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On a Topological Approach for Abstract Systems

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1. *Introduction*

In the theory of general systems, input-output systems and goal-seeking systems are the two primary subjects of the research. This report is to present a new approach for the research of the systems of the former type.

As a generalized model of input-output systems, the concept of an abstract time system was proposed in an attempt to study various systems properties in a generalized framework[1]. *Though the enterprise may have been successful in the original design, it can be pointed out that it has been lacking in geometric or, more precisely, topological notion. When one sees that some systems theoretically important concepts such as stability and approximation are essentially topological ones, it is clear that some appropriate topological notions are indispensable for the full development of the systems theory. In this report, as one trial, we will define a metric for abstract time systems and present some results obtained by a topological approach based on the metric.*

2. *Non-Archimedean Systems and State Spaces*[2]

To put an emphasis on the geometrical aspects of abstract time systems, we take an axiomatic way. Firstly, the concept of a non-Archimedean system is defined, and abstract time systems are then considered to be instances of non-Archimedean systems; by doing so, an abstract time system can be thought of as an geometrical object.

Definition 1.
A metric space (X,d) *is said to be non-Archimedean if the distance function* d: $X\times X\to R$ *satisfies, for all* $x,y,z\in R$,

$$d(x,y)\leq max\{d(x,z),d(z,y)\}.$$

Let (X,d) *and* (Y,d) *be non-Archimedean complete metric spaces, then a subset* S *of* $X\times Y$ *is called a non-Archimedean system or simply a system while* X *and* Y *are called base spaces of* S. *For a system* $S\subset X\times Y$ *and* $x\in X$, $S(x)$ *denotes the vertical* *section of S at x, which is* $S(x)=\{y\in Y \mid (x,y)\in S\}$; *and the domain of S is defined by* $D(S)=\{x\in X \mid S(x)\neq\emptyset\}$. *For the sake of convenience, we consider only systems whose domains are X unless otherwise mentioned. If a system* $f\subset X\times Y$ *defines a function of X to Y, f is said to be functional, or a functional system.* □

Notice that the base spaces of a non-Archimedean system is complete metric space.

Definition 2.

A system $S\subset X\times Y$ *is said to be pre-L-continuous if for any* $x,x'\in X$ *and for any* $y\in S(x)$ *there exists some* $y'\in S(x')$ *such that* $d(y,y')\leq d(x,x')$. *It should be noted that this condition is an analogy of Lipschitz's condition.*

For any system $S\subset X\times Y$, *Cs denote the set*

$$Cs=\{f\subset S \mid f \text{ is a pre-L-continuous functional subsystem}\},$$

where a subsystem of S is a system that is contained in S. □

Definition 3.

A system $S\subset X\times Y$ *is said to be L-continuous if* $\cup Cs = S$. □

It should be noted that for a functional system, the concept of pre-L-continuity is equivalent to that of L-continuity.

As will be stated precisely later, the concept of L-continuity is deeply related to that of causality of time systems; hence the class of L-continuous systems is one of the most important classes of systems, theoretically and practically.

Definition 4.

A metric space (X,d) is said to be strongly-complete if for any linearly ordered non-empty set Λ, *and for any sequence* $\{x_\lambda\}_{\lambda\in\Lambda}$ *in X with* Λ *being its indexing set, there exists some* $x\in X$ *such that for all* $\lambda\in\Lambda$

$$(2)\quad d(x,x_\lambda)\leq \sup_{\lambda\geq\mu} d(x_\mu,x_\lambda).$$

A subset K of (X,d) is said to be strongly-complete if K is a strongly-complete space as a subspace of (X,d). Furthermore, a system $S\subset X\times Y$ *is said to be strongly-complete if S(x) is strongly-complete for all* $x\in X$. □

It is easy to see that strongly-completeness implies completeness; furthermore, it can be shown that the converse implication does not generally hold.

Definition 5.

Let $\varepsilon > 0$ *be given. Then, a subset* K *of a metric space* (X,d) *is said to be* ε-*isolated if any two distinct elements* x *and* x' *of* K *satisfy* $d(x,x') \geqq \varepsilon$. *Notice that a singleton set is* ε-*isolated. Furthermore, a system* $S \subset X \times Y$ *is* ε-*isolated if* $S(x)$ *is* ε-*isolated for all* $x \in X$.□

Definition 6.

Let $S \subset X \times Y$ *be a system. If there exist a set* C *and mapping* $\rho : C \times X \to Y$ *such that* $\{(x, \rho(c,x)) \mid c \in C, x \in X\} = S$, *then* ρ *is called a (an initial)state space representation of* S *and* C *a state space. A state space representation* $\rho : C \times X \to Y$ *is called reduced if the following holds only when* $c = c'$:

$$\rho(c,x) = \rho(c',x) \text{ for all } x \in X;$$

and it is called L-continuous if it satisfies:

$$d(\rho(c,x), \rho(c,x')) \leqq d(x,x') \text{ for all } c \in C \text{ and all } x, x' \in X. \square$$

Notice that if a mapping $\rho : Cs \times X \to Y$ *is defined by* $\rho(f,x) = f(x)$, *one can easily show that this mapping is an L-continuous state space representation of a system* S *if* S *is L-continuous.*

Definition 7.

Let $S \subset X \times Y$ *be a system and* (Y,d) *be bounded, then define*

$$Fc(S) = \{f \subset S \mid f \text{ is a continuous mapping } f : X \to Y\},$$

and a distance function $d_* : Fc(S) \times Fc(S) \to R$ *defined by*

$$d_*(f,g) = \sup\{d(f(x), g(x)) \mid x \in X\}.$$

It should be noted that $Cs \subset Fc(S)$. (Cs, d_*) *denotes the metric space as a subspace of the metric space* $(Fc(S), d_*)$.□

In the rest of this section, X *and* Y *denote non-Archimedean complete metric spaces and* S *a non-Archimedean system defined over* X *and* Y.

The following theorem shows a relationship between pe-L-continuity and L-continuity under the condition of strongly-completeness.

Theorem 1. [1]

A system $S \in X \times Y$ is L-continuous if it is pre-L-continuous and strongly-complete. □

Theorem 2.

Let $S \subset X \times Y$ be a system, and X, Y be compact. Then Cs is a compact subset of $Fc(S)$ if $S(x)$ is closed in Y for each $x \in X$. □

Corollary 2.1.

Let $S \subset X \times Y$ be a system, and X, Y be compact. If S is strongly-complete, then Cs is compact. □

It is easy to see that the ε-isolatedness of a system S is inherited to Cs, and hence it is also easy to see that the following proposition holds.

Proposition 3.

Let $S \subset X \times Y$, and X,Y be compact. If S is ε-isolated for some $\varepsilon > 0$, then Cs is finite set. □

The following proposition gives a criterion against which the ε-isolatedness of a system, or moreover of its state spaces can be tested.

Proposition 4.

Let $S \subset X \times Y$, and X,Y be compact. If S is pre-L-continuous and satisfies the condition (1) *given below, then S is ε-isolated for some $\varepsilon > 0$.*

(1) $\exists N > 0$ *such that* $|S(x)| = N$ *for all* $x \in X$. □

The test, of course, is not perfect one: there are many ε-isolated systems that do not meet the criterion; in other words, the criterion is not strict enough to sift out the ε-isolated systems from others, and therefore a better one is yet to be proposed.

The following proposition states that Cs is, in a sense, essentially the greatest state space associated with a L-continuous system S.

Proposition 5.

Let $S \subset X \times Y$ be an L-continuous system. Then, the state space of any reduced L-continuous state space representation of S can be mapped into Cs by an injection. □

Proposition 6.

Let $S \subset X \times Y$, and X,Y be compact. If S is pre-L-continuous and satisfies the condition (1) *given below, then any state space of L-continuous state representation of S is finite.*

(1) $\exists N > 0$ *such that* $|S(x)| = N$ *for all* $x \in X$. □

3. *Time System and Its Applications* [1], [2], [3]

Non-Archimedean system has been introduced as a generalization of abstract time systems with emphasis on their geometric aspect. In this section an abstract time system will be defined, and it is shown that an abstract time system is a model of non-Archimedean systems; by so doing, the meaning of L-continuity and ε-isolatedness will become clear. Finally, as an example of an abstract time system, a time system associated with an automaton will be discussed.

To define an abstract time system, we must first define what time is, or in practice what we expect time to be. The most conspicuous characteristic we have in mind when we think of time is an order it seems inherently to posses; and another notion of continuous flow of time with a constant speed also seems to be prevalent among us. In consideration of these seemingly natural properties we find in time, the following enumerated three conditions are very natural as characterization of time for our present purpose:

(1) *T^* is an ordered additive group; that is, T^* is an additive group with a linear order defined on it and satisfies the following property:*
For all $x,y,z \in T^*$, $x<y$ implies $x+z<y+z$.

(2) *T^* is Archimedean; that is, for any $u,v \in T^*$ with $0<u$ there exists a positive integer n such that $nu>v$.*

(3) *Every non-empty upper bounded subset of T^* has a least-upper-bound in T^*.*

For the sake of convenience, we assume time has its initial point, thus we define a time set T as the non-negative part of T^* which satisfies above enumerated three conditions. Fortunately, it can be proved that any set satisfying these three conditions is isomorphic to R or Z, and therefore we can take the sets $R^*=\{x\in R \mid x\geq 0\}$ or $N^*=\{n\in Z \mid n\geq 0\}$ as time sets. A time set $T = N^*$ is called a discrete time set.

Definition 8.

Let A and B be non-empty sets, which will be referred to as input and output alphabet set respectively, and $X=A^T=\{x \mid x:T\to A$ is a mapping$\}$ and $Y=B^T=\{x \mid x:T\to B$ is a mapping$\}$ will be referred to as an input set and an output set, respectively. A subset $S\subset X\times Y(=A^T\times B^T)$ is said to be an abstract time system, or simply a time system. If the time set T is discrete, then S is said to be discrete time system. In this section, we always assume that T denotes a time set and that $X=A^T$ and $Y=B^T$. □

Definition 9.

Let θ: $T\to R$ be such that $\theta(t)=(1+t)^{-1}$, and let $\nu:X\times X\to T\cup\{\infty\}$ and $d:X\times X\to R$ be as follows:

$$\nu(x,x') = \begin{cases} \inf\{t\in T \mid x(t)\neq x'(t)\}, & \text{if } x \neq x' \\ \infty & \text{otherwise,} \end{cases}$$

and

$$d(x,x') = \begin{cases} \theta(\nu(x,x')) & \text{if } x \neq x', \\ 0 & \text{otherwise.} \end{cases}$$

□

It can be shown that the function $d:X\times X\to R$ is a non-Archimedean distance function and that (X,d) is a complete metric space. A similar distance function on Y, which will be denoted also by d, can be defined, and then (Y,d) is a non-Archimedean and complete metric space. Hence, time systems over X and Y are instances of non-Archimedean systems.

The following proposition describes the intuitive meaning of the metric defined above; that is, the degree of closeness of two points x and y of X

amounts to how long they as functions coincide with each other from time 0.

Proposition 7.

Let $x,y\in X$, *and* $t\in T$. *Then* $x^t = y^t$ *if and only if* $d(x,y)\leq\theta(t)$,
where for any $z\in X$ *and* $t\in T$, z^t *denotes the restriction of* $z:T\to A$ *to* $T^t=\{u\in T\,|\,u<t\}$. □

Definition 10.

A functional system $f\subset X\times Y$ *is said to be causal if for any* x, $x'\in X$ *and for any* $t\in T$, $x^t = x'^{\,t}$ *implies* $f(x)^t = f(x')^t$. □

Proposition 8.

A functional system $f\subset X\times Y$ *is L-continuous if and only if* f *is causal.* □

This indicates that the notion of causality is some aspect of the wider notion of continuity.

It should be noted that for a time system S, Cs is the set of all causal functional subsystems of S. An L-continuous time system with respect to the metric defined in Definition 8 *is called a causal system.*

Proposition 9.

A time system $S\subset X\times Y$ *is* ε*-isolated for some* $\varepsilon>0$ *if and only if the condition* (1) *is satisfied:*

(1) *There is some positive time* τ *such that for any* $x\in X$ *and for any* y *and* y' *with* $(x,y)\in S$ *and* $(x,y')\in S$, $y^\tau = y'^{\,\tau}$ *implies* $y = y'$. □

This proposition shows that ε*-isolatedness of a system is equivalent to the finitely-observability of the system, where a time system S is called finitely observable if there is a time* $\tau>0$ *such that for any input* x *if two outputs* y *and* y' *are indistinguishable before* τ, *then* y *and* y' *are the same output.*

If we hold a practical point of view that a man can not directly perceive infinity, we see the importance of the case where the input and output alphabet sets are finite and the time set is discrete. In the remainder of the report we will discuss only discrete time systems whose base spaces have alphabet sets of *finite cardinality, and will show some applications in finite automata theory.*

What characterize the case in question where the base spaces X and Y have finite sets as their alphabet sets and the time set is discrete are the facts that base spaces are compact with respect to the metric defined in Definition 8, *and that the notion of strongly-completeness is equivalent to that of completeness for the base spaces.*

Proposition 10.

If A *is a finite set and* T *a discrete time set, then the metric space* (X,d), *where* $X = A^T$ *and* d *is the metric defined in Definition* 8, *is compact.* □

Proposition 11.

For the same space (X,d) *as in Proposition* 10, *any subset* K *of* X *is strongly-complete if and only if it is complete, which is, in a complete space, equivalent to that it is closed.* □

As corollaries to Proposition 3, 4, *and some others, the following Theorem* 12 *and Proposition* 13 *are obtained.*

Theorem 12.

If a time system $S \subset X \times Y$ *is finitely observable, then the state space of any L-continuous state space representation of* S *is a finite set.* □

Proposition 13.

The state space of any L-continuous state space representation of a causal time system S *satisfying the condition* (1) *given below is necessarily finite.*

(1) $\exists N > 0$ *such that* $|S(x)| = N$ *for all* $x \in X$. □

Finally, we discuss automaton type time systems as an application of what have been shown so far.

Definition 11.

Given an automaton $\mathcal{A} = \langle A, B, C, \delta, \mu \rangle$, *where* A *and* B *are finite input and output alphabet sets respectively,* C *a state set, and* δ *and* μ *state transition function and output function respectively. Let* $\delta^* : C \times X \to C^T$ *be the extended state-transition* *function defined by*

$$\delta^*(c,x)(0)=c \quad \text{and} \quad \delta^*(c,x)(t+1)=\delta(\delta^*(c,x)(t),x(t)),$$

and $\mu^*:C\times X\to Y$ *the extended output function defined by*

$$\mu^*(c,x)(0)=\mu(c,x(0)) \quad \text{and} \quad \mu^*(c,x)(t+1)=\mu(\delta^*(c,t),x(t)).$$

The behavior of $S_{\mathcal{A}}$ *is defined by* $S_{\mathcal{A}}=\{(x,\mu^*(c,x))\mid (c,x)\in C\times X\}$.

A time system for which there is an automaton $\mathcal{A}$ *such that* $S_{\mathcal{A}}=S$, *is called a automaton type time system, while such an automaton is said to represent* S. *Notice that automaton type time systems are by definition discrete time systems.*

If there is some finite state space automaton representing S, *then* S *is said to be a finite automaton type, and furthermore, if it is the case that any reduced automaton that represents* S *has a finite state space,* S *is said to be an intrinsically finite automaton type.* □

It is known that the class of finite automaton type time system divides into a class whose members are of intrinsically finite automaton type and that whose members are not; that is to say, there exist finite automaton type time systems of both types.

Concerning what have been stated here, the following theorem is known; the theorem is a corollary to Theorem 12.

Theorem 14.

Any finitely observable time system of automaton type is an intrinsically finite automaton type time system. In other words, if the behavior of an automaton with finite alphabet sets is finitely observable, no equivalent automaton that is reduced has infinite state space. □

Another corollary, the one to Proposition 13, *is obvious.*

4. *Conclusion*

We have seen that abstract time systems posses non-Archimedean metric structure and that by directing our attention to the structure much information on the systems can be obtained. These facts seem to suggest the feasibility and possibilities of topological approach for the abstract systems theory.

References

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[2] *T.Nishikawa and Y.Takahara, "TOPOLOGICAL APPROACH FOR ABSTRACT SYSTEMS" (submitted to Int.J. of Systems Science).*

[3] *T.Nishikawa and Y.Takahara, "CHARACTERIZATION OF A FINITE STATE AUTOMATON— TOPOLOGICAL APPROACH" (submitted to Int.J. of General Systems).*