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Higher cycles of the moduli space of stable curves

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§1. Introduction and results.

We shall denote the moduli space of stable curves of genus \( g \) by \( M_g \) and assume \( g \geq 3 \). \( M_g \) is a compactification of the classical moduli space \( M_g \) of Riemann surfaces of genus \( g \) [D-M]. It is known that \( M_g \) and \( M_g \) are complex V-manifolds of dimension \( 3g-3 \) and the compactification locus \( D = \overline{M}_g - M_g \) is the sum of \( 1 + \lfloor \frac{g}{2} \rfloor \) divisors \( D_0, \ldots, D_{[\frac{g}{2}]} \). Scott Wolpert showed in [W] that \( 2 + \lfloor \frac{g}{2} \rfloor \) analytic 2-cycles on \( \overline{M}_g \) can be constructed and they span \( H_2(\overline{M}_g; \mathbb{Q}) \) from the result of Harer [H]. Similarly Wolpert defined some analytic \( 2k \)-cycles (for \( k < g \)) and showed that they are independent in \( H_{2k}(\overline{M}_g; \mathbb{Q}) \) by calculating their intersection pairing with components of the strata of \( D \). A surface represented by a point of \( D \) necessarily has nodes (double points). \( D \) is stratified by the number of nodes. Hence the Betti number \( b_{2k}(\overline{M}_g) \) is greater than or equal to \( n_k \), the number of the \( 2k \)-cycles Wolpert constructed. Roughly speaking, \( n_k \) is almost equal to \( \frac{1}{2} \binom{g-1}{k} \).

The idea for constructing analytic cycles is as follows: Fix a stable curve \( S \) with nodes such that \( S - \{ \text{nodes} \} \) is not connected and choose some components \( S_1, \ldots, S_k \) of \( S - \{ \text{nodes} \} \), then we can get an analytic cycle \( A \) by letting the conformal structure of each \( S_j \) vary over
all structures represented in its moduli space, while the structure of \( S - \bigcup_j S_j \) is kept fixed. Wolpert considers, for \( S_j \), once punctured tori and quadruply punctured spheres when \( k = 1 \) and considers quadruply punctured spheres when \( 1 < k < g \).

Because these constructions are only from the moduli space of once-punctured tori and the moduli space of quadruply puncture spheres, we have no information of the Betti numbers of degree more than \( 2g - 2 \) and less than \( 4g - 4 \). In this paper, we shall define analytic fibre spaces having tori as generic fibre and improve Wolpert’s estimates. We have estimates of the Betti numbers of all even degrees. We have the following results.

**Theorem A.**

When \( k \geq 2 \), \( b_{2k}(\overline{\mathcal{M}}_g) = b_{6g-6-2k}(\overline{\mathcal{M}}_g) \geq \max(\alpha_{g,k}, \alpha_{g,3g-3-k}) \)

where \( \alpha_{g,k} \) is a certain kind of permutation numbers.

The number \( \alpha_{g,k} \) can be computed by the following formula

**Proposition B.**

\[ f_g(t) = \sum_k \alpha_{g,k} t^k \]

where \( f_g(t) \) is a polynomials defined in the section 5.

Roughly speaking, this estimates are more than the square of Wolpert’s estimates. In fact we get easily the following inequality.

\[ \alpha_{g,k} > \frac{1}{2} \binom{g-1}{k} + \frac{1}{2} \sum_{k', l} \binom{k' - 1}{l} \cdot \binom{k - 2k' + 1}{l + 1} \cdot \binom{g - l - 2}{k - k'} \]

This paper is organized as follows. In \( \S 2 \) we construct analytic fibrations \( \tilde{Q}_t \rightarrow \mathcal{U}_t \) which have two analytic cross sections with no intersection
and whose general fibre is an elliptic curve. In §3 we construct cycles of $\overline{\mathcal{M}}_g$ from three fibrations; one of them constructed in §2 and the other constructed by Wolpert. In §4 we show that they are independent classes in $H_*(\overline{\mathcal{M}}_g; Q)$ by calculating the intersection pairings between the cycles and components of the strata of $\mathcal{D}$ and show Theorem A. In §5 we give algorithm to compute $\alpha_{g,k}$.

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§2. Twice punctured elliptic curves.

To construct the cycles, we need a fibre space of elliptic curves with the base a compact complex surface which has two cross sections. We require that the sections have no intersections and have no values at nodes of singular fibres. First of all we shall review the definition of the fibre spaces $\mathcal{U}_\ell$, $Q_\ell$ from [W]. These are fibre spaces of elliptic curves over $\overline{\mathbb{H}/\Gamma_\ell}$. We assume $\ell \geq 3$. Let $\mathbb{H}$ be the upper half plane, $\Gamma_\ell$ the principal congruence subgroup of level $\ell$, $\overline{\mathbb{H}/\Gamma_\ell}$ the compactification of the quotient of $\mathbb{H}$ by $\Gamma_\ell$, and $\mathcal{U}_\ell$ the compactification of the universal elliptic curve with level $\ell$ structure. The base $\overline{\mathbb{H}/\Gamma_\ell}$ has $\frac{i}{\ell}$ filled-in punctures (we call them cusps), where $i = [\text{PSL}(2; \mathbb{Z}) : \Gamma_\ell]$. Singular fibres of $\mathcal{U}_\ell$ and $Q_\ell$ correspond to cusps of $\overline{\mathbb{H}/\Gamma_\ell}$. The projection $\mathcal{U}_\ell|_{\mathbb{H}/\Gamma_\ell} \to \mathbb{H}/\Gamma_\ell$ is given as follows. Let $\Gamma_\ell L = \Gamma_\ell \ltimes \mathbb{Z}^2$, where for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\ell$, $g(m, n) = (am + cn, bm + dn)$. $\Gamma_\ell L$ acts on $\mathbb{H} \times \mathbb{C}$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(m, n)(z, \xi) = \left(\frac{az + b}{cz + d}, \frac{\xi + m + n}{cz + d}\right), \text{ for } (z, \xi) \in \mathbb{H} \times \mathbb{C}$$

Then we define the map to be the projection

$$\mathcal{U}_\ell|_{\mathbb{H}/\Gamma_\ell} = \mathbb{H} \times \mathbb{C}/\Gamma_\ell L \to \mathbb{H}/\Gamma_\ell.$$
\{ |v_k| \leq |\tau| \} on \mathbf{P}_k^1 and \{ |u_{k+1}| \leq |\tau| \} on \mathbf{P}_{k+1}^1 and attach two annuli \{ |\tau| < |v_k| < 1 \} and \{ |\tau| < |u_{k+1}| < 1 \} according to \( v_k u_{k+1} = \tau \).

From these considerations we can see that \( \mathcal{U}_\ell \) is a complex surface and in particular a neighbourhood of a double point of the fibre \( \{ \tau = 0 \} \) is identified with the complex surface \( \{ uv = \tau \} \cap \{ |\tau| < 1, |u| < 1, |v| < 1 \} \) in \( C^3 = \{ (\tau, u, v) \} \) by regarding \( \infty \) on \( \mathbf{P}_k^1 \) and 0 on \( \mathbf{P}_{k+1}^1 \) as the double point. \( \mathcal{U}_\ell \) has \( \ell^2 \) natural sections \( s_1, \cdots, s_{\ell^2} \), the \( \ell \)-division points. The sections form a group isomorphic to \( \mathbb{Z}/\ell \mathbb{Z} \times \mathbb{Z}/\ell \mathbb{Z} \), acting on \( \mathcal{U}_\ell \) by translation in fibres.

We define \( \mathcal{Q}_\ell \) by \( \mathcal{U}_\ell/(\mathbb{Z}/\ell \mathbb{Z} \times \mathbb{Z}/\ell \mathbb{Z}) \).

Let \( p : \mathcal{U}_\ell \rightarrow \mathcal{U}_\ell/(\mathbb{Z}/\ell \mathbb{Z} \times \mathbb{Z}/\ell \mathbb{Z}) = \mathcal{Q}_\ell \) be the natural projection.

A fibre in \( \mathcal{Q}_\ell \) over a cusp is a projective line \( \mathbb{P}^1 \) whose points 0 and \( \infty \) are identified to form a double point. In particular a neighbourhood of a double point of fibre on \( \{ \tau = 0 \} \) is identified with the analytic space \( \{ uv = \tau^\ell \} \cap \{ |\tau| < 1, |u| < 1, |v| < 1 \} \) in \( C^3 = \{ (\tau, u, v) \} \) by regarding \( \infty \) on \( \mathbf{P}_k^1 \) and 0 on \( \mathbf{P}_{k+1}^1 \) as the double point. There exists an analytic section \( s \) induced from the sections \( s_1, \cdots, s_{\ell^2} \). The section \( s \) does not take value at the double point. If \( T' \) is a line bundle over \( \mathcal{Q}_\ell \) over the double points \{ double points \} which consists of the tangent vectors of the fibers of \( \pi_2 : \mathcal{Q}_\ell \rightarrow \overline{H/T_\ell} \), then \( c_1(s^*T') = -\frac{1}{12} \).

Next we shall construct the fibration
\[
\pi : \widetilde{\mathcal{Q}}_\ell \rightarrow \mathcal{U}_\ell.
\]

We denote \( \widetilde{\mathcal{Q}}_\ell \) by \( \pi_1^* \mathcal{Q}_\ell \).
We shall define two analytic sections \( \hat{s}_1 \) and \( \hat{s}_2 \) on \( \overline{\mathcal{Q}}_\ell \). The section \( \hat{s}_1 \) is the pull-back of \( s \) by \( \pi_1 \). The section \( \hat{s}_2 \) is defined by

\[
\hat{s}_2 : \mathcal{U}_\ell \to \overline{\mathcal{Q}}_\ell : x \mapsto (x, p(x)),
\]

where \( \overline{\mathcal{Q}}_\ell = \{(x, \xi) \in \mathcal{U}_\ell \times Q_\ell | \pi_1(x) = \pi_2(\xi)\} \) and \( \hat{\pi}(x, \xi) = x \). But if \( s\pi_1(x) = p(x) \), (that is, \( x \in \bigcup_j s_j(H\overline{\Gamma}_\ell) \subset \mathcal{U}_\ell \)) these two sections have intersections. Furthermore, on a double point of a singular fibre of \( \mathcal{U}_\ell \), \( \hat{s}_2 \) takes value at the node point of a singular fibre. We shall modify this fibration so that the sections do not have any intersection, nor does \( \hat{s}_2 \) take value at any node point.

Consider a fibre space which is given by removing \( s_j(H\overline{\Gamma}_\ell) \) from \( \mathcal{U}_\ell \), that is,

\[
\hat{\pi} : \overline{\mathcal{Q}}_\ell|_{\mathcal{U}_j} \to \mathcal{U}_\ell = \mathcal{U}_\ell - \bigcup_j s_j(H\overline{\Gamma}_\ell)
\]

Let \( (\hat{\rho}) := s^*T' \). A metric \( \| \cdot \| \) is fixed for the line bundle \( (\hat{\rho}) \), and in local coordinates the absolute value \( | \cdot | \) is well defined. Let \( U = \{x \in (\hat{\rho}); \|x\| < 1\} \) and we may assume that \( U \) gives coordinates in a neighbourhood of \( s(H\overline{\Gamma}_\ell) \) by the section \( s \). Furthermore \( p^{-1}(U) = U_1 \amalg \cdots \amalg U_{\ell^2} \) where \( U_j \) is a neighbourhood of \( s_j(H\overline{\Gamma}_\ell) \) in \( \mathcal{U}_\ell \) for each \( j \), and \( U_j \) can be identified with \( U \) by the map \( p|_{U_j} : U_j \to U \). Fix \( j \) and identify \( U_j \) with \( U \). We shall describe an auxiliary fibre space \( \mathcal{F} \) over \( U_j \) by attaching \( \hat{\pi}|_{U_j} : \overline{\mathcal{Q}}_\ell|_{U_j} \to U_j \) and \( U_j \times P^1 \to U_j \) as follows.

In the fibres over \( s_j(H\overline{\Gamma}_\ell) \), attach \( \overline{\mathcal{Q}}_\ell|_{s_j(H\overline{\Gamma}_\ell)} \) to a trivial bundle \( s_j(H\overline{\Gamma}_\ell) \times P^1 \to s_j(H\overline{\Gamma}_\ell) \) by identifying \( \hat{s}_1(s_j(H\overline{\Gamma}_\ell)) \subset \overline{\mathcal{Q}}_\ell|_{s_j(H\overline{\Gamma}_\ell)} \) and \( s_j(H\overline{\Gamma}_\ell) \times \{u = 0\} \subset s_j(H\overline{\Gamma}_\ell) \times P^1 \), where \( u \) is one of the inhomogeneous coordinates of \( P^1 \). (The other one is denoted by \( v \) and
satisfies $uv = 1$.) Let $(z, \xi)$ be local coordinates of $U$ (where $z \in \hat{H}/\Gamma_{\ell}$ and $\xi \in \mathcal{C}$).

We define $D_1, D_2, h$ as follows.

\[ D_1 := \left\{ ((z, \zeta), (z, \xi)) \in U_j \times U \subset \mathcal{U}_{\ell} \times \mathcal{Q}_{\ell} \mid \zeta \neq 0, |\xi| \leq |\zeta| \right\} \]

\[ \subset \mathcal{Q}_{\ell}|_{U_j - s_j(\hat{H}/\Gamma_{\ell})} \]

\[ D_2 := \left\{ ((z, \zeta), u) \in U_j \times \mathbb{P}^1 \mid \zeta \neq 0, |u| \leq ||\zeta|| \right\} \]

\[ \subset (U_j - s_j(\hat{H}/\Gamma_{\ell})) \times \mathbb{P}^1 \]

\[ h : \left\{ ((z, \zeta), u) \in U_j \times \mathbb{P}^1 \mid \zeta \neq 0, ||\zeta|| < |u| < 1 \right\} \]

\[ \rightarrow \left\{ ((z, \zeta), (z, \xi)) \in U_j \times U \subset \mathcal{U}_{\ell} \times \mathcal{Q}_{\ell} \mid \zeta \neq 0, |\zeta| < |\xi| < \frac{|\zeta|}{||\zeta||} \right\} \]

\[ ((z, \zeta), u) \mapsto ((z, \zeta), (z, \frac{\zeta}{u})) \]

Then we attach $\mathcal{Q}_{\ell}|_{U_j - s_j(\hat{H}/\Gamma_{\ell})} - D_1$ and $(U_j - s_j(\hat{H}/\Gamma_{\ell})) \times \mathbb{P}^1 - D_2$ by the attaching map $h$, and we have a fibre space $\mathcal{F} \rightarrow U_j$. It is easy to check that $\mathcal{F}$ is an analytic fibre space and the map $\mathcal{F} \rightarrow U_j$ is holomorphic.

Identify $\mathcal{F}|_{U_j - s_j(\hat{H}/\Gamma_{\ell})}$ with $\mathcal{Q}_{\ell}|_{U_j - s_j(\hat{H}/\Gamma_{\ell})}$ by

\[ (U_j - s_j(\hat{H}/\Gamma_{\ell})) \times \mathbb{P}^1 - D_2 \rightarrow (U_j - s_j(\hat{H}/\Gamma_{\ell})) \times U_j \]

\[ (((z, \zeta), u) \mapsto ((z, \zeta), (z, \frac{\zeta}{u})) \]

\[ (((z, \zeta), v) \mapsto ((z, \zeta), (z, \zeta v)) \]

and denote the identification map by $\iota_j$. For all $j = 1, 2, \ldots, \ell^2$, attach
\(\mathcal{F}\) and \(\overline{\mathcal{Q}_\ell}|_{\mathcal{U}_{p^\circ}}\) by \(\iota_j\), and denote the analytic fibre space constructed by \(\hat{\pi}': \widehat{\mathcal{Q}_\ell} \rightarrow \mathcal{U}_\ell\).

Next let \(U'\) be a neighbourhood of a node on a singular fiber of \(\mathcal{U}_\ell\). We shall construct an auxiliary fibre space \(\mathcal{F}'\) over \(U'\). We consider coordinates of \(U'\) as follows.

\[
U' := \{(u, v) \in \mathbb{C}^2; |u| < 1, |v| < 1\} \quad \pi_1(u, v) = uv
\]

Let \(P\) be a projective line and \(D_1, D_2\) be two disks, that is,

\[
P := \mathbb{P}^1 : \text{with inhomogeneous coordinates } u_1, v_1(u_1v_1 = 1)
\]

\[
D_1 := \{v_2 \in \mathbb{C} \mid |v_2| < 1\}
\]

\[
D_2 := \{u_2 \in \mathbb{C} \mid |u_2| < 1\}
\]

Attach \(U' \times D_1, U' \times D_2\) and \(U' \times P\) as follows.

(i) On \((0, 0) \in U'\), identify

\[
(0, 0) \times \{v_2 = 0\} \in (0, 0) \times D_1 \quad \text{and} \quad (0, 0) \times \{u_1 = 0\} \in (0, 0) \times P,
\]

\[
(0, 0) \times \{u_2 = 0\} \in (0, 0) \times D_2 \quad \text{and} \quad (0, 0) \times \{v_1 = 0\} \in (0, 0) \times P.
\]

(ii) On \((z_1 \neq 0, 0) \in U'\), remove \(\{|v_2| \leq |z_1|^{t}\}\) from \(D_1\), remove \(\{|u_1| \leq |z_1|^{t}\}\) from \(P\) and attach \(\{|z_1|^{t} < |v_2| < 1\} \subset D_1\) to \(\{|z_1|^{t} < |u_1| < 1\} \subset P\) by identifying \(v_2u_1 = z_1^{t}\). Identify

\[
(0, 0) \times \{u_2 = 0\} \in (0, 0) \times D_2 \quad \text{and} \quad (0, 0) \times \{v_1 = 0\} \in (0, 0) \times P.
\]

(iii) On \((0, z_2 \neq 0) \in U'\), remove \(\{|u_2| \leq |z_2|^{t}\}\) from \(D_2\), remove \(\{|v_1| \leq |z_2|^{t}\}\) from \(P\) and attach \(\{|z_2|^{t} < |u_2| < 1\} \subset D_1\) to \(\{|z_2|^{t} < |v_1| < 1\} \subset P\) by identifying \(v_1u_2 = z_2^{t}\). Identify

\[
(0, 0) \times \{v_2 = 0\} \in (0, 0) \times D_1 \quad \text{and} \quad (0, 0) \times \{u_1 = 0\} \in (0, 0) \times P.
\]
(iv) On \((z_1 \neq 0, z_2 \neq 0) \in U'\), remove \(\{|v_2| \leq |z_1|^t\}\) from \(D_1\), remove \(\{|u_1| \leq |z_1|^t\}\) from \(P\) and attach \(\{|z_1|^t < |v_2| < 1\} \subset D_1\) to \(\{|z_1|^t < |u_1| < 1\} \subset P\) by identifying \(v_2u_1 = z_1^t\). Remove \(\{|u_2| \leq |z_2|^t\}\) from \(D_2\), remove \(\{|v_1| \leq |z_2|^t\}\) from \(P\) and attach \(\{|z_2|^t < |u_2| < 1\} \subset D_2\) to \(\{|z_2|^t < |v_1| < 1\} \subset P\) by identifying \(u_2v_1 = z_2^t\).

This fibre space \(\mathcal{F}'\) is an analytic space and the map \(\mathcal{F}' \to U'\) is holomorphic.

Finally we shall construct a fibre space \(\tilde{\pi} : \mathcal{Q}_\ell \to \mathcal{U}_\ell\) by attaching \(\mathcal{F}'\) to \(\mathcal{Q}_\ell |_{\mathcal{U}_\ell - \{\text{double points}\}}\).

Let \(B\) be defined as follows.

\[
B = \left\{ ((z_1, z_2), \pi(u, v)) \in U' \times \pi(U') \mid z_1z_2 = uv \right\}
- \left\{ ((0, 0), \pi(0, 0)) \right\} \subset \mathcal{Q}_\ell'
\]

Also we define \(\iota' : B \to \mathcal{F}'\) as follows.

\[
\iota'((z_1, z_2), \pi(u, v))
= \begin{cases} 
(z_1, z_2, z_1^t) & \text{in the } (x, y, u_1)-\text{coordinates (if } z_1 \neq 0, u \neq 0) \\
(z_1, z_2, z_2^t) & \text{in the } (x, y, u_1)-\text{coordinates (if } z_2 \neq 0, v \neq 0) \\
(z_1, z_2, u^t) & \text{in the } (x, y, v_2)-\text{coordinates (if } |u| \geq |z_1|^t) \\
(z_1, z_2, v^t) & \text{in the } (x, y, u_2)-\text{coordinates (if } |v| \geq |z_2|^t) 
\end{cases}
\]

This map is well defined and biholomorphic. Attach \(\mathcal{F}'\) to \(\mathcal{Q}_\ell |_{\mathcal{U}_\ell - \{\text{nodes}\}}\) by the map \(\iota'\) for all nodes on \(\mathcal{U}_\ell\) and we have a new fibre space \(\tilde{\pi} : \mathcal{Q}_\ell \to \mathcal{U}_\ell\).
Lemma 2.1.

\( \tilde{\pi} : \tilde{Q}_\ell \to \mathcal{U}_\ell \) have two analytic cross sections \( \tilde{s}_1, \tilde{s}_2 \) which have no intersection and have no values at nodes on fibers.

Proof

The sections \( \tilde{s}_1 \) and \( \tilde{s}_2 \) are defined as follows:

\[
\tilde{s}_1 = \begin{cases} 
\hat{s}_1 & \text{in } \tilde{Q}_\ell|_{U_\ell - \cup s_j(H/\Gamma_\ell)} \\
(z, \xi) \mapsto (z, \xi, 0) \in U_j \times P^1 & \text{in } \mathcal{F}
\end{cases}
\]

\[
\tilde{s}_2 = \begin{cases} 
\hat{s}_2 & \text{in } \tilde{Q}_\ell|_{U_\ell'} \\
(z, \xi) \mapsto (z, \xi, 1) \in U_j \times P^1 & \text{in } \mathcal{F}
\end{cases}
\]

It is easy to show that \( \tilde{s}_1 \) and \( \tilde{s}_2 \) are well defined and that they have no intersection. Also they do not take values at nodes on fibers.

Removing the two sections \( \tilde{s}_1, \tilde{s}_2 \) from \( \tilde{Q}_\ell \) we obtain a family \( \tilde{Q}_\ell^o \) of twice punctured tori. Each singular fibres is one of the following four types.
Figure 1.

Type 1 On $s(\text{cusps})$

Type 2 On $s(\overline{H/\Gamma_\ell} - \text{cusps})$

Type 3 On $\pi_1^{-1}(\text{cusps}) - s(\text{cusps}) - \text{nodes}$
§3. Construction of the cycles.

The locus $\mathcal{D} \subset \overline{\mathcal{M}}_g$ of surfaces having nodes (double points) is a divisor and is stratified by the number of nodes. That is to say, the $k$-strata is the locus of surfaces with $k$ nodes. Counting the number of components in the $k$-strata is a combinatorial problem. (The number of 1-stratum components is $1 + \lfloor \frac{g}{2} \rfloor$.) The closure of a $k$-stratum component represents a $(6g - 6 - 2k)$-dimensional homology class. Wolpert [W] showed that Poincaré dual $[\omega]$ of Weil-Petersson Kähler form $\omega$ on $\overline{\mathcal{M}}_g$ and the $1 + \lfloor \frac{g}{2} \rfloor$ cycles above represented by the closure of 1-stratum components span $H_{6g-8}(\overline{\mathcal{M}}_g; \mathbb{Q})$. $n_k$ components of the $k$-strata of the specific pattern for the nodes are mutually independent in $H_{6g-6-2k}(\overline{\mathcal{M}}_g; \mathbb{Q})$ for $k < g$.

Considering another specific pattern for the nodes we construct more homology classes for all even degrees.

In this section first we shall define "$k$-selections" ($k \leq 2g - 2$). After that we shall construct cycles and subvarieties of $\mathcal{M}_g$. To construct cycles we use three fibrations

$$\tilde{\pi} : \Delta^o \to \mathcal{U}_t$$
$$\pi_2 : \Delta^o \to \mathbb{H}/\Gamma_t$$
$$\pi : \mathcal{U}^o_2 \to \mathbb{H}/\Gamma_2.$$
where $Q_t^o$ is $Q_t - Im(s)$. $U_2$ is defined by the compactification of $H \times C/\Gamma_2 L \rightarrow H/\Gamma_2$ in the same way as $U_t$. Generic fibre of $U_2$ is a complex projective line with four distinguished points $s_1, s_2, s_3, s_4$. Singular fibres on filled in cusps are two projective lines connected by a double point.

$\mathcal{U}_2$ is $U_2 - \cup s_j(H \overline{\Gamma}_2)$. Note that $c_1(s_i^* T') = -1$ for $i = 1, 2, 3, 4$.

Fix a surface $S$ and we shall consider the "pattern" for the curves $c_1, c_2, \cdots, c_{g-1}, d_1, d_2, d'_2, \cdots, d'_{g-1}, d_{g-1}, d_g$ as indicated below. (Figure 2)
Definition 3.1. A $k$-selection $\sigma$ is a choice of $k$ homotopy classes from the free homotopy classes $[c_1], \cdots, [c_{g-1}], [d_1], [d_2], [d'_2], \cdots, [d_{g-1}], [d'_{g-1}], [d_g]$ satisfying the following two conditions

1) $[d_i]$ is in the $k$-selection if and only if $[d'_i]$ is in the $k$-selection.

2) If $[d_i], [d'_i]$ are in the $k$-selection, then neither $[c_{i-1}]$ nor $[c_i]$ is not in the $k$-selection (for $2 \leq i \leq g - 1$)

The classes $[c_1], \cdots, [c_{g-1}], [d_1], [d_2], [d'_2], \cdots, [d_{g-1}], [d'_{g-1}], [d_g]$ may be permuted by a homeomorphism of $S$. We denote the number of distinct $k$-selections modulo the action of homeomorphisms by $\alpha_{g,k}$. ($\alpha_{g,k}$ is the number of conjugate classes of $k$-selections. (see definition 4.2))

Fix a $k$-selection $\sigma$, if $d_j, d'_j$ occur in the $k$-selection $\sigma$ then collapse $c_{j-1}, c_j$ to nodes and replace the component containing $d_j, d'_j$ by the fibre of a family $\bar{\mathcal{Q}}^o_{\ell}$: remove the component containing $d_j, d'_j$, identify the node to which $c_{j-1}$ collapses with the first puncture in the fibre of $\bar{\mathcal{Q}}^o_{\ell}$, and identify the node to which $c_j$ collapses with the second puncture in the fibre of $\bar{\mathcal{Q}}^o_{\ell}$.

If $c_j (2 \leq j \leq g - 2)$ is in the $k$-selection $\sigma$, we collapse $d_j, d'_j, d_{j+1}, d'_{j+1}$ to nodes and replace the component containing $c_j$ by the fibre of the family $\mathcal{U}^o_{\ell}$. If both $d_1$ and $c_1$ are in the $k$-selection $\sigma$, we collapse $d_2, d'_2$ to nodes and replace the component containing $c_1, d_1$ by the fibre of the family $\bar{\mathcal{Q}}^o_{\ell}$. The construction is similar for the case when both $d_g, c_{g-1}$ are in $\sigma$. If $c_1$ is in the $k$-selection $\sigma$ but $d_1$ is not, we collapse $d_1, d_2, d'_2$ to nodes and replace the component containing $c_1$ by the fibre of the family $\mathcal{U}^o_{\ell}$. The construction is similar for the case when $c_{g-1}$ is in $\sigma$ but $d_g$ is not. If $d_1$ is in the $k$-selection $\sigma$ but $c_1$ is not, we collapse $c_1$ to a node and replace the component containing $d_1$ by the fibre of the
family $Q^*_t$. The construction is similar for the case when $d_g$ is in $\sigma$ but $c_{g-1}$ is not.

Thus we have defined an analytic fibre space $A_\sigma$. $A_\sigma$ has the Cartesian product $(U_t)^a \times (H/\Gamma_\ell)^b \times (H/\Gamma_2)^c$ as the base and is the connected sum along punctures of the fibres of $(Q^*_\ell)^a$, $(Q^*_2)^b$, $(U^2)^c$ and a fixed surface $R$ for some non-negative integers $a$, $b$, $c$. The analytic fibre space $A_\sigma$ determines a mapping from $(U_t)^a \times (H/\Gamma_\ell)^b \times (H/\Gamma_2)^c$ to $\overline{M}_g$. Let $[A_\sigma]$ be the homology class determined by $A_\sigma$. In this way, we can define a $2k$-cycle $[A_\sigma]$ for each k-selection $\sigma$.

§4. Counting the intersection numbers.

In the Section 3 we showed that for each k-selection we can define a $2k$-cycle on $\overline{M}_g$. On the other hand, for each k-selection $\tau$ we can define a subvariety $V_\tau$ of dimension $6g-6-2k$ of $\overline{M}_g$ which consists of surfaces whose nodes correspond to the curves selected in the k-selection. Let $V^\circ_k \subset D$ be the locus of surfaces having precisely $k$ nodes. We divide $V^\circ_k$ into connected components. The closure of them are distinct subvarieties of $\overline{M}_g$. They are determined by the choice of $k$ disjoint simple geodesics (which are to be collapsed to nodes) on a surface of genus $g$.

We shall compute an intersection number of the $2k$-cycle $A_\sigma$ and $6g-6-2k$ subvariety $V_\tau$ for k-selections $\sigma, \tau$ by pushing off $[A_\sigma]$ from $D$. That is, we open up the attaching nodes of replaced components. We shall show that the nodes cannot be opened up for all fibres. To open up the nodes we smoothly perturb sections

\[s : \overline{H}/\Gamma_\ell \to Q_\ell \]
\[s_j : \overline{H}/\Gamma_2 \to U_2 \]
\[\tilde{s}_1, \tilde{s}_2 : U_\ell \to \tilde{Q}_\ell. \]
The perturbation of $s, s_j$ is given in [W]. Since $c_1(s^*T') = -\frac{i}{12}$, we can choose $s'$, a smooth perturbation of $s$ in $U$, such that at only one point $p \in Q_\ell$ it intersects $s$ and $s'(z) = (z, z^{-\frac{i}{12}})$ in $(z, \xi)$ coordinate around $p$. Since $c_1(s_j^*T') = -1$, we can choose $s_j'$, a smooth perturbation of $s_j$, such that at only one point $q \in Q_\ell$ it intersects $s_j$ and $s_j'(z) = (z, \overline{z})$ in $(z, \xi)$ coordinate around $q$.

Before defining the perturbation of the sections $\tilde{s}_1, \tilde{s}_2$, we give some remarks according to [W].

A $V$-manifold such as $\overline{\mathcal{M}}_g$ is a rational homology manifold and if cycles intersect at manifold points then their pairing is determined standardly in the intuitive way. Next we shall describe coordinates for the local manifold covers of $\overline{\mathcal{M}}_g$. Let $S$ be a Riemann surface with nodes $p_1, \ldots, p_m$ such that each component of $S - \{p_1, \ldots, p_m\}$ is hyperbolic. Suppose that at the node $p_i$ punctures $a_i$ and $b_i$ are paired. Choose disjoint neighborhood $D^1_i, D^2_i$ $(i = 1, 2, \ldots, m)$ of the punctures $a_i$ and $b_i$ and let $z_i : D^1_i \to D = \{u \in C; |u| < 1\}$ and $w_i : D^2_i \to D$ be local coordinates with $z_i(a_i) = w_i(b_i) = 0$. Fixing an suitable open set $\mathcal{O}$ disjoint from $D^1_i$ and $D^2_i$, Beltrami differentials $\mu_j$ are chosen with support in $\mathcal{O}$ spanning the Teichmüller space of $S - \{p_1, \ldots, p_m\}$ (the dimension $3g - 3 - m$). If $t = (t_1, \ldots, t_{3g-3-m}) \in C^{3g-3-m}$ is sufficiently close to the origin, the sum $\mu(t) = \sum_j t_j \mu_j$ satisfies $\|\mu(t)\|_{\infty} < 1$ and thus a $\mu$-conformal solution $\omega^{\mu(t)}$ of the Beltrami equation exists. The Riemann surface $\omega^{\mu(t)}(S) = S_t$ is a quasiconformal deformation of $S$. The map $\omega^{\mu(t)}$ is conformal on $D^1_i$ and $D^2_i$; therefore $z_i$ and $w_i$ serve as coordinates for $\omega^{\mu(t)}(D^1_i)$ and $\omega^{\mu(t)}(D^2_i) \subset S_t$ respectively. Given $\tau = (\tau_1, \ldots, \tau_m) \in D^m$, we construct a surface $S_{\tau,t}$ as follows. Remove the discs $\{z_i; |z_i| \leq |\tau_i|\}$ and $\{w_i; |w_i| \leq |\tau_i|\}$ from $S_\tau$. Attach $\{z_i; |\tau_i| < |z_i| < 1\}$ to $\{w_i; |\tau_i| < |w_i| < 1\}$ by identifying $z_i$ and $\tau_i/w_i$. 


to obtain \( S_{\tau, t} \). The couple \((\tau, t)\) gives holomorphic coordinates for the local manifold cover of \( \mathcal{M}_{\theta} \) around the point represented by \( S \). The automorphism group \( \text{Aut}(S) \) acts locally on these coordinates. (see also [B])

Now we construct perturbations of \( \tilde{s}_{1}, \tilde{s}_{2} \). Let \( s'_{1}|_{U_{t}} : U_{t}^{\circ} \rightarrow \tilde{Q}_{t} \) be the pull back of \( s' \) by \( \pi_{1} \), and let \( s'_{1}|_{U_{j}} : U_{j} \rightarrow \tilde{Q}_{t} \) be defined by \( s'_{1}(z, \zeta) = (z, \zeta, s'(z)(\bar{\zeta}) \) in the \((z, \zeta, v)\) coordinates of \( \tilde{Q}_{t} \). By a partition of unity subordinate to \( \{U_{t}^{\circ}, U_{1}, \ldots, U_{t}\} \), we can construct \( \tilde{s}'_{1} : U_{t} \rightarrow \tilde{Q}_{t} \). Note that in \( U' \), \( \tilde{s}'_{1} \) is a map to \( \tilde{Q}_{t} \). The intersection of \( \tilde{s}_{1} \) and \( \tilde{s}'_{1} \) represents the homology class \(-\frac{i}{12}[\text{fiber}] - \sum_{j}[s_{j}(H/\Gamma_{\ell})] \). That is, let \( \tilde{\rho}_{1} \) be the pull back of the tangent vector field along the fibres on \( \tilde{Q}_{t} \) by \( \tilde{s}_{1} \), then the Poincaré dual of the Euler class of \( \tilde{\rho}_{1} \) is \(-\frac{i}{12}[\text{fiber}] - \sum_{j}[s_{j}(H/\Gamma_{\ell})] \). We can define \( \tilde{s}'_{2} \) in the same way, then for the pull back \( \tilde{\rho}_{2} \) by \( \tilde{s}_{2} \), the Poincaré dual of the Euler class of \( \tilde{\rho}_{2} \) is \(-\frac{i}{12}[\text{fiber}] - \sum_{j}[s_{j}(H/\Gamma_{\ell})] \).

Using \( \tilde{s}'_{1} \) and \( \tilde{s}'_{2} \) we open up the punctures. Fixing a k-selection \( \sigma \) such that \( d_{j}, d'_{j} \) are selected but \( d_{j-1}, d'_{j-1} \) are not, we open up the first puncture in the fibre of \( \tilde{Q}_{t} \) using \( \tilde{s}'_{1} \). Let a metric \( \| \cdot \| \) be fixed for the line bundle \( \tilde{\rho}_{1} \). In the local coordinate \((z_{1}, z_{2}, \zeta)\) of \( \tilde{Q}_{t} \) the absolute value \( |\zeta| \) is well defined. Choose a neighborhood \( \mathcal{U} \) of the 0-section in \( \tilde{\rho}_{1} \) mapping injectively to the fibre space \( \tilde{Q}_{t} \). We identify \( \mathcal{U} \) with its image in \( \tilde{Q}_{t} \). Now \( \mathcal{U} \) may be chosen to intersect each fibre in a disk centered at the origin and the section \( \tilde{s}_{1} \) may be contained in \( \mathcal{U} \) and \( \| \tilde{s}_{1} \| < 1 \). Choose a local coordinate chart \((z_{1}, z_{2}, \zeta)\) of \( \mathcal{U} \) in \( \tilde{Q}_{t} \). The section \( \tilde{s}_{1} \) is represented in \((z_{1}, z_{2}, \zeta)\) as \( \zeta = \tilde{s}_{1}'(z_{1}, z_{2}) \). Let \( \omega \) be the coordinate disk of neighborhood of fixed side of a node \( c_{j} \) collapsing. We assume the \( \omega \) maps the neighborhood to the unit disk. Remove a disk \( \{ |\omega| \leq \| \tilde{s}_{1}'(z_{1}, z_{2}) \| \} \) from \( S \), remove a disk \( \{ |\zeta| \leq \| \tilde{s}_{1}'(z_{1}, z_{2}) \| \} \) from
a fibre $F(z_1, z_2)$ over a point $(z_1, z_2)$ of $\tilde{Q}_\ell$, and form a connected sum of the resulting surfaces by identifying

$$\{||\tilde{s}'_1(z_1, z_2)|| < |w| < 1\} \subset S$$

$$\rightarrow \{||\tilde{s}'_1(z_1, z_2)|| < |z| < ||\tilde{s}'_1(z_1, z_2)||/||s'_1(z_1, z_2)||\} \subset F(z_1, z_2)$$

by setting $w\zeta = \tilde{s}'_1(z_1, z_2)$. It is easy to check that this construction does not depend on the choice of coordinate. In the case when $d_j, d'_j$ are selected but $d_{j+1}, d'_{j+1}$ are not, we can open up the second puncture by using $\tilde{s}'_2$ in the same way as above.

In the case when both $d_j, d'_j$ and $d_{j+1}, d'_{j+1}$ are selected, choose local coordinates $(z_1, z_2, \zeta)$ around the first puncture of $d_{j+1}$-component, $(x_1, x_2, \zeta_2)$ around the second puncture of $d_j$-component. Now as above, remove the disc neighborhoods

$$\{||\zeta|| \leq ||\tilde{s}'_1(z_1, z_2)||/||s'_1(z_1, z_2)||\} \text{ and } \{||\zeta_2 || \leq ||\tilde{s}'_2(x_1, x_2)||/||s'_2(z_1, z_2)||\}$$

of punctures, and form a connected sum of the resulting surfaces by identifying

$$\{||\tilde{s}'_1||/||s'_2|| < \zeta < ||\tilde{s}'_1||/||s'_1||\} \text{ with } \{||\tilde{s}'_2||/||s'_1|| < \zeta_2 < ||\tilde{s}'_2||/||s'_2||\}$$

by setting $\zeta_1\zeta_2 = \tilde{s}'_1\tilde{s}'_2$. It is easy to show that this construction is independent of the choice of coordinates. In the case when $d_1$ is selected or $c_j$ is selected, we can open up the node by using $s'$ or $s'_1, \ldots, s'_4$ respectively.

Let $\mathcal{A}_\sigma^+$ be a smooth fibre space of stable curves of genus $g$ constructed from a $k$-selection $\sigma$ as above, then $\mathcal{A}_\sigma$ and $A_\sigma$ determine homotopic cycles on $\overline{\mathcal{M}}_g$.

We will consider intersections of $\mathcal{A}_\sigma^+$ and $k$-selection subvarieties.
Lemma 4.1.

Let $\sigma$ be a $k$-selection. Assume that if $g \geq 4$, not all $[d_2], [d_2'], \ldots, [d_{g-1}], [d'_{g-1}]$ are selected simultaneously. For any $k$-selection cycle $[A_\tau]$, the cycle $[A^\dagger_\tau]$ can be chosen to intersect the $k$-selection subvariety $\mathcal{V}_\sigma$ in manifold points of $\overline{M}_g$.

Proof Consider $\text{Aut}(S_\infty)$, where $S_\infty$ is a fibre of $\mathcal{A}_r^\dagger$ over an intersection with $\mathcal{V}_\sigma$. Assume the complement of the $k$-selection nodes in $S_\infty$ has $m$ connected components, $S_1, \ldots, S_m$. From the description of $\mathcal{A}_r^\dagger$ each components may be varied arbitrary in a open set of its Teichmüller space. We may divide these components into three types,

i) $S_j$; once punctured torus or thrice punctured sphere with two of the punctures identified,

ii) $S_j$; twice punctured torus, or quadruply punctured sphere with two of the punctures identified, or quadruply punctured sphere,

iii) $S_j$; once punctured surface of genus at least 2 or twice punctured surface of genus at least 2 or thrice punctured surface of genus at least 1 or quadruply punctured surface of genus at least 1.

In the case i) we may assume that $\text{Aut}(S_j)$ is generated by an elliptic involution, in the case ii) $\text{Aut}(S_j)$ is not trivial but we may assume that the group $\text{Aut}_f(S_j)$ of automorphism fixing the punctures is trivial, and in the case iii) we may assume that $\text{Aut}(S_j)$ is trivial.

By topological considerations the only homeomorphism of $S_\infty$ which might permute the components $S_1, \ldots, S_m$ is the left-right switch map. Certainly we may assume that the conformal structures for $S_1, \ldots, S_m$ are distinct and hence this homeomorphism cannot be represented in $\text{Aut}(S_\infty)$.

When $g \geq 4$, since $\sigma$ does not contain at least one of the curves
There exists a component of the case iii) in $S_{\infty}$. Hence any element of $\text{Aut}(S_{\infty})$ fixes all punctures and $\text{Aut}(S_{\infty}) = \Pi_{j}\text{Aut}_{f}(S_{j})$. Hence we only consider for the case components $S_{j}$ of case i). There are three possibilities for $\text{Aut}(S_{\infty})$, that is, 1) $\text{Aut}(S_{\infty})$ is trivial, or 2) $\text{Aut}(S_{\infty}) = \mathbb{Z}/2\mathbb{Z}$, or 3) $\text{Aut}(S_{\infty}) = (\mathbb{Z}/2\mathbb{Z})^{2}$. These three cases correspond respectively to 1) none of $c_{1}, c_{g-1}, 2)$ exactly one of $c_{1}, c_{g-1}, 3)$ both $c_{1}$ and $c_{g-1}$ occur in the k-selection $\sigma$. For case 1) $S_{\infty}$ certainly represents a manifold point. For case 2) assume $c_{1}$ is selected in $\sigma$, a non-trivial element $k \in \text{Aut}(S_{\infty})$ is from an elliptic involution. We introduce local manifold cover coordinates $(\tau_{1}, t)$ where $\tau_{1}$ is for the $c_{1}$ node. The elliptic involution is generic for an elliptic curve so that $k$ acts as $k(\tau_{1}, t) = (-\tau_{1}, t)$. Hence $(\tau_{1}^{2}, t)$ give coordinates of $\overline{\mathcal{M}}_{g}$ around intersection point. For case 3) consider local manifold cover coordinates $(\tau_{1}, \tau_{g-1}, t)$, where $\tau_{1}$ is for the $c_{1}$ node, $\tau_{g-1}$ is for the $c_{g-1}$ node. $\text{Aut}(S_{\infty})$ has two generators

$$(\tau_{1}, \tau_{g-1}, t) \rightarrow (-\tau_{1}, \tau_{g-1}, t)$$

$$(\tau_{1}, \tau_{g-1}, t) \rightarrow (\tau_{1}, -\tau_{g-1}, t).$$

Hence $(\tau_{1}^{2}, \tau_{g-1}^{2}, t)$ give coordinates of $\overline{\mathcal{M}}_{g}$ around the intersection points.

When $g = 3$, and $[d_{2}]$ and $[d'_{2}]$ are not selected in $\sigma$ we can define coordinates as above. When $g = 3$, and $[d_{2}]$ and $[d'_{2}]$ are selected, each component is twice punctured torus, or quadruply punctured sphere with two of the punctures identified. In this case $\text{Aut}(S_{\infty}) = \mathbb{Z}/2\mathbb{Z}$ and if $(\tau_{1}, \tau_{2}, t)$ is a local manifold cover coordinate such that $\tau_{1}$ is for the $d_{2}$ node, $\tau_{2}$ is for the $d'_{2}$ node, a non-trivial element $k \in \text{Aut}(S_{\infty})$ acts as

$$k(\tau_{1}, \tau_{2}, t) = (\tau_{2}, \tau_{1}, t).$$
Hence \((\tau_1 + \tau_2, (\tau_1 - \tau_2)^2, t)\) gives a coordinate of \(\overline{\mathcal{M}}_g\) around intersection points. This completes a proof of lemma 4.1. \(\square\)

When \(g \geq 4\) and all \([d_2], [d_2'], \ldots, [d_{g-1}], [d_{g-1}']\) occur in a \(k\)-selection \(\sigma\), the \(k\)-selection cycle \([\mathcal{A}_k^\tau]\) cannot be chosen to intersect \(\mathcal{V}_\sigma\) at manifold points of \(\overline{\mathcal{M}}_g\). So we need a smooth perturbation \(\mathcal{V}_\sigma^#\) of \(\mathcal{V}_\sigma\). We consider only the case \(\sigma = \{[d_2], [d_2'], \ldots, [d_{g-1}], [d_{g-1}']\}\). The other cases are similar.

Let \(S_\infty\) be a fibre of \(\mathcal{A}_k^\tau\) over an intersection with \(\mathcal{V}_\sigma\), we may assume \(Aut(S_\infty) = Z/2Z\). We introduce a local manifold cover \((V, \psi : V \rightarrow \overline{\mathcal{M}}_g, \psi(V))\) and coordinates \((\tau_2, \tau_2', \ldots, \tau_{g-1}, \tau_1', \ldots, t_m)\) of \(V\) where \(\tau_j\) is for the \(d_j\) node, \(\tau_j'\) is for the \(d_j'\) node. A non-trivial element \(k \in Aut(S_\infty)\) acts as

\[k(\tau_2, \tau_2', \ldots, \tau_{g-1}, \tau_1', t_1, \ldots, t_m) \rightarrow (\tau_2', \tau_2, \ldots, \tau_{g-1}', \tau_1, t_1, \ldots, t_m).\]

In this local manifold cover, \(\mathcal{V}_\sigma\) is given as the locus \(\{\tau_2 = \tau_2' = \cdots = \tau_{g-1} = \tau_{g-1}' = 0\}\) and this locus is mapped injectively to \(\overline{\mathcal{M}}_g\). We introduce a local coordinate neighborhood \(U\) and a local coordinate chart \((x_1, \ldots, x_m)\) of \(\mathcal{V}_\sigma\) around the point represented by \(S_\infty\). \((x_1, \ldots, x_m)\) is mapped \((0, \ldots, 0, x_1, \ldots, x_m)\) in the local manifold cover. Let \(\epsilon(x)\) be a function on \(\mathcal{V}_\sigma\) such that \(U\) contains the support of \(\epsilon(x)\) and in a small neighborhood of \(S_\infty\) \(\epsilon(x) = \epsilon\) where \(\epsilon\) is a small constant. We set \(\mathcal{V}_\sigma^#\) in \(\psi(V)\) as follows

\[(x_1, \ldots, x_m) \rightarrow \psi(\epsilon(x), 0, \ldots, 0, x_1, \ldots, x_m) \in \psi(V).\]

\(\mathcal{V}_\sigma^#\) is homotopic to \(\mathcal{V}_\sigma\) and we may assume that the cycle \(\mathcal{A}_k^\tau\) intersects \(\mathcal{V}_\sigma^#\) in manifold points.
For 1-selections, Wolpert [W] showed the intersection pairing between cycles \( \{A_{\sigma}\}_{\sigma} \) and varieties \( \{V_{\tau}\}_{\tau} \) are full rank.

Here we remark the following; if \( k \)-selections \( \sigma \) and \( \tau \) are in the same class modulo the action of homeomorphisms then the associated subvarieties \( V_{\sigma} \) and \( V_{\tau} \) are equal. Thus the number of subvarieties \( \{V_{\sigma}\}_{\sigma} \) is \( \alpha_{g,k} \). (see §3)

**Definition 4.2.**

1. For a \( k \)-selection \( \sigma \), we define its *conjugate* \( \bar{\sigma} \) as a \( k \)-selection which contains \( [d_j] \)(resp. \( [c_j] \)) iff \( \sigma \) contains \( [d_{g-j+1}] \)(resp. \( [c_{g-j}] \)).
2. We call a \( k \)-selection \( \sigma \) is *symmetric* iff \( \sigma = \bar{\sigma} \).
3. Assume \( k \geq 2 \). For two \( k \)-selections \( \sigma, \tau \), we define the \( \wedge \)-intersection number \( [A_{\tau}] \cdot [V_{\sigma}] \) as the number of intersections where just selected curves in \( \sigma \) are collapsed to nodes.

Let \([\quad] \cdot [\quad] \) be an intersection pairing in \( \overline{\mathcal{M}}_{g} \), then it is easy to show the following lemma.

**Lemma 4.3.**

1. \([A_{\tau}] \cdot [V_{\sigma}] = [A_{\tau}] \cdot [V_{\overline{\sigma}}]\)
2. If \( \sigma \) is not symmetric then

\[
[A_{\tau}] \cdot [V_{\sigma}] = [A_{\tau}] \cdot [V_{\sigma}] = [A_{\tau}] \cdot [V_{\sigma}] + [A_{\tau}] \cdot [V_{\overline{\sigma}}]
\]

3. If \( \sigma \) is symmetric then

\[
[A_{\tau}] \cdot [V_{\sigma}] = [A_{\tau}] \cdot [V_{\sigma}] = [A_{\tau}] \cdot [V_{\sigma}]
\]

If and only if \( \tau \) is equal to either \( \sigma \) or \( \bar{\sigma} \), then \( \sigma \) and \( \tau \) are the same \( k \)-selection modulo homeomorphism action, and then \( \alpha_{g,k} \) is the number of
conjugate classes of k-selections. However, for the present we distinguish them and consider a matrix of $\sim$-intersection pairing.

**Lemma 4.4.**

If the $\sim$-intersection pairing matrix is not degenerate then the intersection pairing matrix is not degenerate.

**Proof** Let $\sigma_1, \ldots, \sigma_m$ be all symmetric k-selections and $\tau_1, \ldots, \tau_n$, $\bar{\tau}_1, \ldots, \bar{\tau}_n$ be all non-symmetric k-selections. From lemma 4.3 (3), the $\sim$-intersection pairing matrix is

$$
\begin{pmatrix}
[A_{\sigma_1}] \cdot [V_{\sigma_1}] & [\overline{A_{\sigma_1}}] \cdot [V_{\tau_1}] & [A_{\sigma_1}] \cdot [V_{\tau_1}] \\
[A_{\tau_1}] \cdot [V_{\sigma_1}] & [A_{\tau_1}] \cdot [V_{\tau_1}] & [\overline{A_{\tau_1}}] \cdot [\overline{V_{\tau_1}}] \\
[A_{\tau_1}] \cdot [V_{\bar{\tau}_1}] & [A_{\tau_1}] \cdot [\overline{V_{\bar{\tau}_1}}] & [\overline{A_{\tau_1}}] \cdot [\overline{V_{\bar{\tau}_1}}]
\end{pmatrix}.
$$

By elementary transformations with respect to the column and lemma 4.3, we can transform it to

$$
\begin{pmatrix}
[A_{\sigma_1}] \cdot [V_{\sigma_1}] & [\overline{A_{\sigma_1}}] \cdot [V_{\tau_1}] & [A_{\sigma_1}] \cdot [V_{\tau_1}] \\
[A_{\tau_1}] \cdot [V_{\sigma_1}] & [A_{\tau_1}] \cdot [V_{\tau_1}] & [\overline{A_{\tau_1}}] \cdot [\overline{V_{\tau_1}}] \\
[A_{\tau_1}] \cdot [V_{\bar{\tau}_1}] & [A_{\tau_1}] \cdot [\overline{V_{\bar{\tau}_1}}] & [\overline{A_{\tau_1}}] \cdot [\overline{V_{\bar{\tau}_1}}]
\end{pmatrix}.
$$

By elementary transformations with respect to the row, we can transform this to

$$
\begin{pmatrix}
[A_{\sigma_1}] \cdot [V_{\sigma_1}] & [\overline{A_{\sigma_1}}] \cdot [V_{\tau_1}] & * \\
[A_{\tau_1}] \cdot [V_{\sigma_1}] & [A_{\tau_1}] \cdot [V_{\tau_1}] & * \\
0 & 0 & *
\end{pmatrix}.
$$
Note that the intersection pairing matrix is

\[
\begin{pmatrix}
[A_{\sigma_1} \cdot [V_{\sigma_j}] & [A_{\tau_1} \cdot [V_{\tau_j}] \\
[A_{\tau_1} \cdot [V_{\sigma_j}] & [A_{\sigma_1} \cdot [V_{\tau_j}]]
\end{pmatrix}
\]

and the lemma follows.

From Lemma 4.4, it is sufficient to show that the \^\(^\wedge\)-intersection pairing matrix is non-degenerate. We prove this by induction. First, we compute the intersection pairing matrix for the two 2-selections \(D = \{[d_j], [d'_j]\}\) and \(C = \{[c_{j-1}], [c_j]\}\).

**Lemma 4.5.**

\[
\begin{pmatrix}
\frac{1}{m}V_D & \frac{1}{n}V_C \\
\frac{i}{\ell A_{\sigma}} & 1 & \frac{1}{12} \\
A_C & 2 & 1
\end{pmatrix}
\]

where

\[
i = [PSL(2, \mathbb{Z}) : \Gamma_{\ell}]
\]

\[
m = \begin{cases} 
2 & \text{if } g = 3 \\
1 & \text{if } g \neq 3
\end{cases}
\]

\[
n = \begin{cases} 
4 & \text{if } g = 3 \\
2 & \text{if } C \text{ contains exactly one of } [c_1], [c_{g-1}] \\
1 & \text{otherwise}
\end{cases}
\]
Proof

Intersection points between the perturbation $\mathcal{A}_D^\#$ of $\mathcal{A}_D$ and $\mathcal{V}_D$ correspond to nodes of singular fibres of $\mathcal{U}_\ell$. Let $(z_1, z_2)$ be coordinates of $\mathcal{U}_\ell$ around a node and let $(\tau, \tau', t) \in \mathcal{O}^{3g-5}$ be local manifold cover coordinates of $\overline{\mathcal{M}}_g$ around the intersection point $p$. Then from § 2 we can represent $[\mathcal{A}_D^\#]$ as follows.

$$ [\mathcal{A}_D^\#]: \mathcal{U}_\ell \to \overline{\mathcal{M}}_g $$

$$(z_1, z_2) \mapsto [(\tau = z_1^\ell, \tau' = z_2^\ell, t = f(z_1, z_2))]$$

for some smooth function. When $g \geq 4$, $(\tau, \tau', t)$ are coordinates of $\overline{\mathcal{M}}_g$ and $\mathcal{V}_D$ is given locally as the locus $\{\tau = \tau' = 0\}$. Since the intersection number at $p$ is $\ell^2$ and $\mathcal{U}_\ell$ has $i/\ell$ singular fibres and one singular fibre has $\ell$ nodes, the intersection number of $\mathcal{A}_D$ and $\mathcal{V}_D$ is given by $\ell^2 \times i/\ell \times \ell = i\ell^2$. In case $g = 3$, $(\sigma_1, \sigma_2, t) := (\tau + \tau', (\tau - \tau')^2, t)$ give coordinates of $\overline{\mathcal{M}}_g$ around $p$ and only in this case the intersection number is $2i\ell^2$.

Next we shall calculate the intersection of $\mathcal{A}_D$ and $\mathcal{V}_D$. The Poincaré duals of the Euler classes of $\tilde{s}_1$, $\tilde{s}_2$ are both

$$ -\frac{i}{12} [\text{fibre}] - \sum_j [s_j(\mathcal{H}/\Gamma_\ell)] $$

and

$$ [s_j(\mathcal{H}/\Gamma_\ell)] \cdot [s_k(\mathcal{H}/\Gamma_\ell)] = \begin{cases} -\frac{i}{12} & (j = k) \\ 0 & (j \neq k) \end{cases} $$

Any intersection points of $\mathcal{A}_D^\#$ and $\mathcal{V}_C$ correspond to intersections of zero points of $\tilde{s}_1'$ and those of $\tilde{s}_2'$ in $\mathcal{U}_\ell$. Let $(\tau_1, \tau_2, t)$ be local coordinates of
a local manifold cover over an intersection point $p$ in $\overline{\mathcal{M}}_g$ and $(z_1, z_2)$ be local coordinates of corresponding point in $\mathcal{U}_\ell$. Then we have

$$\mathcal{A}_D^\ell : \mathcal{U}_\ell \rightarrow \overline{\mathcal{M}}_g$$

$$(z_1, z_2) \mapsto [(\tau_1 = \tilde{s}_1'(z_1, z_2), \tau_2 = \tilde{s}_2'(z_1, z_2), t = f(z_1, z_2))]$$

for some smooth function $f(z_1, z_2)$.

If $C$ does not contain $c_1$ nor $c_{g-1}$, the intersection number of $A_D$ and $V_C$ is equal to that of $-\frac{i}{12}[\text{fibre}] - \sum[s_j(H\overline{\Gamma}_\ell)]$ and itself in $\mathcal{U}_\ell$ since $(\tau_1, \tau_2, t)$ give coordinates of $\overline{\mathcal{M}}_g$.

$$\left( -\frac{i}{12}[\text{fibre}] - \sum_j [s_j(H\overline{\Gamma}_\ell)] \right)^2$$

$$= \frac{i}{6} \sum_j [\text{fibre}] \cdot [s_j(H\overline{\Gamma}_\ell)] + \sum_j [s_j(H\overline{\Gamma}_\ell)]^2$$

$$= \frac{i}{6} \ell^2 - \frac{i}{12} \ell^2 = \frac{i}{12} \ell^2.$$ 

In the same way, we obtain that in the case when exactly one of $c_1$ and $c_{g-1}$ is contained in $C$, $[A_D] \cdot [V_C] = \frac{it^2}{6}$, and that in the case when both $c_1$ and $c_{g-1}$ are contained in $C$, $[A_D] \cdot [V_C] = \frac{it^2}{3}$. We can calculate the intersection number of $A_C$ and $V_C$ using $c_1(s_j^*T') = -1$, that is

$$[A_C] \cdot [V_D] = \begin{cases} 2 & (g \geq 4) \\ 4 & (g = 3) \end{cases}$$

For the intersection number of $A_C$ and $V_C$, note that their intersection
points correspond to \((\infty, \infty)\) in \((\mathcal{H}/\Gamma_2)^2\), and we have

\[
[A_C] \cdot [V_C] = \begin{cases} 
1 & \text{(if no } c_1, c_{g-1} \text{ is contained in } C) \\
2 & \text{(if exactly one is contained in } C) \\
4 & \text{(if both are contained in } C) 
\end{cases}
\]

This completes the proof of Lemma 4.5. \(\square\)

The \(\wedge\)-intersection number for 1-selections are not well-defined generally (for instance, for \([V_{d_1}]\) and \([V_{d_2}]\)). We define them by induction as follows.

**Definition 4.6.**

\[
[A_\sigma \wedge V_{d_1}] := \begin{cases} 
[A_\sigma] \cdot [V_{d_1}] & (\sigma \neq d_g) \\
0 & (\sigma = d_g) 
\end{cases}
\]

\[
[A_\sigma \wedge V_{d_g}] := \begin{cases} 
0 & (\sigma \neq d_g) \\
[A_\sigma] \cdot [V_{d_g}] & (\sigma = d_g) 
\end{cases}
\]

We define \(A_{k,d_j}\) as the \(\wedge\)-intersection matrix of all \(k\)-selections from \([d_1], [c_1], [d_2], [d'_2], [c_2], \ldots, [d_j], [d'_j]\) and \(A_{k,c_j}\) as that of all \(k\)-selections from \([d_1], [c_1], [d_2], [d'_2], \ldots, [c_j]\). We shall prove inductively that all \(A_{k,d_j}\) and \(A_{k,c_j}\) are non-degenerate.

**Lemma 4.7.**

\(A_{1,d_j}, A_{1,c_j}\) are all non-degenerate.

This is immediately from the consequences of §5 in [W].
Lemma 4.8.

$A_{2,d_j}, A_{2,c_j}$ are all non-degenerate.

Proof If $\sigma = \{[d_1],[c_2]\}$ then the $\widehat{\cdot}$-intersection number of $\mathcal{A}_\sigma$ and $\mathcal{V}_\sigma$ is

$$2\left(\sum_j s_j(\overline{H/\Gamma})\right) \cdot [\ell(singular fibres)] = 2i\ell^2$$

and we get

$$A_{2,c_1} = (2i\ell^2)$$

and it is non-degenerate.

Considering $\sigma = \{[d_1],[c_1]\}$ and $\tau = \{[d_2],[d_d']\}$ we obtain

$$A_{2,d_2} = \left\{ \begin{array}{ll} 
(2i\ell^2 & 0) \\
\frac{i\ell^2}{12} & i\ell^2 
\end{array} \right. \quad (g \geq 4)$$

$$\left\{ \begin{array}{ll} 
(2i\ell^2 & \frac{i\ell^2}{6}) \\
0 & 2i\ell^2 
\end{array} \right. \quad (g = 3)$$

Moreover considering $\eta_1 = \{[c_1],[c_2]\}, \eta_2 = \{[d_1],[c_2]\}$ and applying
Lemma 4.5 we have

$$A_{2,c_2} = \begin{pmatrix} A_{2,d_2} & * \\ * & A_{1,c_1} \end{pmatrix}$$

$$= \begin{cases} \begin{pmatrix} 2i\ell^2 & \frac{it^2}{12} & 0 & 0 \\ 0 & \ell^2 & \frac{it^2}{6} & 0 \\ 0 & 2 & 2 & -2 \\ 0 & 0 & -\frac{i}{6} & i \end{pmatrix} & (g \geq 4) \\ \begin{pmatrix} 2i\ell^2 & \frac{it^2}{6} & 0 & 0 \\ 0 & 2i\ell^2 & \frac{it^2}{3} & 0 \\ 0 & 4 & 4 & -4 \\ 0 & 0 & -\frac{i}{3} & 2i \end{pmatrix} & (g = 3) \end{cases}$$

It is easy to check that $A_{2,d_2}$ and $A_{2,c_2}$ are all non-degenerate.

We consider in the case $g \geq 4$. Assume that up to $A_{2,c_j}$ the statement holds ($2 \leq j \leq g - 2$). $A_{2,d_{j+1}}$ is given by

$$A_{2,d_{j+1}} = \begin{pmatrix} A_{2,c_j} & 0 \\ 0 & i\ell^2 \end{pmatrix}$$

and this is also non-degenerate.

Next assume that up to $A_{2,d_j}$ the statement holds ($3 \leq j \leq g - 1$).
Using lemma 4.5, we have

\[
A_{2,c_j} = \left( \begin{array}{c|c}
A_{2,d_j} & * \\
\hline
* & A_{1,c_{j-1}} \\
\end{array} \right)
\]

\[
= \left\{ \begin{array}{c|c|c|c|c|c|c|c|c}
A_{2,c_j} & 0 & 0 & 0 & 0 & \ldots \sigma & \ldots \tau \\
0 & i\ell^2 & \frac{i\ell^2}{12} & 0 & \ldots \sigma & \ldots \tau \\
0 & 2 & 1 & * & \ldots \sigma & \ldots \tau \\
0 & 0 & 0 & A_{1,d_{j-1}} & \ldots \sigma & \ldots \tau \\
\end{array} \right)
\]

\[
= \left\{ \begin{array}{c|c|c|c|c|c|c|c|c}
A_{2,c_j} & 0 & 0 & 0 & 0 & \ldots \sigma & \ldots \tau \\
0 & i\ell^2 & \frac{i\ell^2}{6} & 0 & \ldots \sigma & \ldots \tau \\
0 & 2 & 2 & * & \ldots \sigma & \ldots \tau \\
0 & 0 & 0 & 2A_{1,d_{j-1}} & \ldots \sigma & \ldots \tau \\
\end{array} \right)
\]

where \( \sigma = \{[d_j], [d_j']\} \), \( \tau = \{[c_{j-1}], [c_j]\} \). This implies that \( A_{2,c_j} \) is non-degenerate.

Finally we prove that \( A_{2,d_g} \) is non-degenerate. We have two steps for that. Let \( \eta = \{[c_{g-1}], [d_g]\} \) and let \( A_{2,d_g'} \) be a \( \wedge \)-intersection matrix of 2-selections except \( \eta \). We define an equivalent relation \( \sim \) to be generated by all elementary transformations.

\[
A_{2,d_g'} = \left( \begin{array}{c|c|c|c|c|c|c|c}
A_{2,c_{g-1}} & 0 & 0 & \ldots \sigma & \ldots \tau \\
0 & i\ell^2 & 0 & \ldots \sigma & \ldots \tau \\
0 & 2 & 2A_{1,c_{g-2}} & \ldots \sigma & \ldots \tau \\
0 & 0 & -2A_{1,c_{g-2}} & \ldots \sigma & \ldots \tau \\
\end{array} \right)
\]
\[ \begin{pmatrix} A_{2,c_{g-2}} & 0 & 0 & 0 & * & * \\ 0 & i\ell^2 & \frac{i\ell^2}{6} & 0 & -i\ell^2 & * \\ 0 & 2 & 2A_{1,c_{g-2}} & -2A_{1,c_{g-2}} & \ldots [d_{g-1}] \\ 0 & 0 & -\frac{i}{6}A_{1,c_{g-2}} & iA_{1,c_{g-2}} & \ldots [c_{g-2}, c_{g-1}] \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \]

\[ \begin{pmatrix} A_{2,c_{g-2}} & 0 & 0 & 0 & * & * \\ 0 & i\ell^2 & 0 & * & -i\ell^2 & * \\ 0 & 2 & 5A_{1,c_{g-2}} & -2A_{1,c_{g-2}} & \ldots \\ 0 & 0 & 0 & 0 & 0 & iA_{1,c_{g-2}} \end{pmatrix} \]

\[ \begin{pmatrix} A_{2,c_{g-2}} & 0 & 0 & 0 & * & * \\ 0 & i\ell^2 & 0 & * & \ldots \\ 0 & 2 & 5A_{1,d_{g-2}} & -2A_{1,c_{g-2}} & \ldots \\ 0 & 0 & 0 & 0 & iA_{1,c_{g-2}} \end{pmatrix} \]

\[ \begin{pmatrix} A_{2,c_{g-2}} & * & 0 & 0 & * \\ 0 & i\ell^2 & * & * \end{pmatrix} \]

This matrix is non-degenerate and hence $A_{2,d'_{g}}$ is also non-degenerate.

$A_{2,d_{g}}$ is given by

\[ A_{2,d_{g}} = \begin{pmatrix} A_{2,d'_{g}} & 0 \\ 0 & 2i\ell^2 \end{pmatrix} \]

and $A_{2,d_{g}}$ is also non-degenerate.

In case $g = 3$, we can prove that all $A_{2,d_{j}}$, $A_{2,c_{j}}$ are non-degenerate in the same way. This completes the proof of Lemma 4.8. \qed
Lemma 4.9.

$A_{k,c_j}$ and $A_{k,d_j}$ are all non-degenerate for all $k, j$.

Proof

We already prove in the cases $k = 1, 2$. Assume that the statement holds up to the case $k = p - 1$.

If $\{[d_1], [c_1], [d_2], [d'_2], \ldots, [c_{j-1}]\}$ does not have any $p$-selections but $\{[d_1], [c_1], [d_2], [d'_2], \ldots, [c_{j-1}], [d_j], [d'_j]\}$ has $p$-selections then

$$A_{p,d_j} = i\ell^2 A_{p-2,d_{j-1}}$$

and hence $A_{p,d_j}$ is non-degenerate.

If $\{[d_1], [c_1], [d_2], [d'_2], \ldots, [d_j], [d'_j]\}$ does not have any $p$-selections but $\{[d_1], [c_1], [d_2], [d'_2], \ldots, [d_j], [d'_j], [c_j]\}$ has $p$-selections then

$$A_{p,c_j} = A_{p-1,c_{j-1}}$$

and hence $A_{p,c_j}$ is non-degenerate.

When $A_{p,c_{j-1}}$ is non-degenerate $A_{p,d_j}$ is given by

$$A_{p,d_j} = \begin{pmatrix} A_{p,c_{j-1}} & 0 \\ 0 & i\ell^2 A_{p-2,d_{j-1}} \end{pmatrix}$$

and hence $A_{p,d_j}$ is non-degenerate.

When $A_{p,d_j}$ is non-degenerate, $A_{p,c_j}$ is given as follows. If there is no $p$-selection containing both $[c_{j-1}],[c_j]$ then

$$A_{p,c_j} = \begin{pmatrix} A_{p,d_j} & 0 \\ 0 & A_{p-1,c_j} \end{pmatrix}$$
and is non-degenerate. Otherwise

\[ A_{p,c_{j}} = \begin{pmatrix} A_{p,d_{j}} & * \\ * & A_{p-1,c_{j-1}} \end{pmatrix} \]

and using lemma 4.5 this matrix is conjugate to

\[
\begin{pmatrix}
B & 0 & 0 & 0 \\
0 & i \ell^2 A_{p-2,c_{j-2}} & \frac{i \ell^2}{12} A_{p-2,c_{j-2}} & 0 \\
0 & 2 A_{p-2,c_{j-2}} & A_{p-2,c_{j-2}} & C \\
0 & 0 & D & A_{p-1,d_{j-1}}
\end{pmatrix}
\]

where

\[
\begin{pmatrix}
B & 0 \\
0 & i \ell^2 A_{p-2,c_{j-2}}
\end{pmatrix} = A_{p,d_{j}}
\]

and, from the assumption, \( B \) is nondegenerate. By elementary transformations we can transform \( A_{p,c_{j}} \)

\[
\begin{pmatrix}
B & 0 & 0 & 0 \\
0 & A_{p-2,c_{j-2}} & 0 & 0 \\
0 & 0 & \frac{5}{6} A_{p-2,c_{j-2}} & C \\
0 & 0 & D & A_{p-1,d_{j-1}}
\end{pmatrix}
\]

When there does not exist any \( p \)-selections containing all of \([c_{j-2}], [c_{j-1}], [c_{j}]\), we have \( C = 0, D = 0 \) and hence \( A_{p,c_{j}} \) is non-degenerate. Other-
wise,

\[
\begin{pmatrix}
\frac{5}{6} A_{p-2,c_{j-2}} & C \\
D & A_{p-1,d_{j-1}}
\end{pmatrix}
\]

\[
= \begin{pmatrix}
\frac{5}{6} A_{p-2,d_{j-2}} & \frac{5}{6} E & 0 & 0 \\
\frac{5}{6} F & \frac{5}{6} A_{p-3,c_{j-3}} & 2 A_{p-3,c_{j-3}} & 0 \\
0 & \frac{i\ell^2}{12} A_{p-3,c_{j-3}} & i\ell^2 A_{p-3,c_{j-3}} & 0 \\
0 & 0 & 0 & G
\end{pmatrix}
\]

for some $E, F, G$. We transform this matrix by elementary transformations and have

\[
\begin{pmatrix}
\frac{5}{6} A_{p-2,d_{j-2}} & \frac{5}{6} E & 0 & 0 \\
\frac{5}{6} F & \frac{4}{6} A_{p-3,c_{j-3}} & 0 & 0 \\
0 & 0 & 0 & A_{p-3,c_{j-3}} \\
0 & 0 & 0 & G
\end{pmatrix}
\]

Since $A_{p-1,d_{j-1}} = \begin{pmatrix} i\ell^2 A_{p-3,c_{j-3}} & 0 \\ 0 & G \end{pmatrix}$, is non-degenerate. Then it is sufficient to show that \(\begin{pmatrix} A_{p-2,d_{j-2}} & E \\ F & \frac{4}{5} A_{p-3,c_{j-3}} \end{pmatrix}\) is non-degenerate.

When there exists no $p$-selection containing all of $[c_{j-3}], [c_{j-2}], [c_{j-1}], [c_j]$, we have $E = 0$ and $F = 0$ and hence $A_{p,c_j}$ is non-degenerate.
Otherwise (by a certain numbering)

\[
\begin{pmatrix}
    A_{p-2,d_{j-2}} & E \\
    F & \frac{4}{5} A_{p-3,c_{j-3}}
\end{pmatrix}
\]

\[
= \begin{pmatrix}
    H & 0 & 0 & 0 \\
    0 & i t^2 A_{p-4,c_{j-4}} & \frac{i t^2}{12} A_{p-4,c_{j-4}} & 0 \\
    0 & 2 A_{p-4,c_{j-4}} & \frac{4}{5} A_{p-4,c_{j-4}} & \frac{4}{5} I \\
    0 & 0 & \frac{4}{5} J & \frac{4}{5} A_{p-3,d_{j-3}}
\end{pmatrix}
\sim \begin{pmatrix}
    H & 0 & 0 & 0 \\
    0 & A_{p-4,c_{j-4}} & 0 & 0 \\
    0 & 0 & \frac{19}{24} A_{p-4,c_{j-4}} & I \\
    0 & 0 & J & A_{p-3,d_{j-3}}
\end{pmatrix}
\]

for some $H, I, J$. When there exists no $p$-selection containing all of $[c_{j-4}]$, $[c_{j-3}]$, $[c_{j-2}]$, $[c_{j-1}]$, $[c_{j}]$, then we have $I = 0$, $J = 0$ and hence $A_{p,c_{j}}$ is non-degenerate.

Repeating this step it is sufficient to prove that a sequence $\{a_n\}_{n=1,2,\ldots}$ such that

\[ a_1 = 1, \quad a_{n+1} = 1 - \frac{1}{6a_n} \]

does not contain zero. It is easy to check $a_n \neq 0$ for any $n$ and hence $A_{p,c_{j}}$ is non-degenerate for $j \leq g - 1$.

To show $A_{p,d_{g}}$ is non-degenerate we have two steps. Let $A_{p,d'_{g}}$ be a $\wedge$-intersection matrix of $p$-selections containing not both of $[c_{g-1}]$ and $[d_{g}]$.

\[
A_{p,d'_{g}} = \begin{pmatrix}
    A_{p,c_{g-1}} & * \\
    * & i A_{p-1,d_{g-1}}
\end{pmatrix}
\]
From Lemma 4.4 and Lemma 4.9 we get the following theorem.

**Theorem A.**

When $k \geq 2$, $b_{2k}(\overline{\mathcal{M}}_g) = b_{6g-6-2k}(\overline{\mathcal{M}}_g) \geq \max(\alpha_{g,k}, \alpha_{3g-3-k})$

**Remark** When $k = 1$, Harer’s result shows the following equality.

$$b_2(\overline{\mathcal{M}}_g) = b_{6g-8}(\overline{\mathcal{M}}_g) = 2 + \left\lfloor \frac{g}{2} \right\rfloor (= \alpha_{g,1} + 1)$$
§5. Number of distinct k-selections.

In this section we introduce certain algorithm to calculate of $\alpha_{g,k}$, the number of the $2k$-cycles we construct. $\alpha_{g,k}$ is a number of distinct $k$-selections i.e. distinct $k$ homotopy classes of the free homotopy classes $[c_1], \ldots, [c_{g-1}], [d_1], \ldots, [d_{g-1}], [d_g]$ in Figure 2 satisfying 1), 2) in Definition 3.1 modulo the action of homeomorphisms. In other words $\alpha_{g,k}$ is the number of conjugate classes of $k$-selections. The final goal in this section is proposition B.

Let matrices $A$, $B$, $C$ in $M_2(Z[t])$ as follows

$$A = \begin{pmatrix} 1 & 1 \\ t & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ t^2 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 1 \\ t^4 & 0 \end{pmatrix}. $$

Let polynomials $a_g(t)$, $b_g(t)$, $c_g(t)$, $d_g(t)$, $f_g(t)$ be as follows.

$$\begin{pmatrix} a_g(t) & b_g(t) \\ c_g(t) & d_g(t) \end{pmatrix} := \begin{cases} \frac{1}{2} \left\{ A(AB)^{g-2}A^2 + t^2(AB)^{g-3}A^2 + t^2 A(AB)^{g-3}A + t^4(AB)^{g-4}A + B(BC)^{A}2^{-1}A + t^2(A(AB)^{g-3}A) \right\} & (g: \text{even}) \\
\frac{1}{2} \left\{ A(AB)^{g-2}A^2 + t^2(AB)^{g-3}A^2 + t^2 A(AB)^{g-3}A + t^4(BC)^{B^2} + t^4(BC)^{B^2} \right\} & (g: \text{odd}) \end{cases} $$

$$f_g(t) := a_g(t) + c_g(t)$$

where for a matrix $X$ and a negative integer $q$ we denote $X^{-1}$ by the inverse matrix of $X$ in $M_2(Z[t])$ and define $X^q = (X^{-1})^{-q}$. It is easy to see that $\begin{pmatrix} a_g(t) & b_g(t) \\ c_g(t) & d_g(t) \end{pmatrix}$ belongs to $M_2(Z[t])$. 
Proposition B.

\[ f_{g}(t) = \sum_{k} \alpha_{g,k} t^{k} \]

Now for a matrix \( X = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix} \in M_{2}(\mathbb{Z}[t]) \) we denote \( a(t) + c(t) \) by \( f(X) \).

Lemma 5.1.

Forgetting homeomorphism action on \( S \), let \( \alpha'_{g,k} \) be a number of \( k \)-selections containing not both of \([d_{1}]\) and \([c_{1}]\) and containing not both of \([d_{g}]\) and \([c_{g-1}]\). Then

\[ f(A(AB)^{g-2}A^{2}) = \sum_{k} \alpha'_{g,k} t^{k} \]

Proof

For each \( j \), we can prove the next claim (\( \# \)) by induction.

CLAIM (\( \# \)).

(1) Let \( p_{j,k} \) and \( p'_{j,k} \) be defined by

\[
(1,1)A(AB)^{j-1} = \left( \sum_{k} p_{j,k} t^{k}, \sum_{k} p'_{j,k} t^{k} \right) \quad (1 \leq j \leq g - 1)
\]

then

\[
p_{j,k} = \#\{ \sigma : k - \text{selection}\mid \{[d_{1}], [c_{1}]\} \not\subset \sigma \subseteq \{[d_{1}], [c_{1}], \ldots, [d_{j}], [d'_{j}]\}\}
\]

\[
p'_{j,k} = \#\{ \sigma : k - \text{selection}\mid \{[d_{1}], [c_{1}]\} \not\subset \sigma \subseteq \{[d_{1}], [c_{1}], \ldots, [c_{j-1}]\}\}
\]
(2) Let \( q_{j,k} \) and \( q'_{j,k} \) be defined by

\[
(1, 1)A(AB)^{j-1}A = \left( \sum_{k} q_{j,k} t^{k}, \sum_{k} q'_{j,k} t^{k} \right) \quad (1 \leq j \leq g-1)
\]

then

\[
q_{j,k} = \#\{ \sigma : \text{k-selection} \mid \{[d_1],[c_1]\} \not\subset \sigma \subseteq \{[d_1],[c_1],\ldots,[c_j]\}\}
\]

\[
q'_{j,k} = \#\{ \sigma : \text{k-selection} \mid \{[d_1],[c_1]\} \not\subseteq \sigma \subseteq \{[d_1],[c_1],\ldots,[d_j],[d'_j]\}\}
\]

Notice that \( \alpha_{g,k}' = q_{g-1,k} + q_{g-1,k-1} \) and

\[
(1, 1)A(AB)^{g-2}A^2 = \left( \sum_{k} q_{g-1,k} t^{k}, \sum_{k} q'_{g-1,k} t^{k} \right) \begin{pmatrix} 1 & 1 \\ t & 0 \end{pmatrix}
\]

\[
= \left( \sum_{k} \alpha'_{g,k} t^{k}, \sum_{k} q_{g-1,k} t^{k} \right)
\]

and we have

\[
f(A(AB)^{g-2}A^2) = \sum_{k} \alpha'_{g,k} t^{k}.
\]

\[\square\]

\textbf{Proof of Proposition B}

In the similar way to the previous lemma, we have that the number of all \( k \)-selections are represented by the coefficients of

\[
f(A(AB)^{g-2}A^2 + t^2(AB)^{g-3}A^2 + t^2A(AB)^{g-3}A + t^4(AB)^{g-4}A)
\]

and we have that the number of all symmetric \( k \)-selections are represented by the coefficients of

\[
\begin{cases}
  f(B(BC)^{g-1}A + t^4(BC)^{\frac{g}{2}-1}A) & (g : \text{even}) \\
  f(B(BC)^{\frac{g+3}{2}}B^2 + t^4(BC)^{\frac{g-5}{2}}B^2) & (g : \text{odd})
\end{cases}
\]
and we complete the proof. \hfill \square

We can easily get the number of k-selections only from \([c_1], \ldots, [c_{g-1}], [d_2], [d'_2], \ldots, [d_{g-1}], [d'_{g-1}]\). In fact we can get the following equality by induction.

\[
f((AB)^{g-2}A) = \sum_k \binom{g-1}{k} + \sum_{k',l} \binom{k'-1}{l} \cdot \binom{k-2k'+1}{l+1} \cdot \binom{g-l-2}{k-k'} t^k.
\]

Then we get the next inequality.

\[
\alpha_{g,k} > \frac{1}{2} \binom{g-1}{k} + \frac{1}{2} \sum_{k',l} \binom{k'-1}{l} \cdot \binom{k-2k'+1}{l+1} \cdot \binom{g-l-2}{k-k'}
\]

Finally some examples of \(f_g\) shall be described.

(1) When \(g = 3\)

\[
f_3(t) = 1 + 2t + 5t^2 + 3t^3 + 2t^4
\]

and we have

\[
b_2 = b_{10} = 3
\]
\[
b_4 = b_8 \geq 5
\]
\[
b_6 \geq 3
\]

(2) When \(g = 4\)

\[
f_4(t) = 1 + 3t + 7t^2 + 9t^3 + 7t^4 + 3t^5 + t^6
\]

and we have

\[
b_2 = b_{16} = 4
\]
\[
b_4 = b_{14} \geq 7
\]
\[
b_6 = b_{12} \geq 9
\]
\[
b_8 = b_{10} \geq 7
\]
(3) When $g = 5$

$$f_5(t) = 1 + 3t + 11t^2 + 16t^3 + 21t^4 + 13t^5 + 8t^6 + 2t^7 + t^8$$

and we have

$$b_2 = b_{22} = 4 \quad b_4 = b_{20} \geq 11$$
$$b_6 = b_{18} \geq 16 \quad b_8 = b_{16} \geq 21$$
$$b_{10} = b_{14} \geq 13 \quad b_{12} \geq 8$$

(4) When $g = 6$

$$f_6(t) = 1 + 4t + 14t^2 + 29t^3 + 43t^4 + 43t^5 + 31t^6 + 16t^7 + 7t^8 + 2t^9 + t^{10}$$

and we have

$$b_2 = b_{28} = 5 \quad b_4 = b_{26} \geq 14$$
$$b_6 = b_{24} \geq 29 \quad b_8 = b_{22} \geq 43$$
$$b_{10} = b_{20} \geq 43 \quad b_{12} = b_{18} \geq 31$$
$$b_{14} = b_{16} \geq 16$$

References


