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# TOWARD AN AGGREGATION THEORY OF LARGE-SCALE SYSTEMS

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## 1. SYSTEMS' PROPERTY AND HOMOMORPHISM

In mathematical general systems theory , a model of a system is defined by a homomorphic image of the system [7,14,18], given their mathematical structures. Of course, a concept of "homomorphism" depends on system's specification and mathematical structures of system's attributes. If a model is a homomorphic image of a system, then this model is a valid model for the system. In the discussions of system models, homomorphisms of algebraic structures or relations have been studied relatively well. Here, homomorphism means preservation of algebraic and relational structures and system's behaviors. Under the homomorphic condition, if there exists a homomorphism from the model onto the system, then the model is isomorphic to the system.

According to N.Y.Foo's discussion on homomorphisms [7], we will study a problem of preservation of systems' properties. Systems' properties are not necessarily specified by algebraic operations and relational structures. Systems properties not specified algebraically or relationally must be also studied in the homomorphism concept. These properties can be represented by logical predicates, in general. B.P.Zeigler [18] raises the following problem.

If a system $S_2$ is provably a homomorphic image of $S_1$, under what conditions can we infer that a property of $S_1$ holds in $S_2$, or vice versa? The first question is called a direct inference, and the latter a converse inference for the given property. Moreover, if some conditions are imposed on systems

$S_1$ and $S_2$ in order to make direct or indirect inferences, then those conditions are called justifying conditions.

In some formal language, which includes

1) symbols for variables, functions, relations and quantifiers and

2) logical connectives (and, or, not),

suppose that a property is formalized by writing down a sentence which expresses that property. A systems' property is first order if it is represented by a sentence of the first order. Moreover, logical sentences are first order if functions and relations involved do not contain the quantifiers. And the sentences are positive if they do not contain the not connective.

Now, Lyndon's theorem, which will be shown below, gives a partial answer to the Zeigler's problem.

Lyndon's theorem [7]. "If A is an algebra and P is a first-order property true of A and P has defining sentence $\alpha(P)$. Then, the property P is also true of any homomorphic image of A if and only if $\alpha(P)$ is logically equivalent to a positive sentence."

Lyndon's theorem can be transformed into the following equivalent one [7].

"Properties of the first order can be reflected in the preimages of the homomorphic images if and only if they can be represented by the negative sentences of first order."

Apparently, the limitation of Lyndon's theorem is that it can investigate only the system's properties which can be represented by the positive sentences of the first order. And, Zeigler studies the justification conditions, however, it is somewhat difficult to construct concrete procedures of the justification. Therefore, it is desirable to use already established methods for studying systems' properties of the higher order without using logic or model theory, as pointed out by Foo[7]. Foo and Ziegler's contribution to systems theory is in the point that they show the possibility of general treatment of preserving systems properties under homomorphisms by introducing logical structures.

## 2. MODELS OF LARGE-SCALE SYSTEMS

In large-scale systems theory, there are several system specifications. Here, we restrict our attention to systems of the I-O (input-output) representation. Then, we have the following two representations of large-scale systems.

a) $S \subseteq X \times Y$ (global system) $X = X_1 \times \ldots \times X_n$ (set of inputs)

$Y = Y_1 \times \ldots \times Y_n$ (set of outputs)

$S_i \subseteq X_i \times Z_i \times Y_i$ (subsystem)

$Z = Z_1 \times \ldots \times Z_n$ (set of interconnecting variables)

$K_i \subseteq Y \times Z_i$ (interconnecting subsystems)

where

$(x,y) \in S \Leftrightarrow (\exists z)(\forall i)((x_i, z_i, y_i) \in S_i \ \& \ (y, z_i) \in K_i))$.

b) $S \subseteq X \times Y$

$S_i \subseteq X_i \times Y_i$

where

$(x,y) \in S \Leftrightarrow (\forall i)((x_i, y_i) \in S_i)$.

Now, information required for the system representation $S \subseteq X \times Y$ is called global (or macro) information; information for the representations $S_i \subseteq X_i \times Z_i \times Y_i$, $K_i \subseteq Y \times Z_i$ and $S_i \subseteq X_i \times Y_i$ is called local (or micro) information. And $S_i$ is called a separated (or decoupled) subsystem and $K_i$ called an interconnecting subsystem. Moreover, in the separated subsystem $S_i \subseteq X_i \times Z_i \times Y_i$, we can define an isolated subsystem

$$S_i\hat{} \subseteq X_i \times \{\{\phi\}\} \times Y_i\hat{} \equiv X_i \times Y_i\hat{}, \text{ where } Y_i\hat{} \subseteq Y_i,$$

by eliminating influences from the other separated subsystems. In general, micro-information for the isolated subsystem representation can not be gained from micro-information for the separated subsystem representation.

So as to connect an isolated subsystem and a separated subsystem, we need the following two representations.

$$S_i' \subseteq \{\{\phi\}\} \times Z_i \times Y_i' \text{ ,where } Y_i' \subseteq Y_i,$$

$$S_i'' \subseteq Y_i\hat{} \times Y_i' \times Y_i.$$

In these representation, the following relation must be satisfied.

$$(x_i, z_i, y_i) \in S_i \Leftrightarrow$$
$$(\exists (y_i\hat{}, y_i'))((x_i, y_i\hat{}) \in S_i\hat{} \quad \& \quad (z_i, y_i') \in S_i'$$
$$\& \quad (y_i\hat{}, y_i', y_i) \in S_i'').$$

For example, in a functionally represented subsystem

$S_i = f_i : X_i \times Z_i \rightarrow Y_i$,

the following relations must hold.

$S_i\hat{} = f_i\hat{} : X_i \times \{0\} \rightarrow Y_i$ (influence-free response function)

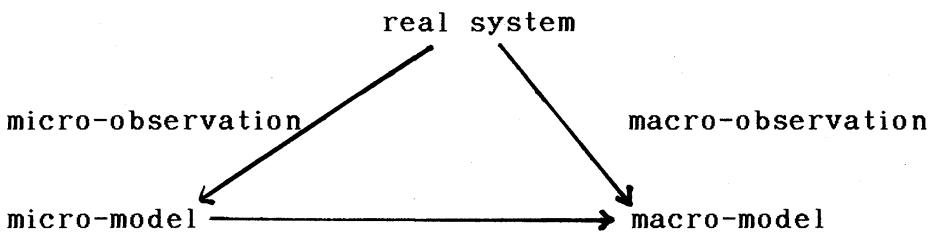
$S_i' = f_i' : \{0\} \times Z_i \rightarrow Y_i$ (input-free response function)

$S_i'' = f_i'' = f_i\hat{} + f_i'$ (synthesis function),

where $(y_i\hat{}, y_i') \rightarrow y_i\hat{} + y_i' = y_i$.

## 3. AGGREGATION MODELS IN SOCIAL SCIENCES

Aggregation theoretical treatments in social systems are observed in economics (market equilibrium theory, input-output analysis, social welfare function and so on) or accounting. However, as pointed out by H.Leibenstein [13], we lack explicit and effective theory of aggregation in social sciences. Also, natural sciences lack an integrated aggregation theory except for physics, where a concept of "average" makes aggregation simple and trivial. Generally, we can show the theoretical structure of aggregation in social sciences as follows.




aggregation model

Here, we must point out that concepts in macro-models are not necessarily those which exist in the real world. Namely, macro-concepts are enlarged or developed form micro-concepts, formed or

observed from interactions among micro-subsystems, or constructed as hypothetical concepts. That is, macro-concepts are secondary, interactive or hypothetical. In this sense, macro-models are extremely conceptual models.

One of the most conceptually simple but mathematically complex examples may be Arrow's construction of a social welfare function (mechanism of social choice). Let M be a choice set (or set of alternatives) to be selected. And suppose that each individual in the society has a preference order of his own, which is represented by

$$\leqq_i \subseteq M \times M.$$

Then , Arrow's problem can be stated as follows.

"Given each individual's preference $\leqq_i$, does there exist a social welfare function

$$f: \Pi (M \times M) \rightarrow M \times M ,$$

where $\Pi$ shows a direct product over a set of individuals in the society, which aggregates individuals values $\leqq_i$ to overall social values $\leqq$ under socially desirable conditions."

That is , a problem of social welfare functions is a problem of aggregation of individual preference relations. In this case, it is assumed that any special relation does not exist among individual preferences (behavioral interactions). This makes Arrow's discussion very simple, except that an aggregation structure must choose a desirable alternative under socially desirable conditions.

In the above-mentioned discussions on Arrow's impossibility theorem, it is very important that there does not exist any special relation among individual preferences. If such a relation does not exist, any separated subsystem can be viewed as an isolated subsystem. In economics, if any economic transaction is done in a completely competitive market, then individual separated subsystem can be considered as an isolated subsystem.

Now, let review social indicators research. It has been considered as an aggregation theory for social phenomena. Social

indicators research has so many difficulties in theory and measurement. We will not discuss such difficulties but point out several problems concerning to aggregation for social measurement. In social indicators research, we do not have both micro-models and macro-models to be measured. Also, we do not have aggregation models. Therefore, aggregation in social indicators research is aggregation of micro-models (individuals or institutions) using separated models and statistical treatment. Naturally, in the research, the validity of the aggregation is not tested or questioned. (Only one model in social indicators research is K.C.Land's model [12]. But his model lacks operationality and reality from aggregation theoretical viewpoints.)

Evaluating social indicators research positively, we reconstruct its aggregation structure. Let $J=\{1,\ldots,m\}$ be a set of social members, and $X_1,\ldots,X_n$ be sets of individual attributes. Here, we assume that each individual has the same sets of social attributes. Then, each individual's attribute space can be represented as follows.

$$S_i \subseteq X_1 \times \ldots \times X_n .$$

Of course, only one point in $S_i$ is observed for each individual in measurement. Moreover, let define a macro-attribute space $S^\wedge$ as follows.

$$S^\wedge \subseteq X_1{}^\wedge \times \ldots \times X_n{}^\wedge ,$$

where $X_i{}^\wedge$ is a set of aggregated attributes. Next, let define an attribute identification function

$$\rho_i : X_i \rightarrow I_i .$$

It is not necessary to exist this function for each i, however, we assume the existence for simplicity. (For example, this function classifies the income level.) At a measurement, let $s_i \in S_i$, then

$$\rho(s_i)=(\rho_1,\ldots,\rho_n)(s_i)$$

classifies individuals' attributes (partition of J). Given a partition $J^\wedge$ of J, there must exist an aggregation structure, shown below, on the social indicators space.

$$\begin{tikzcd}
S \subseteq \Pi S_i \arrow[d, "\rho"] \arrow[ddl, "\Phi"'] \arrow[ddr, dashed] & \subseteq \Pi X \\
\Pi I \arrow[d, "\Phi"] & \\
S^{\wedge} & J^{\wedge} \arrow[r, dashed] & S^{\sim} \arrow[ul, dashed]
\end{tikzcd}$$

where a direct product is applied on J or $J^{\wedge}$. Here, $\Phi$ is an aggregation function. Now, we have at least two aggregation concepts.

1) Overall aggregation ($S^{\wedge}$): aggregation over J.

2) Local aggregation (disaggregation) ($S^{\sim}$): aggregation by local society over the partition $J^{\wedge}$ or, disaggregation from overall aggregation.

These aggregations have several meanings as below.

1) To each subset of the partition $J^{\wedge}$, a local society or local macro model can be constructed. Then, we can infer macro-properties of each local society.

2) On the basis of local but macro models mentioned above, we may investigate relationships among local societies.

3) By detecting the differences among local societies, which are not socially negligible, we can choose a social policy which mitigates the differences.

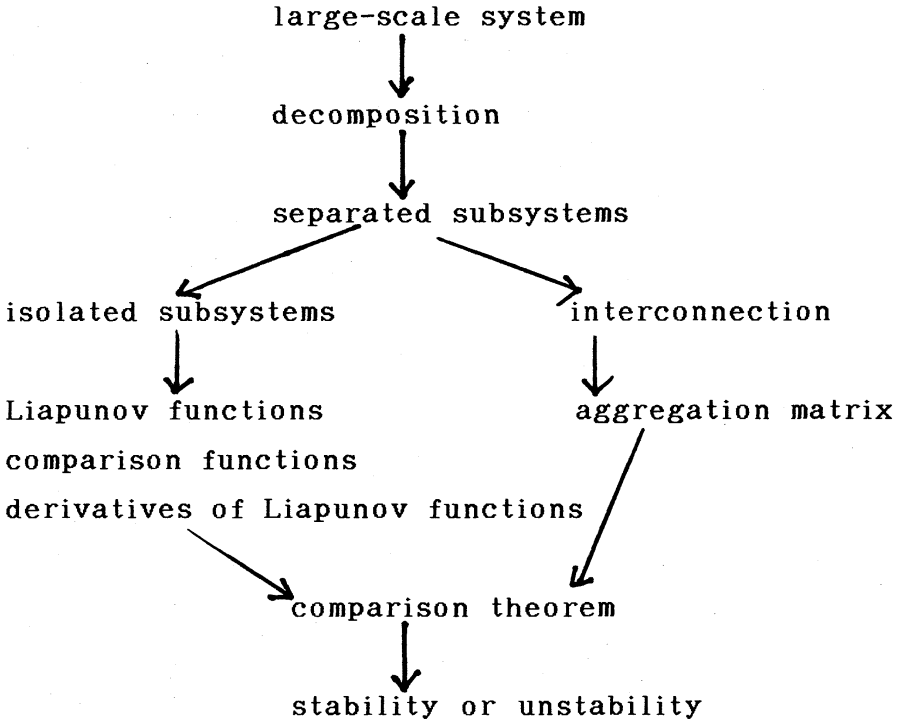
In order to construct an aggregation theory for social sciences, it is necessary to scrutinize micro-models, macro-models and aggregation models. Actually, there exists no integrated theory for social systems as a whole. At least, in social indicators research, we lack all of micro-models, macro-models and aggregation models.

## 4. AGGREGATION MODELS IN LARGE-SCALE SYSTEMS THEORY

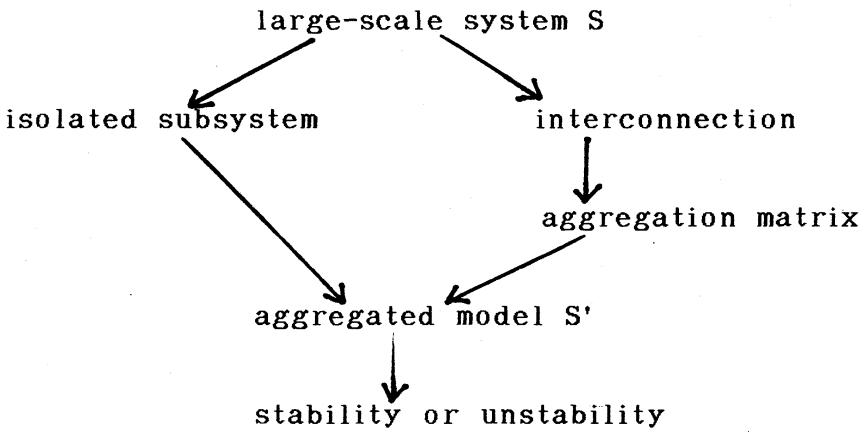
Concepts which do not exist in unit systems but have important meanings in large-scale complex systems, for example, conflict, structural perturbation, competition, coalition, coordination, connective stability, connective reliability and so on, are macro-concepts. Coordination concepts are relatively well studied in hierarchical systems theory, and these concepts are aggregated concepts for behavioral interactions among subsystems. In large-scale systems theory, discussions on aggregation of micro-properties into macro-properties are relatively restricted to stability concepts. According to D.D.Siljak [16], we review his decomposition-aggregation analysis of stability. In his discussion, he uses a state-transition matrix representation, however, we use enlarged concepts mentioned in Section 2.

| | |
|---|---|
| global system | $dx/dt=f(t,x)$ |
| separated subsystems | $dx_i/dt=g_i(t,x_i)+h_i(t,x)$ |
| isolated subsystems | $dx_i/dt=g_i(t,x_i)$, |

where x is a state vector, t is a time variable, f and $g_i$ are state transition matrices, and $h_i$ is a connection function which shows interaction among states variables. Here, he assumes that information on stability is already given for each isolated subsystem, for example, each Liapunov function is given. And it is assumed that a pair of a dominating function and a dominated function for each Liapunov function is known. Moreover, it is assumed that a derivative of each Liapunov function is dominated by some negative function. In such a case, if each connecting function is involved in some class of functions, then the stability of the global system can be studied through an $n \times n$ matrix. His approach to a stability analysis can be visualized as below.




This approach has several important meanings. Firstly, given some information on each isolated subsystem and connecting function, we can judge whether the global system is stable or not if we have enough information to construct an aggregation matrix. Secondly, he does not explicitly state the aggregation process, but this process can be show as below.




In the decomposition-aggregation analysis, $S \subseteq S'$ is satisfied in

general. That is, a set of the aggregated model's behaviors is larger than that of the original system's behaviors. In this sense, in developing several aggregation theories for large-scale systems, aggregated models are constructed in a way that their behaviors are wider than original systems' behaviors. We can call this way of model construction a "conservative law" in large-scale systems research.

Next, El-Attar and Vidyasagar's decomposition-simplification-aggregation analysis of stability [6] may play an important role in analysis and design of large-scale systems. They use a transfer function representation of large-scale systems. However, we use our I-O representation for simplicity. Their problem is stated as follows. "If a large-scale S is unstable, then can we find a feedback-system F which stabilizes a global system?" To solve this problem, they apply an approach as shown in the next figure.

large-scale system
↓
decomposition
↓
isolated subsystems
↓
simplification of isolated subsystems
↓
design of feedback subsystems
(decentralized stabilization)
↓
aggregation
↓
(aggregated model)
↓
overall stability or unstability

In their approach, simplification plays a very important role. (

A simplified model must faithfully realizes all the unstable behaviors of an isolated subsystem.)

In El-Attar and Vidyasagar's approach, simplification must satisfy a conservative law in a sense of faithful realization of unstable behaviors. Assumably, the conservative law is at least a necessary condition for aggregation models of large-scale systems. The conservative law of large-scale systems research implies that systems' properties must be inferred from a safety side of discussions. The larger systems are, the more important the conservative law is in application. However, we may not infer meaningful systems' properties if we extremely apply the conservative law to large-scale systems research. In this sense, we must strike the balance between safety inference and "real" inference.

## 5. STRUCTURE OF AGGREGATION THEORY

An aggregation structure consists of micro-system(s), macro-system and aggregation system (or aggregation function). We represent them as follows.

$S_m \subseteq X_m \times Y_m$

$S_M \subseteq X_M \times Y_M$

$S_A \subseteq X_m \times X_M \times Y_m \times Y_M$,

where $S_m$;micro-system

$S_M$;macro-system

$S_A$;aggregation system

and

$(x_m, x_M, y_m, y_M) \in S_A \ \& \ (x_m, y_m) \in S_m \Rightarrow (x_M, y_M) \in S_M$.

In general, it is assumed that there exists some decomposition $(S_{AX}, S_{AY})$ of the aggregation system $S_A$.

$S_{AX} \subseteq X_m \times X_M$ ;input aggregation system

$S_{AY} \subseteq Y_m \times Y_M$ ;output aggregation system.

This decomposition is based on an assumption that inputs and outputs can be aggregated independently on each other. Here, The following relation must be satisfied.

$$(x_m, x_M, y_m, y_M) \in S_A \Leftrightarrow (x_m, x_M) \in S_{AX} \ \& \ (y_m, y_M) \in S_{AY}.$$

Now, in order to make a meaningful aggregation for the aggregation structure, it is necessary for the following figure to be commutative, in a set theoretical sense.

$$\begin{tikzcd} X_m \arrow[r, "S_m"] \arrow[d, "S_{AX}"'] & Y_m \arrow[d, "S_{AY}"] \\ X_M \arrow[r, "S_M"'] & Y_M \end{tikzcd}$$

That is, the following must hold.

$$(x_m, y_m) \in S_m \ \& \ (x_m, x_M) \in S_{AX} \ \& \ (y_m, y_M) \in S_{AY} \Rightarrow (x_M, y_M) \in S_M$$

Moreover, several relations on the micro-system(s) must be preserved. For example, over micro-systems or between micro-systems, let assume that there exist the following relations.

$$R(\alpha) \subseteq X_m \times \ldots \times X_m \times Y_m \times \ldots \times Y_m$$

$$\alpha \in I$$

where direct products over $X_m$ and $Y_m$ are applied on appropriate index sets $kx(\alpha)$ and $ky(\alpha)$.

Then, $S_A = (S_{AX}, S_{AY})$ is totally consistent with respect to relations $R(\alpha)$ if and only if the following conditions are satisfied [10].

1) $a \in R(\alpha) \Rightarrow S_A(a) \in S_A(R(\alpha))$

2) $\neg(b \in R(\alpha)) \Rightarrow \neg(S_A(b) \in S_A(R(\alpha))$.

The first condition corresponds to the homomorphism condition. The second condition requires that relations which do not exist over micro-systems must not be represented on the macro-system. Apparently, the second condition is equivalent to the next one.

$$S_A(b) \in S_A(R(\alpha)) \Rightarrow b \in R(\alpha).$$

Therefore, total consistency implies isomorphisms of relations between micro-systems and macro-systems. This requirement is too strong in application. Hence, we can define that an aggregation

system is consistent with respect to relations if it satisfies the first conditions.

## 6. ROLES OF AGGREGATION THEORY IN LARGE-SCALE SYSTEMS RESEARCH

Social systems are extremely large-scale systems. In this sense, we may find so many systems models in social sciences for large-scale systems research. In social sciences, aggregation theories , even if really useful aggregation theories do not exist, play very important roles because of following reasons.

1) In general, we can not find social states or changes as a whole only through the micro-observation, because we can not deal with so huge micro-data or we lack appropriate macro-models. In order to analyze social systems as a whole, we need macro-concepts, macro-observations and macro-models which operationalize macro-concepts. Moreover, construction of macro-models is feasible only through aggregating micro-data.

2) If we can explain or predict social changes through macro-models, we will use them to orientate the changes toward socially desirable directions. (Policy scientific perspective) [4].

3) From a view point of fairness of social policies, it is not necessarily desirable to control social members individually based on micro-observations. And individual controls often contradicts democratic principles. Such individual controls are called hard-controls.

4) Hard-controls, even if economically or physically feasible, are very time and cost consuming means, because of their individual properties.

Social controls on the basis of macro-models are generally comprehensive controls. We call such controls soft-controls. Soft-controls aims to control societies or sub-societies as a whole. Soft-controls have an important property that they can effectively use a relatively small number of control variables ( aggregated information). That is, we can comprehensively understand social states and changes, and enhance

the possibility of soft-controls, on the basis of macro-models. However, we must point out that there exists no aggregation theory, even in social sciences, which is theoretically and practically complete.

In large-scale systems research, aggregation theoretical treatment is relatively restricted to hierarchical systems theory, team theory or reliability theory. However, if we study general large-scale systems both in social and natural sciences and in engineering constructs, we must develop a general framework for aggregation. To go to this direction, we must classify present aggregation models and clarify the relationships among them.

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