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Universal Maps for Binary Preference Structure Generating Decision Principles

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Abstract

A concept of binary preference structure generating decision principle is proposed as a device to treat extreme complex decision situations. We argue two types of binary preference structure generating decision principles: The first is for complexity management of dynamic environment by constructiong a model of the environment. The second is the satisfaction decision principle which is for complexity management by increase of internal complexity.

For the both cases we formulate the decision principles as decision principle functors in the category-theoretic terms to reveal new aspects of the decision principles. We also identify universal maps for the functors and investigate decision-theoretic meaning of them.

1 Introduction

Decision making is essential and distinctive behavior of human being and organizations. It becomes crucial especially when decision making involves extreme *complexity* in some sense.

Usually complexity involved with decision making situation is represented by a pay-off matrix shown in Fig.1.

In Fig. 1 each row corresponds to an action available to the decision maker while each column represents an action by the *environment* or the nature. The matrix shows that if the decision maker and the environment take action i and j, respectively, then the pay-off for the decision maker is a_{ij}.

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{pmatrix}$$

Fig.1 Payoff Matrix

This is the most usual interpretation of the pay-off matrix and such a decision situation is often called *decision making under uncertainty.*

However, the matrix has much variety of interpretation. If we consider that i is an action taken by a decision maker and that he evaluates it from various viewpoints, a_{ij} can be interpreted as evaluation of the action with respect to the j-th attribute. In this case the matrix represents *multiattribute decision making situation.* There is (internal) complexity in the decision situation in the sense that the decision maker is not sure which attribute he should put emphasis on.

If we assume i an action available to society and a_{ij} evaluation of i by a member j in the society, the matrix can be interpreted to show *group decision situation.* Since in the society there are many people whose value system are very different, a sort of complexity should exists in it.

Since we consider a pay-off matrix can represent a fairly general situation of complex decision making, this paper uses the idea and denotes a complex decision situation by $S = (M, U, g)$. M is a set of alternatives available to the decision maker and U a set of uncontrollable variables while $g: M \times U \to R$ is a pay-off function. R denotes the set of all reals. Hence $g(i, j)$ stands for a_{ij} in the matrix above. We assume the decision makers are maximizers.

When we tackle a complex decision situation we are forced to introduce a decision principle since a simple optimization is meaningless and impossible. In the following sections we formulate, in a category theoretic framework, a decision principle as a decision principle functor and reveal the principle's characteristics based on the formulation. we also identify universal map for the functor and argue the decision theoretic meaning of it. A universal map constitutes one of the most important materials in category theory since for a given object it identifies essential approximation of the object, *i.e.,* it gives a minimal (or maximal) object preserving properties the given object satisfies with [3]. We believe that investigation on the universal map for the functor provides a new view point from which we can throw light on essential aspects of the decision principle.

2 Decision Principle Functors

Now let us represent a class of decision situations by

$$\mathbf{D} = \{S = (M, U, g) | M \subset \mathbf{M},\ U \subset \mathbf{U},\ g: M \times U \to R\},$$

where $\mathbf{M}$, $\mathbf{U}$ denote super sets of M and U, respectively.

A decision principle is intuitively understood as a way by which the decision maker determines *rational solutions.* However, it is quite crucial which decision principle should be taken for dealing with a particular decision situation *rationally.* For instance, the well known "max-min" decision principle may be applied to a wide class of decision situations, but not to all the situations since it represents pessimistic attitude of the decision maker to the situation. That is why we are interested in clarifying the nature of decision principles in a unified framework [4].

This paper defines a decision principle δ by a mapping which assigns to each decision situation an ordered alternative set; *i.e.*,

$$\delta: \mathbf{D}_\delta \to \wp(\mathbf{M} \times \mathbf{M})$$

$$S = (M, U, g) \mapsto (M, \geq_{\delta(S)}).$$

$\mathbf{D}_\delta \subset \mathbf{D}$ is a set of decision situations which the principle can deal with while $\geq_{\delta(S)} \subset M \times M$ denotes a preference ordering on M induced by δ. We assume $\geq_\delta$ is a partial ordering.

With a decision principle δ we can associate a solution function H_δ

$$H_\delta: \mathbf{D}_\delta \to \wp(\mathbf{M})$$

$$S = (M, U, g) \mapsto H_\delta(S),$$

where we have

$$H_\delta(S) = \{m \in M \mid m \text{ is a maximal element with respect to } \geq_{\delta(S)}\}.$$

$H_\delta(M, U, g)$ is not always a singleton set but possibly a subset of M.

We pointed out that, in general, rational decision principles are, compactly to some surprising extent, characterized by introducing an associated functor [5][6]:

Let $S = (M, U, g)$ and $S' = (M', U', g')$ be in $\mathbf{D}$. $h = (h_1, h_2, h_3)$ is called an affine modelling morphism if

1. h_1, h_2, h_3 are such that $h_1 \colon M \to M'$ and $h_2 \colon U \to U'$ while $h_3 \colon R \to R$ is a positive affine transformation, *i.e.*, h_3 is of the form $h_3(r) = pr + q$ for all $r \in R$, where p, $q \in R$ and $p > 0$.

2. The diagram is commutative (Refer to Fig.2).

$$\begin{tikzcd}
M \arrow[d, "h_1"'] & \times & U \arrow[d, "h_2"'] \arrow[r, "g"'] & R \arrow[d, "h_3"] \\
M' & \times & U' \arrow[r, "g'"'] & R
\end{tikzcd}$$

Fig.2 Modelling Morphism

The affine modelling morphisms are adopted to define domain categories of associated decision principle functors. That is:

Let $\mathcal{D}$ be such that

$$\mathrm{Ob}\mathcal{D} = \mathbf{D},$$

and

$$\mathrm{Mor}_{\mathcal{D}}(S, S') = \{h = (h_1, h_2, h_3) | h \text{ is an affine modelling morphism}\}.$$

We define a $\mathcal{D}$-composition by

$$(h'_1, h'_2, h'_3) \cdot (h_1, h_2, h_3) = (h'_1 h_1, h'_2 h_2, h'_3 h_3)$$

where the compositions on the righthand side is the usual compositions of functions. Let $1 = (1, 1, 1)$ be an $\mathcal{D}$-identity. Then it is clear that $\mathcal{D}$ is actually a category.

For a given decision principle δ how to define a subcategory $\mathcal{D}_\delta$ of $\mathcal{D}$ essentially characterizes δ ([5] refers this property as the similarity condition). The morphisms in the domain category of the associated functor represent what decision situations the principle deals with as the similar.

On the other hand, as a codomain category of an associated functor we will define a category $\mathcal{L}$ by

$$\mathrm{Ob}\mathcal{L} = \{(M, \geq) | (M, \geq) \text{ is a partially ordered structure}\},$$

and

$$\mathrm{Mor}_{\mathcal{L}}((M, \geq), (M', \geq')) = \{k_1 | \ k_1 \text{ is a strict order homomorphism}\}.$$

$k_1 \colon M \to M'$ is called a strict order homomorphism from $(M, \geq)$ to $(M', \geq')$ if

1. $(\forall m, m' \in M)(m \geq m' \Rightarrow k_1(m) \geq' k_1(m'))$,

2. $(\forall m, m' \in M)(m > m' \Rightarrow k_1(m) >' k_1(m'))$

hold.

We define an $\mathcal{L}$-composition by the usual composition of functions. An $\mathcal{L}$-identity is an identity function. Then $\mathcal{L}$ is also a category.

Definition 1 *Let* $\delta\colon \mathbf{D}_\delta \to \wp(\mathbf{M} \times \mathbf{M})$ *be a decision principle. A decision principle functor associated with* δ *is a functor* $\mathcal{F}_\delta\colon \mathcal{D}_\delta \to \mathcal{L}$ *such that*

$$\mathcal{F}_\delta(S) = (M, \geq_{\delta(S)}) \ \ \textit{for every } S = (M, U, g) \in \ Ob\mathcal{D}_\delta,$$

and

$$\mathcal{F}_\delta(h) = h_1 \ \ \textit{for every } h = (h_1, h_2, h_3) \in \ Mor\mathcal{D}_\delta,$$

where $\mathcal{D}_\delta$ *is a subcategory of* $\mathcal{D}$ *such that* $Ob\mathcal{D}_\delta = D_\delta$.

This paper especially interested in ways of managing complexity in decision situations. When tackling complex decision situation the decision maker may not care about precise or rigid decision; indeed, such decision making is impossible as well as meaningless. He would rather like to seek a rough but reasonable decision. In order to describe this we introduce a concept of binary preference structure. Intuitively, in a binary preference structure every alternative is simply either *good* or *bad*; the evaluation is not continuous but discrete. Formally,

Definition 2 *Let* $S = (M, U, g)$ *and* δ *be a decision principle.* $\delta(S) = (M, \geq)$ *is called a binary preference structure if*

$$(\forall m, m' \in M)(m, m' \in H_\delta(S) \Rightarrow m \sim_{\delta(S)} m')$$

and

$$(\forall m, m' \in M)(m, m' \notin H_\delta(S) \Rightarrow m \sim_{\delta(S)} m').$$

We call a decision principle δ a binary preference structure generation decision principle (hereafter we simply write it by bpsg decision principle) if it generates a binary preference structure to every decision situation in $\mathbf{D}_\delta$.

This paper investigates two ways of generating a binary preference structure on the set of the alternatives. The first is to eliminate uncertainty contained in the situation by observing it. This approach may be referred to as complexity management by reduction of external or world complexity [1]. The second goes to the opposite direction; the external complexity is not necessarily adjusted, but the decision maker adopts a simple method to cope with the external complexity. This attitude is referred to as complexity management by generating internal complexity [1].

In the following sections we will discuss two types of bpsg decision principles, each of which corresponds to the decision attitudes above: In section 3 we will argue bpsg decision principle of the first type while in section 4 we will do the second type.

3 Universal Map for Reduced Decision Principle Functor

When a decision situation $S = (M, U, g)$ is given, a way of dealing with complexity is to reduce it to $(M, \{u\}, g_u)$, where $g_u = g|M \times \{u\}$, by observing the environment and collecting information about $S = (M, U, g)$ to identify that the environment is a particular $u \in U$. In this case the decision making load is considerably removed. This approach is what we call complexity management by reduction of external complexity.

This type of decision making is important particularly when the environment the decision maker has to manage dynamically changes [2]. To describe dynamics of environmental change we introduce *environmental state transition function* $\alpha: U \to U$. That is, if an initial environment state is $u \in U$, then the environment deterministic changes by $\alpha(u), \alpha^2(u), \alpha^3(u), \cdots$ according to α.

If the observation of the environment is exact, decision making in this case is basically decision making under certainty and quite simple. However, sometimes complexity stems from limitation on the capability of observing the environment. That is, even when a decision situation really happens is $(M, \{u\}, g_u)$, the decision maker may mistake it as $S' = (M, \{u'\}, g_{u'})$, where $u \neq u'$. This possibility makes the problem rather complex.

In this paper we assume that (1)the decision maker knows the whole set of uncertainty U and (2)he can exactly monitor the uncertainty at each moment he observes the environment. The assumptions seem rather strict. However, later we will derive conditions under which the second assumption becomes unnecessary.

Now, we fix M and U and let us define

$$\mathbf{D}_{M,U} = \{S_u = (M, \{u\}, g) \mid u \in U,\ g: M \times \{u\} \to R\}.$$

It gives a class of the decision situations observed by the decision maker.

3.1 Formulation of Internal Modelling Principle

First we will consider a bpsg dp δ such that $D_\delta = \mathbf{D}_{M,U}$ and for each $S_u = (M, \{u\}, g) \in \mathbf{D}_{M,U}$ $H_\delta(S_u)$ is a singleton set. That is, δ and H_δ are of the forms

$$\delta: \mathbf{D}_{M,U} \to \wp(M \times M)$$

and

$$H_\delta : \mathbf{D}_{M,U} \to M,$$

respectively.

In this case for each observed decision situation $(M, \{u\}, g)$ δ determines as a solution such an alternative $m^* \in M$ that attains the maximum value of $g(m, u)$. Since in this case we can identify $H_\delta : \mathbf{D}_{M,U} \to \wp(S)$ such that $(M, \{u\}, g_u) \mapsto \{m\}$ with a function from U into M such that $u \mapsto m$ in an obvious way, we will write H_δ for the function as well.

In general, to manage environmental complexity successfully the decision maker has to construct a model reflecting the dynamics of the external world (Internal Modelling Principle [2]). Formally, an internal model of environmental dynamics α is defined by

Definition 3 **([2])** *An internal model of α is a mapping $\beta: M \to M$ such that the diagram is commutative (Refer to Fig.3).*

$$\begin{tikzcd} U \arrow[r, "\alpha"'] \arrow[d, "H_\delta"'] & U \arrow[d, "H_\delta"] \\ M \arrow[r, "\beta"'] & M \end{tikzcd}$$

Fig.3 Internal Model

By our notational convenience we may write the commutative diagram by

$$\beta H_\delta(S_u) = H_\delta(S_{u'})$$

for each $u \in U$ where $u' = \alpha(u)$.

Lemma 1 *If β is an internal model of α if and only if for each $u \in U$ we have*

$$\beta^t H_\delta(u) = H_\delta \alpha^t(u)$$

for any $t = 0, 1, 2, \ldots$.

For the proof refer to [2].

The Lemma implies that if an internal model exists, once a solution m is found for an initial environmental state u then the decision maker can anticipatively generate a solution $\beta^t(m)$ for each environmental state $\alpha^t(u)$ for $t = 1, 2, \ldots$ without real-time observation of the environment.

One of the authors showed a condition for existence of such an internal model β for a given α [2]. To give the condition we need the following definition.

Definition 4 *Let $f: A \to B$ be a function. Ker f is an equivalence relation on A induced by f , i.e.,*

$$(\forall a, a' \in A)((a, a') \in Ker\ f \Leftrightarrow f(a) = f(a'))$$

We will simply write f for Ker f for notational convenience. Then we have,

Proposition 2 *Let δ be a decision principle such that $H_\delta: \mathbf{D}_\delta \to M$.*

$$H_\delta \alpha \subset H_\delta$$

holds if and only if there is an internal model β of α.

For the proof refer to [2].

The condition implies that the environment does not change so rapidly with respect to the decision attitude H_δ. We should notice the condition depends on the nature of δ as well as on that of α.

Let $\mathcal{D}_\delta$ be a subcategory of $\mathcal{D}$ such that

$$\text{Ob}\mathcal{D}_\delta = \mathbf{D}_{M,U},$$

and

$$\text{Mor}_{\mathcal{D}_\delta}(S_u, S_{u'}) = \{h = (\beta, \alpha|\{u\}, h_3) \in \ \text{Mor } \mathcal{D}|\ \beta \text{ is an injective internal model of } \alpha\},$$

if $S_u = (M, \{u\}, g)$ and $S_{u'} = (M, \{u'\}, g)$.

Lemma 3 *$\mathcal{D}_\delta$ is a category.*

Proof: Let $h = (\beta, \alpha|\{u\}, h_3)$ and $h' = (\beta', \alpha'|\{u'\}, h'_3)$ be $\mathcal{D}_\delta$-morphisms from $S_u = (M, \{u\}, g)$ to $S_{u'} = (M, \{u'\}, g')$ and from $S_{u'} = (M, \{u'\}, g')$ and $S_{u''} = (M, \{u''\}, g'')$, respectively.

It is clear that $h'h = (\beta'\beta, \alpha'\alpha|\{u\}, h'_3 h_3)$ is a $\mathcal{D}_\delta$-morphism from $S_u = (M, \{u\}, g)$ to $S_{u''} = (M, \{u''\}, g'')$: $h'h$ makes the diagram of Fig. 2 commutative. Since both β and β' are injective, so is $\beta'\beta$.

To show $\beta'\beta$ is an internal model of $\alpha'\alpha$, we will prove $H_\delta\alpha'\alpha(u) = \beta'\beta H_\delta(u)$ holds for any $u \in U$. Indeed, we have $H_\delta\alpha'\alpha(u) = H_\delta\alpha'(\alpha(u)) = \beta' H_\delta(\alpha(u)) = \beta'\beta H_\delta(u)$. Consequently, $\beta'\beta$ is an internal model of α. $(1, 1|\{u\}, 1)$ is an identity from $S_u = (M, \{u\}, g)$ to $S_{u'} = (M, \{u'\}, g')$. □

Proposition 4 *$\mathcal{F}_\delta: \mathcal{D}_\delta \to \mathcal{L}$ such that $\mathcal{F}_\delta(M, \{u\}, g) = (M, \geq_{\delta(M,\{u\},g)})$ and $\mathcal{F}_\delta(h_1, h_2, h_3) = h_1$ is a decision principle functor associated with δ.*

Proof: It is clear that $\mathcal{F}_\delta$ is a function from $\mathrm{Ob}(\mathcal{D}_\delta)$ to $\mathrm{Ob}(\mathcal{L})$.

We will show that $\mathcal{F}_\delta$ is a function from $\mathrm{Mor}(\mathcal{D}_\delta)$ to $\mathrm{Mor}(\mathcal{L})$. Let $(\beta, \alpha|\{u\}, h_3)$ be a $\mathcal{D}_\delta$-morphism from $S_u = (M, \{u\}, g)$ to $S_{u'} = (M, \{u'\}, g')$. Then, by definition we have $\mathcal{F}_\delta(\beta, \alpha|\{u\}, h_3) = \beta$. We will prove β is a strict order homomorphism.

Let $m, m' \in M$ be such that $m \geq_{S_u} m'$ and assume $\beta(m) \geq_{S_{u'}} \beta(m')$ does not hold, *i.e.*, $\beta(m') >_{S_{u'}} \beta(m)$. Since $(M, \geq_{S_{u'}})$ is binary and $|H_\delta(S_{u'})| = 1$, it follow that $\beta(m') = H_\delta(S_{u'})$. Since β is an internal model of α we have $\beta(m') = H_\delta(S_{u'}) = \beta H_\delta(S_u)$. Since β is injective, we have $m' = H_\delta(S_u)$. It follows from $m \geq_{S_u} m'$ that $m = m'$ because of $|H_\delta(S_u)| = 1$. It contradicts to $\beta(m') >_{S_{u'}} \beta(m)$.
Let $m, m' \in M$ be such that $m >_{S_u} m'$. It implies $m = H_\delta(S_u)$ and hence $\beta(m) = \beta H_\delta(S_u)$. Since β is an internal model of α we have $\beta(m) = H_\delta(S_{u'})$. Since $(M, \geq_{S_{u'}})$ is binary and $|H_\delta(S_{u'})| = 1$, $\beta(m) >_{S_{u'}} m''$ for any $m'' \in M$. Hence we have $\beta(m) >_{S_{u'}} \beta(m')$.

Since $\mathcal{F}_\delta(1, 1|\{u\}, 1) = 1$ holds $\mathcal{F}_\delta$ preserves an identity while $\mathcal{F}_\delta$ clearly preserves compositions. Consequently, $\mathcal{F}_\delta$ is a functor. □

3.2 Universal Map for Reduced Decision Principle Functor

In this subsection let us consider a bpsg decision principle δ such that $\mathbf{D}_\delta = \mathcal{D}_{M,U}$ and for each $S_u = (M, \{u\}, g) \in \mathbf{D}_{M,U}$ $H_\delta(S_u)$ is not necessarily a singleton set, *i.e.*, δ determines several admissible alternatives for each decision situation $(M, \{u\}, g)$.

Formally, δ and H_δ are of the forms

$$\delta: \mathbf{D}_{M,U} \to \wp(M \times M)$$

and

$$H_\delta : \mathbf{D}_{M,U} \to \wp(M),$$

respectively.

Let $m \in M$ be arbitrary and set

$$K_m = \{S_u | m \in H_\delta(S_u)\}.$$

Proposition 5 *For any $m \in M$ if*

$$\cap_{S_u \in K_m} H_\delta(S_u) \neq \phi \Rightarrow \cap_{S_u \in K_m} H_\delta(S_{\alpha(u)}) \neq \phi$$

holds, then there is $\theta: M \to M$ such that

$$\theta(H_\delta(S_u)) \subset H_\delta(S_{\alpha(u)})$$

Proof: Let $m \in M$ be arbitrary. (1)Suppose there is u such that $m \in H_\delta(S_u)$, *i.e.*, $K_m = \{S_u | m \in H_\delta(S_u)\} \neq \phi$. Since we have $m \in \cap_{S_u \in K_m} H_\delta(S_{\alpha(u)})$, by the assumption we have $\cap_{S_u \in K_m} H_\delta(S_{\alpha(u)}) \neq \phi$. Let $m' \in \cap_{S_u \in K_m} H_\delta(S_{\alpha(u)})$ be arbitrary and define $\theta(m) = m'$. Then we have $m' \in \cap_{S_u \in K_m} H_\delta(S_{\alpha(u)}) \subset H_\delta S_{\alpha(u)}$. (2) If for all $u \in U$ we have $m \notin H_\delta(u)$, *i.e.*, $K_m = \{S_u | m \in H_\delta(S_u)\} = \phi$, let us define $\theta(m) = m_0$ where $m_0 \in M$ is arbitrary.

(1) and (2) define a map $\theta: M \to M$. Furthermore, for any $m \in H_\delta(S_u)$ we have $\theta(m) \in H_\delta(S_{\alpha(u)})$, that is, $\theta(H_\delta(S_u)) \subset H_\delta(S_{\alpha(u)})$. □

θ is called a generalized internal model of α. The assumption of the proposition, similar to the interpretation of β, implies that α does not change so rapidly. The proposition claims that if m is a solution for S_u then $\theta(m)$ is necessarily a solution for $S_{\alpha(u)}$.

Definition 5 *A generalized internal model θ of α is called to preserve non-solutions if $\theta([H_\delta(S_u)]^c) \subset [H_\delta(S_{\alpha(u)})]^c$ holds, where X^c denotes the complement set of X.*

If θ preserves non-solutions, then any non-solution alternative in the original decision situation equally remains as non-solution in the next decision situation.

Let $\mathcal{D}'_\delta$ be a subcategory of $\mathcal{D}$ such that

$$\mathrm{Ob}\mathcal{D}'_\delta = \mathbf{D}_{M,U},$$

$$\begin{aligned}\mathrm{Mor}_{\mathcal{D}'_\delta}(S_u, S_{u'}) \quad = \quad & \{h = (\theta, \alpha|\{u\}, h_3) \in \ \mathrm{Mor}\ \mathcal{D}| \\ & \theta \text{ is a generalized internal model of } \alpha \text{and preserves non-solutions}\}.\end{aligned}$$

Then,

Lemma 6 *$\mathcal{D}'_\delta$ is actually a category.*

Proof: Let $h = (\theta, \alpha|\{u\}, h_3)$ and $h' = (\theta', \alpha'|\{u'\}, h'_3)$ be $\mathcal{D}'_\delta$-morphisms from $S_u = (M, \{u\}, g)$ to $S_{u'} = (M, \{u'\}, g')$ and from $S_{u'} = (M, \{u'\}, g')$ and $S_{u''} = (M, \{u''\}, g'')$, respectively.

It is clear that $h'h = (\theta'\theta, \alpha'\alpha|\{u\}, h'_3 h_3)$ makes the diagram of Fig.2 commutative.

Since θ and θ' are generalized internal models of α and α', respectively, so is $\theta'\theta$: Indeed, by the assumptions we have $\theta(H_\delta(S_u)) \subset H_\delta(S_{\alpha(u)})$ and $\theta'(H_\delta(S_{u'})) \subset H_\delta(S_{\alpha'(u')})$. Since it follows from $\alpha(u) = u'$ that $H_\delta(S_{\alpha(u)}) = H_\delta(S_{u'})$ we have

$$\theta'\theta H_\delta(S_u) \subset \theta' H_\delta(S_{\alpha(u)}) = \theta'(H_\delta(S_{u'})) \subset H_\delta(S_{\alpha'(u')}) = H_\delta(S_{\alpha'\alpha(u)})$$

and hence $\theta'\theta$ is a generalized internal model of $\alpha'\alpha$.

Since θ and θ' preserve non-solutions, we have $\theta([H_\delta(S_u)]^c) \subset [H_\delta(S_{\alpha(u)})]^c$ and $\theta'([H_\delta(S_{u'})]^c) \subset [H_\delta(S_{\alpha'(u')})]^c$. It implies that $\theta'\theta[H_\delta(S_u)]^c \subset \theta'[H_\delta(S_{\alpha(u)})]^c \subset [H_\delta(S_{\alpha'(u')})]^c$. Hence, $\theta'\theta$ also preserve non-solutions.

$(1, 1|\{u\}, 1)$ is an identity from $S_u = (M, \{u\}, g)$ to $S_{u'} = (M, \{u'\}, g')$.

Consequently, $\mathcal{D}'_\delta$ is a category. □

Proposition 7 *$\mathcal{F}_\delta$: $\mathcal{D}'_\delta \to \mathcal{L}$ such that $\mathcal{F}_\delta(M, U, g) = (M, \geq_{\delta(M,U,g)})$ and $\mathcal{F}_\delta(h_1, h_2, h_3) = h_1$ is a decision principle associated with δ.*

Proof: We will show that $\mathcal{F}_\delta$ is actually a functor. It is clear that $\mathcal{F}_\delta$ is a function from $\mathrm{Ob}(\mathcal{D}'_\delta)$ to $\mathrm{Ob}(\mathcal{L})$.

Now we will prove that $\mathcal{F}_\delta$ is a function from Mor($\mathcal{D}'_\delta$) to Mor($\mathcal{L}$). Let $(\theta, \alpha|\{u\}, h_3)$ be a $\mathcal{D}'_\delta$-morphism from $S_u = (M, \{u\}, g)$ to $S_{u'} = (M, \{u'\}, g')$. Then, we have $\mathcal{F}_\delta(\theta, \alpha|\{u\}, h_3) = \theta$ by definition. We will prove θ is a strict order homomorphism. Let $m, m' \in M$ be such that $m \geq_{S_u} m'$.

1. For the case where we have $m >_{S_u} m'$: Since we have $m \in H_\delta(S_u)$ and $m' \notin H_\delta(S_u)$, it follows $\theta(m) \in H_\delta(S_{u'})$ and $\theta(m') \notin H_\delta(S_{u'})$ because of the condition of θ. It implies that $\theta(m) >_{\delta(S_{u'})} \theta(m')$.

2. For the case where we have $m \sim_{S_u} m'$: If we have $m, m' \in H_\delta(S_u)$, it follows $\theta(m), \theta(m') \in H_\delta(S_{u'})$ by the definition of θ. It implies that $\theta(m) \sim_{\delta(S_{u'})} \theta(m')$. If we have $m, m' \notin H_\delta(S_u)$, it follows $\theta(m)$, $\theta(m') \notin H_\delta(S_{u'})$ because θ preserves non-solutions. Hence we have $\theta(m) \sim_{\delta(S_{u'})} \theta(m')$.

Since $\mathcal{F}_\delta(1, 1|\{u\}, 1) = 1$ holds, $\mathcal{F}_\delta$ preserves an identity while $\mathcal{F}_\delta$ clearly preserves compositions.

Consequently, $\mathcal{F}_\delta$ is a functor. □

Propositions 4 and 7 claim that internal models of the environment constitute the morphisms of the domain categories of the associated functors. In this sense existence of internal model is closely related to similarity among the decision situations.

One of typical examples of a bpsg decision principles of this type is a reduced decision principle, which is defined by:

Definition 6 *Let $\delta: \mathbf{D}_\delta \to \wp(M \times M)$ be an arbitrary decision principle. A decision principle $\delta^*: \mathbf{D}_\delta \to \wp(M \times M)$ is called a reduced decision principle of δ if we have*

$$\delta^*(S) = (M, \geq_{\delta^*(S)}) \quad \textit{for each} \quad S = (M, U, g)$$

where for $m, m' \in M$ $m \geq_{\delta^*(S)} m'$ is defined by

$$(m, m' \in H_\delta(S)) \quad \textit{or} \quad (m \in H_\delta(S) \quad \textit{and} \quad m' \notin H_\delta(S)) \quad \textit{or} \quad (m, m' \notin H_\delta(S)).$$

It is obvious that δ^* is actually a bpsg decision principle.

We can always define a reduced decision principle for a given decision principle. A relationship between them is described by:

Proposition 8 *Let $\delta: \mathbf{D}_\delta \rightarrow \wp(M \times M)$ be a decision principle and $\delta^*: \mathbf{D}_\delta \rightarrow \wp(M \times M)$ be a reduced decision principle of it. Suppose $\mathcal{F}_\delta$ and $\mathcal{F}_{\delta^*}$ are their associated functors, respectively. Then there is a natural transformation $\eta = \{\eta_S | S \in Ob\mathcal{D}_\delta\}$ from $\mathcal{F}_\delta$ to $\mathcal{F}_{\delta^*}$, where $\eta_S: M \rightarrow M$ is an identity function if $S = (M, U, g)$.*

Proof: By the definition of η it is clear that the diagram is commutative for each $\mathcal{D}_\delta$-morphism $k = (k_1, k_2, k_3)$ from $S = (M, U, g)$ to $S' = (M', U', g')$ (Refer to Fig. 4).

Let $S = (M, U, g) \in \mathrm{Ob}\mathbf{D}_\delta$. We will show that η_S is a strict order homomorphism from $(M, \geq_{\delta(S)})$ to $(M, \geq_{\delta^*(S)})$. Suppose $m \geq_{\delta^*(S)} m'$ for $m,\ m' \in M$. It implies $m >_{\delta^*(S)} m'$ or $m \sim_{\delta^*(S)} m'$. For the former case we have $m \in H_\delta(S)$ and $m' \notin H_\delta(S)$ and hence $m >_{\delta^*(S)} m'$. For the latter case, we have $m,\ m' \in H_\delta(S)$ or $m,\ m' \notin H_\delta(S)$. It follows by definition that $m \sim_{\delta^*(S)} m'$. □

$$\begin{tikzcd}
\mathcal{F}_\delta(S) \arrow[r, "\eta_S"'] \arrow[d, "k_1"'] & \mathcal{F}_{\delta^*}(S) \arrow[d, "k_1"] \\
\mathcal{F}_\delta(S') \arrow[r, "\eta_{S'}"'] & \mathcal{F}_{\delta^*}(S')
\end{tikzcd}$$

Fig.4 Natural Transformation

For any reduced decision principle $d^*: \mathbf{D}_d \rightarrow \wp(M \times M)$ of a decision principle $d: \mathbf{D}_d \rightarrow \wp(M \times M)$ and for any decision situation $S = (M, U, g) \in \mathbf{D}_d$ there is a bspg decision principle $\delta: \mathbf{D}_{M,U} \to \wp(M \times M)$ such that $\geq_{d^*(S)} = \geq_{\delta(M,\{u\},g)}$ holds for some $(M, \{u\}, g') \in \mathbf{D}_{M,U}$.

Moreover, we have a stronger fact. We need the following definition.

Definition 7 *A decision principle $\delta: \mathbf{D}_\delta \to \wp(M \times M)$ is called Pareto consistent if for any $(M, U, g) \in \mathbf{D}_\delta$ we have*

1. $(\forall m, m' \in M)(m \geq_{p(S)} m' \Rightarrow m \geq_{\delta(S)} m')$
2. $(\forall m, m' \in M)(m >_{p(S)} m' \Rightarrow m >_{\delta(S)} m')$

where $\geq_{p(S)}$ is the Pareto ordering defined by

$$(\forall m, m' \in M)(m \geq_{p(S)} m' \Leftrightarrow (\forall u \in U)(g(m,u) \geq g(m',u)))$$

and

$$(\forall m, m' \in M)[m >_{p(S)} m' \Leftrightarrow (m \geq_{p(S)} m' \quad \textit{and} \quad (\exists u \in U)(g(m,u) > g(m',u)))]$$

Theorem 9 *Let $d: \mathbf{D}_d \to \wp(M \times M)$ be an arbitrary decision principle and $d^*: \mathbf{D}_d \to \wp(M \times M)$ be a reduced decision principle of it. Let $\delta: \mathbf{D}_{M,U} \to \wp(M \times M)$ be a bpsg decision principle and $\mathcal{F}_\delta: \mathcal{D}'_\delta \to \mathcal{L}$ be its associated functor. Let $S = (M, U, g) \in \mathbf{D}_d$ be arbitrary. If δ is Pareto consistent then for $B = d^*(S)$ there is $A \in \mathbf{D}_{M,U}$ such that $(1, A)$ is a universal map of B with respect to $\mathcal{F}_\delta$. (Refer to Fig. 5).*

Proof: Let $S = (M, U, g) \in \mathbf{D}_d$ be arbitrary and set $B = d^*(S) = (M, \geq_{d^*(S)})$.

$$\begin{tikzcd}
A \arrow[d, "\exists k"] & & B \arrow[r] \arrow[d, "\forall k_1"] & \mathcal{F}_\delta(A) \arrow[dl, "\mathcal{F}_\delta(k)"] \\
\forall A' & & \mathcal{F}_\delta(A') &
\end{tikzcd}$$

Fig.5 Universal Map

Let $A = (M, \{u\}, g')$ be such that

$$g'(m,u) = \begin{cases} L & \text{if } m \in H_d(S) \\ l & \text{otherwise} \end{cases}$$

where L and l are reals such that $L > l$ holds. Then, A is clearly in $\mathbf{D}_{M,U}$.

We also have $1: M \to M$ is a strict order homomorphism from $(M, \geq_{d^*(S)})$ to $(M, \geq_{\delta(A)})$: Indeed, If $m >_{d^*(S)} m'$ holds, we have $m \in H_d(S)$ and $m' \notin H_d(S)$. It implies that $g'(m,u) = L$ and $l = g'(m',u)$ and hence $g'(m,u) = L > l = g(m',u)$. Since δ is Pareto consistent, it implies $m >_{\delta(A)} m'$. If $m \sim_{d^*(S)} m'$ holds, we have $m, m' \in H_d(S)$ or $m, m' \notin H_d(S)$. For the former case, by definition it follows that $g'(m,u) = g'(m',u) = L$, which implies $m \sim_{\delta(A)} m'$ because δ is Pareto consistent. For the latter case we can apply the same argument.

Next, we will show $(A, 1)$ is a universal map of B. Let $A' = (M, \{u'\}, g'') \in \mathbf{D}_{M,U}$ and $k_1: M \to M$ be an arbitrary $\mathcal{L}$- morphism from $(M, \geq_{d^*(S)})$ to $\mathcal{F}_\delta(A') = (M, \geq_{\delta(A')})$.
First we will show $g''(k_1(m), u') = N$ for each $m \in H_\delta(S)$ and $g''(k_1(m), u') = n$ for each $m \notin H_\delta(S)$ and $N > n$:Let $m, m^0 \in H_d(S)$, *i.e.*, $m \sim_{d^*(S)} m^0$, then since $k_1: M \to M$ is a strict order homomorphism, it implies that $k_1(m) \sim_{\delta(A')} k_1(m^0)$. Suppose $g''(k_1(m), u')$ is different from $g''(k_1(m^0), u')$; we can assume $g''(k_1(m), u') > g''(k_1(m^0), u')$ without loss of generality. Since δ is Pareto consistent, it implies $k_1(m) >_{\delta(A')} k_1(m^0)$, which contradicts to the fact that $k_1(m) \sim_{\delta(A')} k_1(m^0)$. It yields that $g''(k_1(m), u') = g''(k_1(m^0), u')$ and we can set it as N. The same proof is applicable to the case of $m, m^0 \notin H_d(S)$.
Now we show $N > n$. We have $N = g''(k_1(m), u')$ and $n = g''(k_1(m), u')$ if $m \in H_d(S)$ and $m^0 \notin H_d(S)$. It follows that $m >_{d^*(S)} m^0$. Since k_1 is a strict order homomorphism, we have $k_1(m) >_{\delta(A')} k_1(m^0)$. Suppose $g''(k_1(m), u') = N \leq n = g''(k_1(m^0), u')$. Since δ is Pareto consistent, we have $k_1(m) \leq_{\delta(A')} k_1(m^0)$, which is a contradiction.
Let $k_3: R \to R$ be such that

$$k_3(r) = \frac{N-n}{L-l} r - \frac{N-n}{L-l} l + n.$$

It is obvious that $k_3(l) = n$ and $k_3(L) = N$ and k_3 is a positive affine transformation and hence a strict order homomorphism.

Now set $k = (k_1, k_2, k_3)$, where k_2 is the unique function from $\{u\}$ to $\{u'\}$. Then k is a $\mathcal{D}'_\delta$ morphism: Since we have

$$k_3 g'(m, u) = \begin{cases} k_3(L) = g''(k_1(m), u') & \text{if } m \in H_d(S) \\ k_3(l) = g''(k_1(m), u') & \text{if } m \notin H_d(S), \end{cases}$$

the commutative diagram of Fig.2 holds. k_1 is a generalized internal model of k_2 since k_1 is a strict order homomorphism from $(M, \geq_{\delta^*(S)})$ to $(M, \geq_{\delta^*(A')})$ and hence it preserves non-solutions.

Finally, since we clearly have $k_1 = \mathcal{F}_\delta(k) \cdot 1$, the triangle of Fig. 5 is commutative. Suppose $k' = (k'_1, k'_2, k'_3)$ also satisfies the conditions. Then, we must have $k_1 = k'_1$ and $k_2 = k'_2$ because of the definition. $k_3 = k'_3$ also holds since k_3 is of the positive affine transformation. Hence we complete the proof. □

This theorem insists that any reduced decision principle can be approximated by a Pareto consistent bpsg dp as far as the induced preference structures are concerned.

4 Universal Map of Satisfaction Decision Principle Functor

The second way to handle complexity around decision activity and to generate a binary preference structure is to increase the internal complexity using a simple decision principle. One of such decision principles is the satisfaction decision principle [4]. Since the principle contains a parameter called an aspiration level, it seems similar to the linear weighted sum decision principle. However, the weighting vector of the linear weighted sum decision principle precisely represents the relative importance of each attribute and it generates a fine preference structure on the set of alternatives. On the other hand, the aspiration level simply gives a rough tolerance level and the satisfaction decision principle only induces a binary preference structure. Hence we consider the satisfaction decision principle is quite flexible and more adequate to manage complex decision situation.

We assume that

1. Each alternative for the satisfaction decision principle is a *decision rule, i.e.,* a function $m: U \to A$ from a set of uncertainty into some set A of outcomes. A set of alternative M is, then, $M \subset A^U$.

2. An aspiration level τ for handling a set of alternatives $M \subset A^U$, is also a function $\tau: U \to A$ such that $\tau \in M$

So we will represent a decision situation for the satisfaction decision principle by

$$\hat{g}: [M, \tau] \times U \to R$$

where $M \subset A^U$ and $[M, \tau]$ denotes a pointed set. Each alternative $m \in M$ is evaluated by $\hat{g}(m, u) = g(m(u), u)$ by some function $g: A \times U \to R$. We will simply write $\tau(u)$ for $g(\tau(u), u)$. We call $([M, \tau], U, \hat{g})$ a pointed decision situation.

Let D_s be such that

$$D_s = \{S = ([M, \tau], U, \hat{g}) | M \subset A^U, U \subset \mathbf{U}, \ \ S \text{ is a pointed decision situation}\}.$$

Contrasting to the cases of Section 3 the satisfaction decision principle does not reduce the external complexity. Rather, it chooses a satisfactory alternative which necessarily guarantees admissible performance under any uncertainty. This essential spirit of the principle is represented by the definition of H_s as follows:

Definition 8 *The satisfaction decision principle is a function $s: D_s \to L_s$ such that*

$$s([M, \tau], U, \hat{g}) = (M, \geq_{s([M,\tau],U,\hat{g})})$$

where for m and m' in M we define $m \geq_{s(S)} m'$ by

$$(m,\ m' \in H_s(S)) \ or \ (m,\ m' \notin H_s(S)) \ or \ (m \in H_s(S) \ and \ m' \notin H_s(S))$$

while

$$H_s(S) = \cap_{u \in U} \{m \in M | (g(m, u) \geq \tau(u))\}$$

for each $S = ([M, \tau], U, \hat{g})$.

$(M, \geq_{s(S)})$ is clearly a binary preference structure.

Now let us discuss relationship between the reduced decision principle and the satisfaction decision principle. Let $d^*: \mathbf{D}_\delta \to \wp(M \times M)$ be a reduced decision principle of a decision principle $d: \mathbf{D}_d \to \wp(M \times M)$.

Definition 9 *d^* is called essentially equivalent to the satisfaction decision principle if for each decision situation $(M, U, g) \in \mathbf{D}_d$ there is $\tau: U \to R$ such that*

$$d^*(M, U, g) = s([M, \tau], U, \hat{g})$$

where $\hat{g}(m, u) = g(m(u), u)$ for each $(m, u) \in M \times U$.

Reduced decision principles of some well-known decision principles are essentially equivalent to the satisfaction decision principle while others are not:

Proposition 10 ([4]) *The reduced decision principle of the max-min decision principle is essentially equivalent to the satisfaction decision principle while the reduced decision principle of the regret decision principle is not.*

For the proof refer to [4]. □

By definition the reduced decision principle d^* generates the preference structure in two phases; first d^* generates a finer structure by d and then reduces it to a binary preference structure. On the other hand, the satisfactory decision principle generates a binary preference structure to each decision situation straightforward. In this sense the latter is simpler and more flexible than the former.

For each $S = ([M, \tau], U, \hat{g})$ and $S' = ([M', \tau'], U', \hat{g}')$ where $M \subset A^U$ and $M' \subset A^{U'}$ we call $h = (h_1, h_2, h_3)$ a pointed affine modelling morphism if

1. $h_1: M \to M'$, $h_2: U \to U'$ and $h_3: R \to R'$.

2. $h_1(\tau) = \tau'$ holds.

3. h_3 is of the positive linear form; *i.e.*, $(\forall r \in R)(h_3(r) = pr + q)$, where $p, q \in R$ and $p > 0$.

4. The diagram is commutative (Refer to Fig.6).

$$
\begin{tikzcd}
{[M,\tau]} \arrow[d, "h_1"] & \times & U \arrow[d, "h_2"'] \arrow[rr, "\hat{g}"'] & & R \arrow[d, "h_3"] \\
{[M',\tau']} & \times & U' \arrow[rr, "\hat{g}'"'] & & R
\end{tikzcd}
$$

Fig.6 Pointed Modelling Morphism

Let $\mathcal{D}_s$ be such that

$$\mathrm{Ob}\mathcal{D}_s = \{S = ([M,\tau], U, \hat{g}) | M \subset A^U, U \subset \mathbf{U}, \quad S \text{ is a pointed decision situation}\}$$

and

$$\mathrm{Mor}_{\mathcal{D}_s}(S, S') = \{h = (h_1, h_2, h_3) | h \text{ is a pointed affine modelling morphism from } S \text{ to } S' \ \}.$$

It is clear that $\mathcal{D}_s$ is really a category.

We call $(M, \geq, m_0)$ a pointed binary preference structure if $\geq$ is a binary preference structure and $m_0 \in M$. h_1 is referred to as a pointed order homomorphism from $(M, \geq, m_0)$ to $(M', \geq', m_0')$ if

1. $h_1 \colon M \to M'$ is a strict order homomorphism.

2. $h_1(m_0) = m_0'$ holds.

Let $\mathcal{L}_s$ be such that

$$\mathrm{Ob}\mathcal{L}_s = \{L = (M, \geq, m_0) | M \subset A^U, U \subset \mathbf{U}, \quad L \text{ is a pointed binary preference structure}\}$$

and

$$\mathrm{Mor}_{\mathcal{L}_s}(L, L') = \{h_1 | h_1 \text{ is a pointed order homomorphism from } L \text{ to } L'\}.$$

We can also easily check $\mathcal{L}_s$ is indeed a category.

Let $\mathcal{F}_s : \mathcal{D}_s \to \mathcal{L}_s$ be such that

$$\mathcal{F}_s([M, \tau], U, \hat{g}) = (M, \geq_{s([M,\tau],U,\hat{g})}, \tau)$$

and

$$\mathcal{F}_s(h_1, h_2, h_3) = h_1.$$

Then,

Proposition 11 *$\mathcal{F}_s : \mathcal{D}_s \to \mathcal{L}_s$ is a functor.*

The proof is omitted. □

We call $\mathcal{F}_s$ a satisfaction decision principle functor.

Let $\mathcal{F}_s : \mathcal{D}_s \to \mathcal{L}_s$ is a satisfaction decision principle functor. Let $\mathcal{D}_s^*$ be a full subcategory of $\mathcal{D}_s$ such that

$$\mathrm{Ob}\mathcal{D}_s^* = \{([\{m_1, m_2\}, m_1], \{u\}, \hat{g}) \in \mathrm{Ob}\mathcal{D}_s | \hat{g}(m_1, u) > \hat{g}(m_2, u)\}.$$

An object of $\mathcal{D}_s$ is a pointed decision situation with two alternatives, one of which is satisfactory and the other is not.

Let us suppose $\mathcal{F}_s^* = \mathcal{F}_s | \mathcal{D}_s^* : \mathcal{D}_s^* \to \mathcal{L}_s$. Then we have

Theorem 12 *Let $S = ([M, \tau], U, \hat{g}) \in Ob\mathcal{D}_s$ and $B = \mathcal{F}_s(S) = ([M, \tau], \geq_{s(S)}) \in Ob\mathcal{L}_s$. If we have $H_s([M, \tau], U, \hat{g}) \neq M$. there is a universal map (γ, A) for B with respect to $\mathcal{F}_s^*$.*

Proof: Let $S = ([M, \tau], U, \hat{g})$ be such that $H_s([M, \tau], U, \hat{g}) \neq M$ and $B = \mathcal{F}_s(S) = ([M, \tau], \geq_{s(S)}) \in \mathrm{Ob}\mathcal{L}_s$. It follows from $H_s([M, \tau], U, \hat{g}) \neq M$ that there is $m_{-1} \in M$ such that $m_{-1} \notin H_s([M, \tau], U, \hat{g})$. Then, there is at least one $u \in U$ such that $\hat{g}(\tau, u) > \hat{g}(m_{-1}, u)$ by definition.

$$
\begin{tikzcd}
A \arrow[d, "\exists k'"] & B \arrow[r] \arrow[d, "\forall k_1"] & \mathcal{F}_s^*(A) \arrow[dl, "\mathcal{F}_s^*(k')"] \\
\forall A' & \mathcal{F}_s^*(A') &
\end{tikzcd}
$$

Fig.7 Universal Map

Let $A = ([\{\tau, m_{-1}\}, \tau], \{u\}, \hat{g}')$, where we denote $\hat{g}|\{\tau, m_{-1}\} \times \{u\}$ by $\hat{g}'$ for notational convenience. Then, A is clearly in $\mathrm{Ob}\mathcal{D}_s^*$ since $\hat{g}'(\tau, u) > \hat{g}'(m_{-1}, u)$.

Let $\gamma: M \rightarrow \{\tau, m_{-1}\}$ be the unique strictly order homomorphism, *i.e.*,

$$\gamma(m) = \begin{cases} \tau & \text{if } m \in H_s \\ m_{-1} & \text{otherwise.} \end{cases}$$

Now suppose $A' = ([\{m_1, m_2\}, m_1], \{u'\}, \hat{g}'')$ is arbitrary in $\mathrm{Ob}\mathcal{D}_s^*$ and k_1 is a strict order homomorphism from B to $\mathcal{F}_s(A')$. Since A' is in $\mathrm{Ob}\mathcal{D}_s^*$ we have $\hat{g}''(m_1, u') > \hat{g}''(m_2, u')$. Then we have $m_1 \in H_s(A')$ and $m_2 \notin H_s(A')$, *i.e.*, $m_1 >_{s(A')} m_2$. Since k_1 is strictly order homomorphic, it should satisfy

$$k_1(m) = \begin{cases} m_1 & \text{if } m \in H_s(S) \\ m_2 & \text{otherwise.} \end{cases}$$

Now let $k_1' = k_1|\{\tau, m_1\}$. Then we have $k_1'(\tau) = m_1$ and $k_1'(m_{-1}) = m_2$ since $\tau \in H_s(S)$ and $m_{-1} \notin H_s(S)$. Let k_2' be the unique function from $\{u\}$ to $\{u'\}$. Let $k_3': R \rightarrow R$ be a positive affine transformation satisfying

$$k_3'(r) = \begin{cases} \hat{g}''(m_1, u') & \text{if } r = \hat{g}'(\tau, u) \\ \hat{g}''(m_2, u') & \text{if } r = \hat{g}'(m_{-1}, u) \end{cases}$$

Then $k' = (k'_1, k'_2, k'_3)$ is really a $\mathcal{D}_s$-morphism: Indeed, we have $\hat{g}''(k'_1(\tau), k'_2(u)) = \hat{g}''(m_1, u') = k'_3(\hat{g}'(\tau, u))$ and $\hat{g}''(k'_1(m_{-1}), k'_2(u)) = \hat{g}''(m_2, u') = k'_3(\hat{g}'(m_{-1}, u))$.

Finally we will show the triangle in Fig. 7 is commutative and such a k' is unique. Let $m \in M$. Then

$$\mathcal{F}_s^*(k')\gamma(m) = k'_1\gamma(m) = \begin{cases} k'_1(\tau) = m_1 = k_1(m) & \text{if } m \in H_s(S) \\ k'_1(m_{-1}) = m_2 = k_1(m) & \text{otherwise} \end{cases}$$

holds and hence the triangle is commutative.

Suppose $k'' = (k''_1, k''_2, k''_3)$ satisfies the same conditions. The commutativity implies

$$k'_1\gamma = k_1 = k''_1\gamma.$$

Since γ is surjective, it follows $k'_1 = k''_1$. $k'_2 = k''_2$ must hold because of the uniqueness of k'_2. Since k'_3 and k''_3 are positive affine transformations they are of the form $k'_3(r) = pr + q$ and $k''_3(r) = p'r + q'$, where $p, p', q, q' \in R$ and $p, p' > 0$. The commutative diagram implies $k'_3(r_1) = pr_1 + q = p'r_1 + q' = k''_3(r_1)$ and $k'_3(r_2) = pr_2 + q = p'r_2 + q' = k''_3(r_2)$, where $r_1 = \hat{g}(\tau, u)$ and $r_2 = \hat{g}(m_{-1}, u)$ and $r_1 \neq r_2$. Then we have $p(r_1 - r_2) = p'(r_1 - r_2)$. It follows from $r_1 \neq r_2$ that $p = p'$ and so $q = q'$.

Consequently, k' is the unique morphism satisfying the conditions. It completes the proof. □

The theorem claims that as far as the resulting preference structures are concerned, the satisfaction decision principle approximately identifies any decision situation with a decision situation with two "essentially different" alternatives, one uncertainty and a positive affine performance function.

5 Conclusions

We proposed and formulated a concept of binary preference structure generating decision principle as a way to manage complexity of decision making situations. We also clarified meaning of internal modelling principle in category theoretic terms.

The main theorems of the paper show that as far as the resulting preference structure is concerned, both a reduced decision principle and the satisfaction decision principle approximately identify a decision situation with two "essentially different" alternatives, one uncertainty and a positive affine performance function. It rationally supports our intuitive understanding that they are flexible and simple enough to apply to complex decision situations.

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