strictly increases, and this process stops when $Y_1^{(k)}$ contains all systems other than $S^{(k)}$.

Figure 5. $Y_1$ contains all systems other than $S$.

We now prove the independence of MCS by induction on $(MCS)$.

Proposition 6.6. MCS is linearly independent in $I_Y^{(1)}/I_Y^{(2)}$.

Proof. The case #(MCS) = 1. In this case, $Y$ is a graph of the form shown on the left in Figure 6.
We want to prove that \((D_{e_1}D_{e_2}^{-1})^n = 0\) in \(I_Y^{(1)}/I_Y^{(2)}\) holds only if \(n = 0\). It is known that \(\pi_1(G,Y,P)\) is canonically isomorphic to \(\pi_1(G',Y',p_1)\), where \(Y'\) is the graph shown on the right in Figure 6, \(G_{p_i} := \pi_1(G|_{Y_i}, Y_i, q_i)\) with \(q_i \in Y_i\), and \(G_{e_i} \to G_{t(e_i)}\) is induced from the one in \((G,Y)\).

As in Subsection 5.2, \(\pi_1(G|_{Y_i}, Y_i, q_i)\) is isomorphic to the fundamental group of a Riemann surface of genus \(g_i \geq 1\) with two punctures for \(i = 1,2\), and the edge twist \(D_{e_i}\) is compatible with this identification.

Thus, it is enough to prove the independence in the case that \(Y'\) is as above and

\[
G_{p_1} = \langle \alpha_1, \beta_1, \ldots, \alpha_{g_1}, \beta_{g_1}, x, y | [\alpha_1, \beta_1] \cdots [\alpha_{g_1}, \beta_{g_1}]xy = 1 \rangle
\]

\[
G_{p_2} = \langle \alpha'_1, \beta'_1, \ldots, \alpha'_{g_2}, \beta'_{g_2}, x', y' | [\alpha'_1, \beta'_1] \cdots [\alpha'_{g_2}, \beta'_{g_2}]x'y' = 1 \rangle.
\]

We have a projection

\[
G_{p_1} \to \overline{G}_{p_1} := \langle \alpha_1, \beta_1, x, y | [\alpha_1, \beta_1]xy = 1 \rangle
\]

\[
G_{p_2} \to \overline{G}_{p_2} := \langle \alpha'_1, \beta'_1, x', y' | [\alpha'_1, \beta'_1]x'y' = 1 \rangle
\]

inducing a group homomorphism

\[\pi_1(G', Y', p_1) \to \pi_1(\overline{G}, Y', p_1),\]

whose kernel is generated as a normal subgroup by

\[\{\eta \in \pi_1(G', Y', p_1) | \eta \in \text{Ker}\{G_{p_1} \to \overline{G}_{p_1}\} \text{ or } \eta = y\eta_2y^{-1}, \eta_2 \in \text{Ker}\{G_{p_2} \to \overline{G}_{p_2}\}, y = e_1 \text{ or } \tilde{e}_2\}.\]

It is easy to see that this kernel is stable under the action of \(D_{e_i}\). (Skeptical readers may first read Section 6, in particular, the proof of Lemmas 6.8–6.10 in Subsection 6.5 and next return here.)

Thus, similarly to the case of BRG, it is enough to show the independence of \(\{D_{e_1}D_{e_2}^{-1}\}\) in \(I_Y^{(1)}/I_Y^{(2)}\) for \((\overline{G}, Y')\) the surface group of genus 3, as shown in Figure 7, which was settled in Lemma 4.4.
The case \( \#(\text{MCS}) > 1 \). Fix a system \( S \) as in Lemma 6.5, and let \( Y, e_i \) for \( 1 \leq i \leq n \) be as shown in Figure 5. Let \( S_1 = S, S_2, S_3, \ldots, S_l \) be all the systems of \( Y \). We choose \( y_i \in S_i \) as in the definition of MCS. We may assume that \( y_1 \in S_1 \) equals \( e_i \) with \( i \neq n \), by changing the direction of each \( e_j \) if it is necessary.

We may also assume \( y_i^1 = e_{i+1} \) (note that 1 is a superscript, not a power). Suppose that

\[
\prod_{i=1}^l \prod_{j=1}^{\#(S_i)-1} (D_{y_i} D_{y_i^j}^{-1})^{n_{ij}} = 0
\]

in \( I_Y^{(1)}/I_Y^{(2)} \). Then, by applying the graph reduction along \((e_i, e_{i+1})\) and calculating the image of the LHS by \( \varphi \), we have

\[
\prod_{j=2}^{\#(S_1)-1} (D_{y_1} D_{y_1^j}^{-1})^{n_{1j}} \prod_{i=2}^l \prod_{j=1}^{\#(S_i)-1} (D_{y_i} D_{y_i^j}^{-1})^{n_{ij}} = 0
\]

(see Proposition 6.1). By induction hypothesis, \( n_{ij} = 0 \) for \( i \geq 2 \) and \( n_{ij} = 0 \) for \( j \geq 2 \) hold.

Thus, we have

\[
(D_{y_1} D_{y_1^1}^{-1})^{n_{11}} = (D_{e_i} D_{e_{i+1}}^{-1})^{n_{11}} = 0
\]

in \( I_Y^{(1)}/I_Y^{(2)} \). If \( S \) contains another edge \( e_{i-1} \) or \( e_{i+2} \), then the induction hypothesis and the graph reduction along \((e_{i+1}, e_{i-1})\) or \((e_{i+2}, e_i)\), respectively, imply that \( n_{11} = 0 \). If \( S \) contains only \( e_i \) and \( e_{i+1} \), then \( Y_1 \) contains another system \( S' \), since \( \#(\text{MCS}) > 1 \). Choose \( e_i', e_{i+1}' \in S' \) so that in the graph reduction along \((e_i', e_{i+1}')\), \( Y_r \) contains \( Y_2 \) (see Figure 8.)

By induction hypothesis in the reduced graph along \((e_i', e_{i+1}')\), we have \( n_{11} = 0 \).

(q.e.d.)
6.4 Reducing non-compact cases to compact ones. In this subsection we assume $n > 0$. Let $(G, Y)$ be a graph of surface groups associated with a most degenerate stable $n$-pointed curve of genus $g$. We introduce another kind of compactification. We equip each puncture $Q_i$ with a new vertex $P_i$ with a loop $L_i$ and put one edge $y_i$ from $Q_i$ to $P_i$. Let $Y'$ denote the obtained graph. The graph of groups $(G', Y')$ corresponds to the compact Riemann surface obtained from the original punctured Riemann surface by filling up each puncture with a handle (Figure 9).

Figure 9.

There exists a canonical injective homomorphism

$$\phi : \pi_{g,n} = \pi_1(G, Y) \cong \pi_1(G'|_Y, Y) \rightarrow \pi_{g+n,0} = \pi_1(G', Y').$$

By mapping $D_y$ for $y \in \text{Edge} Y$ to $D_y \in I_{Y'}$, we have a homomorphism

$$\varphi : I_Y \rightarrow I_{Y'},$$

since $Y$ is a subgraph $Y'$. It is not difficult to see that $\phi$ preserves the filtration by checking generators. In fact, the only problem is whether $c_i$ is mapped into $\pi(2)$ or not, but this is trivial since $c_i = [\alpha_{g+i}, \beta_{g+i}]$ holds in $\pi_1(G', Y')$. It is easy to see that $\varphi(\sigma)(\phi(\eta)) = \phi(\sigma(\eta))$ for any $\sigma \in I_Y, \eta \in \pi_{g,n}$.

We claim that $\varphi$ is filtration-preserving. To show this, it is enough to prove for any $\sigma \in I_Y^{(m)}$ and for $\eta$ of the form $q_i y_i d y_i q_i^{-1}$ with $d \in <G_{P_i}, L_i>$ (see Subsection 5.1 for $q_i$) that

$$\varphi(\sigma)(\eta)\eta^{-1} \in \pi_{g+n,0}(m + 1),$$

since these $\eta$ of weight $-1$ together with the image of $\pi_1(G, Y)$ generate $\pi_1(G', Y')$.

Recall that $c_i = q_i z_i q_i^{-1}$. So we have

$$\varphi(\sigma)(c_i) = sc_i s^{-1}$$

and

$$\varphi(\sigma)(\eta) = s\eta s^{-1}$$
with the same \( s = \sigma(q_i)q_i^{-1} \). Thus, our claim is reduced to proving that \( s \in \pi_{g,n}(m) \).

To prove this, we have to know the structure of the associated Lie algebra

\[ \mathcal{L} := \oplus \text{gr}_m(\pi_{g,n}). \]

As mentioned in [K], we can embed \( \mathcal{L} \) into a free non-commutative associative algebra \( A \) over \( \mathbb{Z} \) with generators \( \alpha_1, \beta_1, \ldots, \alpha_g, \beta_g, \gamma_1, \ldots, \gamma_{n-1} \), and if we equip \( A \) with a gradation by \( \deg(\alpha_1) = \cdots = \deg(\beta_g) = 1 \) and \( \deg(\gamma_1) = \cdots = \deg(\gamma_{n-1}) = 2 \), then this embedding preserves the gradation.

Let \(-l\) be the weight of \( s \). If \( l \geq m \) including the case \( l = \infty \), we have \( s \in \pi_{g,n}(m) \). Assume \( l < m \). Then, \([s, c_i] \in \pi_{g,n}(m+2) \subset \pi_{g,n}(l+2+1) \) holds since \( \sigma \in \mathcal{I}_{Y}^{(m)} \).

Interpreting this in \( A \), we have \( s \cdot c_i - c_i \cdot s = 0 \) in \( A \). We may assume that \( c_i \) is one of \( \gamma_i \), hence one of the canonical generators of \( A \).

Since \( A \) is free, \( s = k \cdot c_i \) in \( A \) holds for some integer \( k \). However, we have \( s = \sigma(q_i)q_i^{-1} \) and \( q_i \) was a path chosen from a fixed spanning tree \( T \). We have

\[ \pi_1(G|_{q_i}, q_i, P) \subset \pi_1(G|_{T}, T, P) \subset \pi_1(G, Y, P). \]

Then \( s = \sigma(q_i)q_i^{-1} \) is in the left group, hence contained in the middle one, and can be written using only \( \alpha_i \) (see Subsection 4.1). Since \( A \) is free on \( \alpha_i, \beta_i, \gamma_i \), this implies \( s = k \cdot c_i = 0 \) in \( A \), that is, \( s = 1 \), leading to a contradiction.

We have proved that

\[ \varphi : \mathcal{I}_Y \to \mathcal{I}_{Y'} \]

is filtration preserving. It is easy to see that bridges and maximal systems of cut pairs in \( Y \) are mapped to ones in \( Y' \) respectively. Hence, the independence of BRG, MCS in \( Y \) follows from the ones in \( Y' \), which was proved previously. (q.e.d.)

### 6.5 Details on graph reduction.

In this subsection we assume \( n = 0 \).

(6.5.1) Precise definition of graph reduction.

Let \( G, Y, G', Y', Y_r, Y_t, e_1 = z_1, z_2, \ldots, z_k = e_2, P \) be as in Proposition 6.1. We shall define a homomorphism

\[ \phi : F(G, Y) \to F(G', Y'), \]

which restricts to

\[ \phi : \pi_1(G, Y, P) \to \pi_1(G', Y', P). \]

We may consider \( F(G, Y) \) as the group generated by

\[ \{ y, t_y | y \in \text{Edge}(Y) \} \]
with relations
\[ y\overline{y} = 1, \quad yt_y\overline{y} = 1, \quad \text{and} \quad t_{y_1}t_{y_2}t_{y_3} = 1, \]
for three edges with a common terminal vertex \( y_1, y_2, y_3 \) in the fixed order (see Subsection 4.1).

The edge twist \( D_y \) maps
\[ y \mapsto yt_y, \quad \overline{y} \mapsto \overline{y}t_{\overline{y}} \]
and leaves the other generators unchanged.

On generators, \( \phi \) is defined as follows:

if \( y \in \text{Edge}(Y_r) \) then
\[ y, t_y \mapsto y, t_y, \]

if \( y \in \text{Edge}(Y_d) - \{z_1, \ldots, z_k, \overline{z}_1, \ldots, \overline{z}_k\} \) then
\[ y, t_y \mapsto 1, \]

if \( y = z_i \) or \( \overline{z}_i \) then for \( i = 1 \)
\[ z_1 \mapsto e, \quad \overline{z}_1 \mapsto \overline{e}, \quad t_{z_1} \mapsto t_e, \quad t_{\overline{z}_1} \mapsto t_{\overline{e}} \]

and for \( 2 \leq i \leq k, \)
\[ z_i, \overline{z}_i \mapsto 1, \quad t_{z_i} \mapsto t_e, \quad t_{\overline{z}_i} \mapsto t_{\overline{e}}^{-1}. \]

We define
\[ \phi_v : \text{Vert}(Y) \to \text{Vert}(Y') \]
by
\[ p \mapsto p \quad \text{if} \quad p \in \text{Vert}(Y_r), \]
\[ p \mapsto t(e) \quad \text{otherwise}. \]

It is easy to see that \( \phi \) maps \( G_p \) into \( G_{\phi_v(p)} \). It is a tedious but easy task to check that the relations in \( F(G, Y) \) are compatible with the ones in \( F(G', Y') \) through \( \phi \); that is, \( \phi : F(G, Y) \to F(G', Y') \) is a group homomorphism. For this, it is enough to check that
\[ \phi(y)\phi(\overline{y}) = 1, \quad \phi(y)\phi(t_y)\phi(y^{-1})\phi(\overline{t}_y) = 1, \quad \phi(t_{y_1})\phi(t_{y_2})\phi(t_{y_3}) = 1 \]
for \( y_1, y_2, y_3 \) with the same target vertex. All are easy; for example, we check only the third identity. Suppose \( p \in Y_r \). If \( y_1, y_2, y_3 \in Y_r \), then \( \phi(y_i) = y_i \) and there is no
problem. Otherwise, $p = o(e_1)$ or $t(e_2)$. In the first case, suppose for example that $y_1 = \overline{e}_1$. Then
\[
\phi(t_{y_1})\phi(t_{y_2})\phi(t_{y_3}) = t_\overline{e}t_{y_2}t_{y_3} = 1
\]
holds. The latter case can be checked similarly. Next, suppose $p \in Y_d$. If $p$ is not on the path $\{z_1, \ldots, z_k\}$, then $\phi(t_{y_i}) = 1$ holds and no problem exists. If $p$ is on the path, suppose for example that $y_1 = z_i$ and $y_2 = \overline{z}_{i+1}$. It is clear that $y_3$ is different from any $z_i$, and thus we have
\[
\phi(t_{y_1})\phi(t_{y_2})\phi(t_{y_3}) = t_\overline{e}t_\overline{e}^{-1} \cdot 1 = 1.
\]
Now we shall check that
\[
\phi(\pi_1(G, Y, P)) \subset \pi_1(G', Y', P).
\]
Let $x$ be an element of $\pi_1(G, Y, P)$. Then, $x$ can be represented by an admissible word (see Definition 2.2)
\[
w = r_0y_1r_1y_2\cdots r_{n-1}y_nr_n,
\]
where $\{y_1, \ldots, y_n\}$ is a path from $P$ to $P$, $r_i \in G_{o(y_{i+1})}$ for $i = 0, \ldots, n - 1$, and $r_n \in G_{t(y_n)}$.

Regard $e_1, \overline{e}_2$ as left parentheses and $e_2, \overline{e}_1$ as right parentheses, and decompose $w$ with respect to these parentheses as follows. Since $Y - |e_1| - |e_2|$ is not connected and $P \in Y_r$, it is clear that $w$ decomposes as
\[
w = A_1B_1A_2B_2\cdots A_sB_sA_{s+1},
\]
where $A_i \in F(G|_{Y_r}, Y_r)$ and $B_i$ is one of the following four types with $z \in F(G|_{Y_d}, Y_d)$:
\[
(1) \ e_1z\overline{e}_1, \ (2) \ \overline{e}_2ze_2, \ (3) \ e_1ze_2, \ (4) \ \overline{e}_2z\overline{e}_1.
\]
We call this decomposition the *decomposition of $w$ along $(e_1, e_2)$*.

**Lemma 6.7.** Let $B$ be an admissible word of one of the above four types. Let us denote by $s(B)$ the sum of the numbers of occurrence of $t_{z_i}$ for $1 \leq i \leq k$ in $B$ minus the one of $t_{\overline{z}_i}$ for $1 \leq i \leq k$. Then we have for each type:
\[
(1) \ \phi(B) = e(t_\overline{e})^{s(B)}\overline{e} = t_\overline{e}^{-s(B)} \in G_{o(e)}
\]
\[
(2) \ \phi(B) = (t_\overline{e})^{s(B)} \in G_{t(e)}
\]
\[
(3) \ \phi(B) = e(t_\overline{e})^{s(B)}
\]
\[
(4) \ \phi(B) = (t_\overline{e})^{s(B)}\overline{e}.
\]
Proof. Straightforward.

It is easy to see that for an admissible word $w$ as above, its image
\[ \phi(w) = A_1 \phi(B_1) A_2 \phi(B_2) \cdots A_s \phi(B_s) A_{s+1} \]
is also admissible and belongs to $\pi_1(G', Y', P)$ under the identification $t_{\overline{e}_1} = t_\ell$ and $t_{e_2} = t_e$. Thus, we have proved that
\[ \phi(\pi_1(G, Y, P)) \subset \pi_1(G', Y', P). \]

(6.5.2) Stability of $\text{Ker}(\phi)$.

Lemma 6.8. Let $w$ be an admissible word representing an element in $\pi_1(G, Y, P)$. Suppose that $w$ contains only
\[ t_{\overline{z}_1}, t_{z_k}, z_i, \overline{z}_i, \text{ and } y, \overline{y}, t_y, t_{\overline{y}} \text{ for } i = 1 \cdots k \text{ and } y \in \text{Edge}(Y_r). \]
Then, $w \neq 1$ in $\pi_1(G, Y, P)$ implies $\phi(w) \neq 1$.

Proof. Let $w$ be in the form of $w = r_0 y_1 r_1 y_2 \cdots r_{n-1} y_n r_n$ and suppose that $w \neq 1$. Suppose that this word is reducible; that is, this word contains a subword in the form of
\[ y_i(t_{y_i})^l \overline{y}_i. \]
If $y_i \neq \overline{z}_1$ nor $z_k$, we may replace this part of $w$ with $(t_{\overline{y}_i})^{-l}$ without influence on the assumption on $w$. We reduce $w$ in this way as far as possible. If $w$ is still reducible, then $w$ contains
\[ \overline{z}_1 t_{\overline{z}_1} z_1 \]
or
\[ z_k t_{z_k} \overline{z}_k. \]
In the former case, since $Y - |z_1| - |z_k|$ is not connected and back-tracking inside $\{z_1, \ldots, z_k\}$ was already removed, $w$ must contain the subword
\[ \overline{z}_k \cdots \overline{z}_1 t_{\overline{z}_1} z_1 \cdots z_k, \]
which can be replaced with $t_{z_k}^{-l}$. In the latter case, $w$ contains
\[ z_1 \cdots z_k t_{z_k} \overline{z}_k \cdots \overline{z}_1, \]
which can be replaced with $t_{\overline{z}_1}^{-l}$. By iterating these operations, we may assume that $w$ is reduced and $w \neq 1$. We want to prove that $\phi(w) = \phi(r_0) \phi(y_1) \cdots \phi(y_n) \phi(r_n) \neq 1.$
Let us delete all the occurrence of $\phi(z_i)$ for $i = 2, \ldots, k$ from the word $\phi(w)$ since they equal 1, and obtain a new word

$$\phi'(w) := A_0 \phi(y_{i_1}) A_1 \phi(y_{i_2}) \cdots \phi(y_{i_s}) A_s$$

where \( \{y_{i_1}, \ldots, y_{i_s}\} = \{y_1, \ldots, y_n\} - \{z_2, \tilde{z}_2, \ldots, z_k, \tilde{z}_k\} \) and $A_j = \phi(r_{i_j} \cdots r_{i_{j+1}-1})$ for $j = 0, \ldots, s$ with $i_0 = 0$ and $i_{s+1} = n + 1$. Then we have $y_{i_1+1}, \ldots, y_{i_{j+1}-1} \in \{z_2, \ldots, z_k\}$. From the restriction on $w$ that $w$ contains only $t_{z_1}$ and $t_{\tilde{z}_k}$ among elements in the form $t_y, y \in \text{Edge}_{y_d}$, it follows that $r_{i_1} = r_{i_2} = \cdots = r_{i_{j+1}-1} = 1$; that is, $A_j = \phi(r_{i_j})$. It is enough to prove that the word $\phi'(w)$ is reduced and not equal to the word \{1\}. Suppose that this word is not reduced. Then it contains a subword

$$\phi(y_{i_j}) A_j \phi(y_{i_{j+1}})$$

with $A_j \in G_{t(\phi(y_{i_j}))}$ and $\phi(y_{i_{j+1}}) = \phi(y_{i_j})$.

Case 1. $y_{i_j} \neq z_1$ nor $\tilde{z}_1$. In this case, we have $\phi(y_{i_j}) = y_{i_j}$, $\phi(y_{i_{j+1}}) = y_{i_{j+1}} = \overline{y}_{i_j}$, and either $i_{j+1} = i_j + 1$ or $y_{i_{j+1}} \in \{z_2, \ldots, z_k\}$ holds. Suppose that $i_{j+1} = i_j + 1$ holds. Since $\phi : G_{t(y_{i_j})} \rightarrow G_{t(w)}$ is an isomorphism and

$$\phi(y_{i_j}) \phi(r_{i_j}) \phi(y_{i_{j+1}})$$

is reducible,

$$y_{i_j} r_{i_j} y_{i_{j+1}}$$

is also reducible, contradicting the assumption. Now we may assume $i_{j+1} \neq i_j + 1$. It is easy to see that $t(y_{i_j}) = t(z_k)$ and $y_{i_{j+1}} = \tilde{z}_k$ hold. Since $Y - |z_1| - |z_k|$ is not connected and $w$ contains only $z_1, \tilde{z}_i$ among $\text{Edge}(Y) - \text{Edge}(Y_r)$, \( \{y_{i_{j+1}} \cdots y_{i_{j+1}-1}\} \) is a walk from $t(z_k)$ to $t(z_k)$ consists of only $z_i, \tilde{z}_i$ for $2 \leq i \leq k$. The assumption on $w$ and irreducibility of $w$ imply that this walk has no back-tracking. Thus, this walk is empty, contradicting the fact that $y_{i_{j+1}} = \tilde{z}_k$.

Case 2. $y_{i_j} = z_1$ or $\tilde{z}_1$. If $y_{i_j} = z_1$, then similarly to Case 1, \( \{y_{i_{j+1}} \cdots y_{i_{j+1}}-1\} \) must be a path from $t(z_1)$ to $t(z_1)$. The corresponding part of $w$ must be

$$z_1 \cdots z_k r_k \tilde{z}_k \cdots \tilde{z}_1$$

for some $r_k \notin \text{Image}(G_{z_k})$. By passage by $\phi$, we have a reduced word

$$er_k \tilde{e},$$

contradicting the assumption. If $y_{i_j} = \tilde{z}_1$, then we have

$$y_{i_j} r_{i_j} y_{i_{j+1}} = \tilde{z}_1 r_{i_j} z_1,$$

whose image by $\phi$ is also reduced.

Consequently, both cases are impossible and every $y_i$ must coincide with one of $z_j$ for some $2 \leq j \leq k$. This implies that $P = t(z_k)$, and similarly to Case 1, the irreducibility and the assumption on $w$ imply that $w = 1$ holds. (q.e.d.)
Lemma 6.9. The kernel of $\phi$

$$\text{Ker}\{\phi : \pi_1(G, Y, P) \to \pi_1(G', Y', P)\}$$

is generated as a normal subgroup by the following four types of elements. Recall that we decomposed a word of $\pi_1(G, Y, P)$ and defined four types in Definition 6.2, and defined a function $s$ in Lemma 6.7. Let $U$ be a path from $P$ to $o(e_1)$ and $V$ be a path from $P$ to $t(e_2)$, which contain no edge in $\text{Edge}(Y_d) - \{z_1, \overline{z}_1, \ldots, z_k, \overline{z}_k\}$. The generators are:

1. $UB(t_{\overline{e}_1})^s(B)U^{-1}$ for $B$ of type 1,
2. $VB(t_{e_2})^{-s(B)}V^{-1}$ for $B$ of type 2,
3. $U(t_{\overline{e}_1})^s(B)\overline{z}_k\overline{z}_{k-1}\cdots\overline{z}_1U^{-1}$ for $B$ of type 3,
4. $VB(t_{\overline{e}_1})^s(B)z_1z_2\cdots z_kV^{-1}$ for $B$ of type 4.

Proof. It is easy to check that these elements belong to $\text{Ker}(\phi)$ by using Lemma 6.7. Let $x$ be an element of $\text{Ker}(\phi)$. Take a reduced word $w$ representing $x$, and decompose it into

$$w = A_1B_1A_2B_2\cdots A_sB_sA_{s+1}$$

as in Definition 6.2. Suppose that $B_1$ is of type 3 for example. Other cases can be handled similarly. In this case,

$$w = A_1(t_{\overline{e}_1})^{-s(B_1)}U^{-1}U(t_{\overline{e}_1})^s(B_1)B_1\overline{z}_k\overline{z}_{k-1}\cdots\overline{z}_1U^{-1}Uz_1z_2\cdots z_kA_2\cdots B_sA_{s+1}$$

holds. Thus, it is enough to prove that

$$w_0 = A_1(t_{\overline{e}_1})^{-s(B)}z_1z_2\cdots z_kA_2\cdots B_sA_{s+1}$$

can be generated by the above four types of elements. Applying this operation on $B_2, \ldots, B_s$ in this order, we obtain $w'$ satisfying the assumption of Lemma 6.8. Since $\phi(w') = 1$, we have $w' = 1$ by Lemma 6.8. (q.e.d.)

Lemma 6.10. For any $y \in \text{Edge}(Y)$, the edge twist $D_y \in \text{Aut}(\pi_1(G, Y, P))$ stabilizes $\text{Ker}(\phi)$.

Proof. It is enough to check that the image by $D_y$ of each generator of $\text{Ker}(\phi)$ belongs to $\text{Ker}(\phi)$ again. This is straightforward; for example, let $w$ be an element of $\text{Ker}(\phi)$ of type 3 in Lemma 6.9. If $y$ does not occur in $B\overline{z}_k\overline{z}_{k-1}\cdots\overline{z}_1$, then we have

$$\phi \circ D_y(U(t_{\overline{e}_1})^s(B)\overline{z}_k\overline{z}_{k-1}\cdots\overline{z}_1U^{-1}) = \phi(D_y(U))\phi((t_{\overline{e}_1})^s(B)\overline{z}_k\overline{z}_{k-1}\cdots\overline{z}_1)\phi(D_y(U^{-1})) = \phi(D_y(UU^{-1})) = 1.$$
Thus, we may assume that $y$ occurs in $Bz_k\bar{z}_{k-1}\cdots \bar{z}_1$, and it follows that $y$ does not belong to $\text{Edge}(Y_r)$. If $y$ is not any of $z_i, \bar{z}_i$, then we have

$$\phi(D_y(v)) = \phi(v)$$

for any word $v$ since $\phi(t_y) = \phi(\bar{t}_y) = 1$, and consequently, $\phi(D_y(w)) = \phi(w) = 1$ holds.

The rest case is $y = z_i$ or $\bar{z}_i$ for some $i$. In this case, observe that the number of occurrence of $z_i$ in $Bz_k\bar{z}_{k-1}\cdots \bar{z}_1$ is the same with that of $\bar{z}_i$ for each $i$. Since $D_y(z_i) = z_it_{z_i}$ and $D_y(\bar{z}_i) = \bar{z}_it_{\bar{z}_i} = t_{z_i}^{-1}\bar{z}_i$, it is easy to see that

$$\phi \circ D_y(Bz_k\bar{z}_{k-1}\cdots \bar{z}_1) = \phi(Bz_k\bar{z}_{k-1}\cdots \bar{z}_1)$$

holds; and consequently, we have $\phi \circ D_y(w) = \phi(w) = 1$. The other types can be checked similarly. (q.e.d.)

Now we can define

$$\varphi : I_Y \to \text{Out}(\pi_1(G', Y', P))$$

by

$$\varphi(D_y) \cdot \phi(x) = \phi(D_y \cdot x),$$

since $D_y$ stabilizes $\text{Ker}(\phi)$.

(6.5.3) Explicit description of $\varphi$.

**Lemma 6.11.** The homomorphism $\varphi$ maps an edge twist to an edge twist or identity as follows:

- $D_y \mapsto D_y$ if $y \in \text{Edge}(Y_r)$,
- $D_y \mapsto \text{id}$ if $y \in \text{Edge}(Y_d) - \{z_1, \bar{z}_1, \ldots, z_k, \bar{z}_k\}$,
- $D_{z_i} \mapsto D_e$ for $1 \leq i \leq n$.

**Proof.** Let $w$ be a word in $\pi_1(G, Y, P)$. Let $D'_y$ denote the RHS of the above table. It is enough to show that

$$\phi(D_y \cdot w) = D'_y \cdot \phi(w)$$

for each case. If $y \in \text{Edge}(Y_r)$, there is no problem.

If $y \in \text{Edge}(Y_d) - \{z_1, \bar{z}_1, \ldots, z_k, \bar{z}_k\}$, it is obvious that $\phi(D_y \cdot w) = \phi(w)$ holds since $D_y \cdot y = yt_y, D_y \cdot \bar{y} = \bar{y}t_{\bar{y}} = t_{y}^{-1}\bar{y}$, and $\phi(t_y) = 1$. Suppose that $y = z_i$ or $\bar{z}_i$ for some $i$. Decompose $w$ into $w = A_1B_1A_2B_2\cdots A_sB_sA_{s+1}$ as shown in Definition 6.2. Since $D_y(A_i) = A_i$ and $D'_y(\phi(A_i)) = \phi(A_i)$, it is enough to show that

$$\phi(D_y \cdot B) = D'_y \cdot \phi(B) = D_e \cdot \phi(B)$$

for $B$ of any one of four types in Definition 6.2. This can easily be checked; for example, suppose that $y = z_i$ and $B$ is of type 3. Then, it is easy to see that

$$\phi(D_y \cdot B) = e(t_{e})^{s(B)+1}$$

and that

$$D'_y(\phi(B)) = D_e(e(t_{e})^{s(B)}) = e(t_{e})^{s(B)+1}.$$ 

Other cases follow similarly. (q.e.d.)
Lemma 6.12. The homomorphism $\varphi$ preserves the filtration.

Proof. Let $\alpha$ be an element of $I_Y^{(m)}$. Then,

$$\varphi(\alpha)(\phi(x)) = \phi(\alpha(x))$$

and

$$(\varphi(\alpha)(\phi(x)))\phi(x)^{-1} = \phi(\alpha(x)x^{-1}) \in \phi(\pi_1(G,Y,P)(m+1)) \subseteq \pi_1(G',Y',P)(m+1)$$

hold for all $x$. Therefore we have $\varphi(\alpha) \in I_Y^{(m)}$. (q.e.d.)

We have proved all statements in Proposition 6.1.
REFERENCES

[A]. Asada, M., Two properties of the filtration of the outer automorphism groups of certain groups, Preprint.


Figure 1.

\[ \prec \]

Figure 2.