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Zeta functions of prehomogeneous vector spaces with coefficients related to periods of automorphic forms

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§0 Introduction

The purpose of this paper is to generalize the theory of zeta functions associated with prehomogeneous vector spaces ([SS], [S1]) to zeta functions whose coefficients involve periods of automorphic forms. We prove the functional equations and the analytic continuations of such zeta functions in the case where the infinitesimal character of an automorphic form is generic and the prehomogeneous vector space in question have a symmetric structure of K_ϵ-type. In [S6], we have dealt with the case where automorphic forms are given by matrix coefficients of irreducible unitary representations of compact groups.

Our results can be applied, for example, to zeta functions considered in [M3] and [Hej] and some special cases of zeta functions in [M1,2,4]; however, to reduce the size of this paper, we do not include any concrete examples. An expanded version of this paper will appear elsewhere.

In §1, we introduce zeta functions and give their integral representation (Zeta integral). In §2, the functional equation of the zeta integral will be proved. In §3, we define the notion of symmetric structure of prehomogeneous vector spaces and establish some elementary properties. In the final §4, the functional equations of zeta functions will be proved under the condition that the infinitesimal character of an automorphic form is generic and a symmetric structure is of K_ϵ-type.

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§1 Definition of zeta functions and their integral representations

1.1 Let $(\mathbf{G}, \rho, \mathbf{V})$ be a prehomogeneous vector spaces (abbrev. p.v.) defined over the rational number field $\mathbf{Q}$ and denote its singular set by $\mathbf{S}$. Then, by definition, $\mathbf{V_C} - \mathbf{S_C}$ is a single $\mathbf{G_C}$-orbit.

Let $\mathbf{S}_1, \ldots, \mathbf{S}_n$ be the $\mathbf{Q}$-irreducible hypersurfaces contained in $\mathbf{S}$ and take $\mathbf{Q}$-irreducible polynomials $P_1, \ldots, P_n$ defining $\mathbf{S}_1, \ldots, \mathbf{S}_n$, respectively. It is known that the polynomial P_i is unique up to a non-zero constant multiple in $\mathbf{Q}$. For each $i = 1, \ldots, n$, there exists a $\mathbf{Q}$-rational character χ_i satisfying

$$P_i(\rho(g)x) = \chi_i(g)P_i(x) \quad (g \in \mathbf{G}, x \in \mathbf{V}).$$

We call $P_1, \ldots, P_n$ *the basic relative invariants over* $\mathbf{Q}$. Any relative invariant of $(\mathbf{G}, \rho, \mathbf{V})$ with coefficients in $\mathbf{Q}$ can be expressed as a product of $P_1, \ldots, P_n$, negative power being allowed.

Denote by $X_\rho(\mathbf{G})_\mathbf{Q}$ the subgroup of $X(\mathbf{G})_\mathbf{Q}$ generated by $\chi_1, \ldots, \chi_n$, which is a free abelian group of rank n.

Let $\mathbf{G}_0$ be the identity component of $\bigcap_{i=1}^{n} \ker \chi_i$ with respect to the Zariski topology. For an $x \in \mathbf{V}$, put

$$\mathbf{G}_x = \{g \in \mathbf{G} \mid \rho(g)x = x\}.$$

In the following, we assume that

(A-1) *for any* $x \in \mathbf{V_Q} - \mathbf{S_Q}$, *the isotropy subgroup* $\mathbf{G}_x$ *is reductive and* $X((\mathbf{G}_x)^\circ)_\mathbf{Q} = \{1\}$;

(A-2) $\mathbf{G}$ *has a semidirect product decomposition* $\mathbf{G} = \mathbf{LU}$, *where* $\mathbf{L}$ *is a connected reductive* $\mathbf{Q}$-*subgroup and* $\mathbf{U}$ *is a connencted normal* $\mathbf{Q}$-*subgroup with* $X(\mathbf{U}) = \{1\}$.

The group $\mathbf{G}$ always has a semi-direct product decomposition satisfying (A-2). Namely $\mathbf{G}$ is a semi-direct product of $\mathbf{U} = R_u(\mathbf{G})$, the unipotent radical, and a Levi subgroup $\mathbf{L}$. In the following we fix a decomposition $\mathbf{G} = \mathbf{LU}$ satisfying (A-2) once for all, which may not be the Levi decomposition (for concrete examples, see §3 and §5).

One of the consequences of the assumption (A-1) is the following:

Lemma 1.1 *The singular set* $\mathbf{S}$ *is a hypersurface.*

Put $\mathbf{L}_0 = \mathbf{L} \cap \mathbf{G}_0$. Then $\mathbf{L}_0$ is connected and we have $\mathbf{G}_0 = \mathbf{L}_0\mathbf{U}$ (semi-direct product).

Lemma 1.2 *The group* $X(\mathbf{L}_0)_\mathbf{Q}$ *is trivial.*

Proof. By (A-1) and [S1, Lemma 4.1], we have

$$\text{rank } X_\rho(\mathbf{G})_{\mathbb{Q}} = \text{rank } X(\mathbf{G})_{\mathbb{Q}} = \text{rank } X(\mathbf{L})_{\mathbb{Q}}.$$

This implies that

$$\text{rank } X(\mathbf{G}_0)_{\mathbb{Q}} = \text{rank } X(\mathbf{L}_0)_{\mathbb{Q}} = 0.$$

Since $\mathbf{L}_0$ is connected, the group $X(\mathbf{L}_0)_{\mathbb{Q}}$ is trivial. ∎

Let $\mathbf{T}$ be the largest $\mathbb{Q}$-split torus of the identity component of the center $Z(\mathbf{L})$ of $\mathbf{L}$. Then $\dim \mathbf{T} = \text{rank } X(\mathbf{G})_{\mathbb{Q}} = \text{rank}\, X_\rho(\mathbf{G})_{\mathbb{Q}}$ and $\mathbf{L}$ is an almost direct product of $\mathbf{T}$ and $\mathbf{L}_0$.

1.2 Let G^+, G_0^+, T^+, L_0^+ and U^+ be the identity components of the real Lie groups $\mathbf{G}_{\mathbb{R}}, \mathbf{G}_{0,\mathbb{R}}, \mathbf{T}_{\mathbb{R}}, \mathbf{L}_{0,\mathbb{R}}$ and $\mathbf{U}_{\mathbb{R}}$, respectively. Then we have

$$G^+ = T^+ L_0^+ U^+, \quad G_0^+ = L_0^+ U^+$$

and the decomposition

$$g = thu \quad (g \in G^+,\ t \in T^+,\ g \in L_0^+,\ u \in U^+)$$

is unique. By (A-2), the groups L_0^+ and U^+ are unimodular.

Let dt, dh and du be (bi-invariant) Haar measures on T^+, L_0^+ and U^+, respectively. Let $d_r g$ be a right invariant measure on G^+ and let $\Delta : G^+ \longrightarrow \mathbb{R}_+^\times$ be the module of $d_r g$. Then we can normalize these measures so that

$$d_r g = d_r(thu) = \Delta(t) dt\, dh\, du.$$

As proved in [S1, §4], the assumption (A-1) assures the existence of $\delta = (\delta_1, \ldots, \delta_n) \in \mathbb{Q}^n$, for which

$$\Omega(x) = |P(x)|^{-\delta} dx = \prod_{i=1}^{n} |P_i(x)|^{-\delta_i} dx, \quad dx = \text{the Lebesgue measure on } \mathbf{V}_{\mathbb{R}}$$

gives a relatively G^+-invariant measure on $\mathbf{V}_{\mathbb{R}} - \mathbf{S}_{\mathbb{R}}$ with multiplier Δ.

Let

$$\mathbf{V}_{\mathbb{R}} - \mathbf{S}_{\mathbb{R}} = V_1 \cup \cdots \cup V_\nu$$

be the decomposition into connected components. Each connected component V_j is a single G^+-orbit. For an $x \in V_i$, put $G_x^+ = \mathbf{G}_x \cap G^+$. By (A-1), the group G_x^+ is a unimodular Lie group. We normalize a (bi-invariant) Haar measure $d\mu_x$ on G_x^+ such that

$$\text{(1.1)} \qquad \int_{G^+} F(g) d_r g = \int_{V_i} \Omega(\rho(\dot{g})x) \int_{G_x^+} F(\dot{g}h) d\mu_x(h) \quad (F \in L^1(G^+, d_r g)).$$

1.3 Let $\phi : L_0^+ \longrightarrow W$ be a function on L_0^+ with values in a finite-dimensional complex vector space W, which is invariant under the right multiplication of some arithmetic subgroup of $\mathbf{L}_{0,\mathbf{Q}} \cap L_0^+$. Later we shall assume that ϕ is an automorphic form on L_0^+; however at the moment we do not assume it.

Now let us associate to ϕ a linear form $Z_\phi(s)$ on $\mathcal{S}(\mathbf{V_R}) \otimes \mathcal{S}(\mathbf{V_Q})$ with complex parameter s in $\mathbf{C}^n$, which we call the *zeta integral attached to* ϕ (for the definition of $\mathcal{S}(\mathbf{V_R})$ and $\mathcal{S}(\mathbf{V_Q})$, see [S5, §4]).

Consider the canonical surjection $p : \mathbf{G}_0 \longrightarrow \mathbf{L}_0 = \mathbf{G}_0/\mathbf{U}$. The map p induces a real analytic mapping

$$p : G_0^+ \longrightarrow L_0^+ = G_0^+/U^+.$$

For an arithmetic subgroup Γ of $\mathbf{G}_{0,\mathbf{Q}} \cap G_0^+$, put $\Gamma_L = p(\Gamma) \subset L_0^+$. Then Γ_L is an arithmetic subgroup of $\mathbf{L}_{0,\mathbf{Q}} \cap L_0^+$ (cf. [Bo, Theorem 6]).

For $f_\infty \otimes f_0 \in \mathcal{S}(\mathbf{V_R}) \otimes \mathcal{S}(\mathbf{V_Q})$, take an arithmetic subgroup Γ of $\mathbf{G}_{0,\mathbf{Q}} \cap G_0^+$ such that f_0 is Γ-invariant, ω is Γ_{T_0}-invariant and ϕ is Γ_L-invarinat. Then we define the zeta integral attached to ϕ and ω by setting

$$
\begin{aligned}
(1.2)\qquad Z_\phi(s)(f_\infty \otimes f_0) &= Z_\phi(s_1, \ldots, s_n)(f_\infty \otimes f_0) \\
&= \frac{1}{v(\Gamma)} \int_{T^+} \prod_{i=1}^{n} \chi_i(t)^{s_i} \Delta(t)\, dt \int_{G_0^+/\Gamma} \phi(h) \sum_{x \in \mathbf{V_Q} - \mathbf{S_Q}} f_0(x) f_\infty(\rho(thu)x)\, dh\, du,
\end{aligned}
$$

where $v(\Gamma) = \int_{\mathbf{G}_0^+/\Gamma} dh\, du$, which is finite by Lemma 1.1. Note that the integral $Z_\phi(s)$ is independent of the choice of Γ.

In the following we assume that

(A-3) *for any* $f_\infty \otimes f_0 \in \mathcal{S}(\mathbf{V_R}) \otimes \mathcal{S}(\mathbf{V_Q})$, *the integral* $Z_\phi(s)(f_\infty \otimes f_0)$ *is absolutely convergent, when* $\Re(s_1), \ldots, \Re(s_n)$ *are sufficiently large.*

In case ϕ is a constant function, the integral $Z_\phi(s)(f_\infty \otimes f_0)$ gives an integral representation of the usual zeta functions associated with $(\mathbf{G}, \rho, \mathbf{V})$ (see [S1,§4], [S5, §4], [SS, §2]). In this case some sufficient conditions for (A-3) are known by [S2, Theorem 1] and [SS, Lemmas 2.2, 2.5]. For example, we have the following criterion of convergence of $Z_\phi(s)$:

Proposition 1.3 *Assume that* $X_\rho(\mathbf{G})_\mathbf{Q} = X_\rho(\mathbf{G})_\mathbf{C}$. *If* $\mathbf{G}_{0,x} = \mathbf{G}_0 \cap \mathbf{G}_x$ $(x \in \mathbf{V} - \mathbf{S})$ *is a connected semisimple algebraic group and* $\phi : L_0^+ \longrightarrow W$ *is bounded, then* $Z_\phi(f_\infty \otimes f_0)$ $(f_\infty \otimes f_0 \in \mathcal{S}(\mathbf{V_R}) \otimes \mathcal{S}(\mathbf{V_Q}))$ *is absolutely convergent for* $\Re(s_1) > \delta_1, \ldots, \Re(s_n) > \delta_n$.

Proof. Proposition is an immediate consequence of [S2, Theorem 1] and the recent result of Kottwith [K] and Chernousov [C]. ∎

Corollary 1.4 *Assume that $X_\rho(\mathbf{G})_{\mathbf{Q}} = X_\rho(\mathbf{G})_{\mathbf{C}}$. If $\mathbf{G}_{0,x} = \mathbf{G}_0 \cap \mathbf{G}_x$ $(x \in \mathbf{V} - \mathbf{S})$ is a connected semisimple algebraic group and ϕ is a cusp form on L_0^+, then $Z_\phi(f_\infty \otimes f_0)$ $(f_\infty \otimes f_0 \in \mathcal{S}(\mathbf{V}_{\mathbf{R}}) \otimes \mathcal{S}(\mathbf{V}_{\mathbf{Q}}))$ is absolutely convergent for $\Re(s_1) > \delta_1, \ldots, \Re(s_n) > \delta_n$.*

1.4 What we must do first is to find a good condition under which the integral $Z_\phi(f_\infty \otimes f_0)$ can be decomposed into product of Dirichlet series (related only to f_0) and local zeta functions (related only to f_∞), as in the case where ϕ is a constant function.

For an $x \in \mathbf{V}_{\mathbf{Q}} - \mathbf{S}_{\mathbf{Q}}$, put $\Gamma_x = \Gamma \cap G_x^+$. By (A-1), the volume $\mu(x) = \int_{G_x^+/\Gamma_x} d\mu_x$ is finite (for $d\mu_x$, see (1)). Also put

$$\begin{aligned} L_{(x)}^+ &= p(G_x^+) \ (\subset L_0^+), & \Gamma_{(x)} &= p(\Gamma_x) \ (\subset L_{(x)}^+), \\ U_x^+ &= G_x^+ \cap U^+, & \Gamma_{U,x} &= \Gamma_x \cap U^+. \end{aligned}$$

Here we note that $G_x^+ \subset G_0^+$. We normalize Haar measures $d\nu_x$ and $d\tau_x$ on $L_{(x)}^+$ and U_x, respectively by

$$\int_{L_{(x)}^+/\Gamma_{(x)}} d\nu_x = 1 \quad \text{and} \quad \int_{U_x^+/\Gamma_{U,x}} d\tau_x = \mu(x).$$

Then we have $d\mu_x = d\nu_x \, d\tau_x$ on G_x^+.

For each connected component V_i of $\mathbf{V}_{\mathbf{R}} - \mathbf{S}_{\mathbf{R}}$, we fix a representative x_i and put $X_i = L_0^+/L_{(x_i)}^+$. For each $x \in V_i$, choose $t_x \in T^+$, $h_x \in L_0^+$ and $u_x \in U^+$ such that $x = \rho(t_x h_x u_x) x_i$. Define a mapping $^-: V_i \longrightarrow X_i$ by $x \longmapsto \bar{x} = h_x \cdot L_{(x_i)}^+ \in X_i$. The point $\bar{x}$ is independent of the choice of h_x and the mapping $^-$ defines a real analytic mapping equivariant under the action of L_0^+.

For $x \in \mathbf{V}_{\mathbf{Q}} \cap V_i$ and $y \in V_i$, set

$$\mathcal{M}_x^{(i)}\phi(\bar{y}) = \int_{L_{(x)}^+/\Gamma_{(x)}} \phi(h_y h_x^{-1} \eta) \, d\nu_x(\eta), \tag{1.3}$$

which we call the *mean value of ϕ at x*. We consider $\mathcal{M}_x^{(i)}\phi$ as a function on X_i. Now it is easy to see that the usual manipulation in the theory of p.v.'s leads to the following lemma:

Lemma 1.5 *If $\Re(s_1), \ldots, \Re(s_n)$ are sufficiently large to ensure the absolute convergence of $Z_\phi(s)(f_\infty \otimes f_0)$, then*

$$Z_\phi(s)(f_\infty \otimes f_0) = \frac{1}{v(\Gamma)} \sum_{i=1}^{\nu} \sum_{x \in \Gamma \backslash \mathbf{V}_{\mathbf{Q}} \cap V_i} \frac{\mu(x) f_0(x)}{\prod_{j=1}^{n} |P_j(x)|^{s_j}} \int_{V_i} \prod_{j=1}^{n} |P_j(y)|^{s_j} f_\infty(y) \mathcal{M}_x^{(i)}\phi(\bar{y}) \Omega(y).$$

1.5 From now on, we assume that ϕ is an automorphic form on L_0^+ with respect to some arithmetic subgroup. To be precise, let K be a maximal compact subgroup of L_0^+ and π an irreducible unitary representation of K on a finite dimensional Hilbert space W_π. Denote by $\mathcal{Z}(L_0^+)$ be the algebra of bi-invariant differential operators on L_0^+. Let $\chi; \mathcal{Z}(L_0^+) \longrightarrow \mathbb{C}$ be an infinitesimal character. Then we call a function $\phi : L_0^+ \longrightarrow W_\pi$ an *automorphic form of type* (χ, π) *with respect to* Γ_L, if it satisfies the conditions

$$\begin{array}{ll} D\phi = \chi(D)\phi & (D \in \mathcal{Z}(L_0^+)), \\ \phi(kh) = \pi(k)\phi(h) & (k \in K, h \in L_0^+), \\ \phi(h\gamma) = \phi(h) & (h \in L_0^+, \gamma \in \Gamma_L), \\ \phi \text{ is slowly increasing.} & \end{array}$$

We denote by $\mathcal{A}(L_0^+/\Gamma_L; \chi, \pi)$ the space of automorphic forms of type (χ, π) with respect to Γ_L. It is known that the dimension of $\mathcal{A}(L_0^+/\Gamma_L; \chi, \pi)$ is finite ([BJ, Theorem 1.7], [H, Theorem 1]).

Any element $D \in \mathcal{Z}(L_0^+)$ induces an L_0^+-invariant differential operator on the homogeneous space $X_i = L_0^+/L_{(x_i)}^+$, which we denote by $\overline{D}$. We call a function $\psi : X_i \longrightarrow W_\pi$ a *spherical function of type* (χ, π), if it satisfies the conditions

$$\begin{array}{rcll} \overline{D}\psi & = & \chi(D)\psi & (D \in \mathcal{Z}(L_0^+)), \\ \psi(k\bar{x}) & = & \pi(k)\psi(\bar{x}) & (k \in K, \bar{x} \in X_i). \end{array}$$

We denote by $\mathcal{E}(X_i; \chi, \pi)$ the space of spherical functions of type (χ, π) on X_i.

Lemma 1.6 *Let ϕ be an automorphic form in $\mathcal{A}(L_0^+/\Gamma_L; \chi, \pi)$. If the integral* (1.3) *converges absolutely, then the mean value $\mathcal{M}_x^{(i)}\phi$ at x is in $\mathcal{E}(X_i; \chi, \pi)$.*

Our final assumption in this section is the following:

(A-4) *the dimension of $\mathcal{E}(X_i; \chi, \pi)$ $(1 \le i \le \nu)$ is finite.*

Put $m_i = \dim \mathcal{E}(X_i; \chi, \pi)$ $(1 \le i \le \nu)$ and take a basis $\left\{\psi_1^{(i)}, \ldots, \psi_{m_i}^{(i)}\right\}$ of $\mathcal{E}(X_i; \chi, \pi)$. By Lemma 1.4, we can express $\mathcal{M}_x^{(i)}\phi$ as a linear combination of $\psi_1^{(i)}, \ldots, \psi_{m_i}^{(i)}$:

$$\mathcal{M}_x^{(i)}\phi = \sum_{l=1}^{m_i} c_l^{(i)}(\phi; x)\, \psi_l^{(i)}. \tag{1.4}$$

The coefficients $c_l^{(i)}(\phi; x)$ can be viewed as functions of x on $\Gamma\backslash \mathbf{V}_{\mathbb{Q}} \cap V_i$.

We define (global) zeta functions $\zeta_l^{(i)}(\phi, f_0; s)$ and local zeta functions $\Phi_l^{(i)}(f_\infty; \pi, \chi, s)$ by

$$\begin{aligned}
\zeta_l^{(i)}(\phi, f_0; s) &= \frac{1}{v(\Gamma)} \sum_{x \in \Gamma \backslash \mathbf{V}_{\mathbb{Q}} \cap V_i} \frac{m u(x) f_0(x) c_l^{(i)}(\phi; x)}{\prod_{j=1}^n |P_j(x)|^{s_j}}, \\
\Phi_l^{(i)}(f_\infty; \pi, \chi, s) &= \int_{V_i} \prod_{j=1}^n |P_j(y)|^{s_j} \psi_l^{(i)}(\bar{y}) f_\infty(y) \Omega(y) \\
& \qquad (1 \le i \le \nu, 1 \le l \le m_i).
\end{aligned}$$

The zeta functions $\zeta_l^{(i)}(\phi, f_0; s)$ are independent of the choice of Γ. By Lemmas 1.5 and 1.6 and the identity (1.4), we easily obtain the following:

Proposition 1.7 *Assume that* $(\mathbf{G}, \rho, \mathbf{V})$ *satisfies* (A-1) – (A-4). *Then the following identity holds for sufficiently large* $\Re(s_1), \ldots, \Re(s_n)$:

$$Z_\phi(s)(f_\infty \otimes f_0) = \sum_{i=1}^{\nu} \sum_{l=1}^{m_i} \zeta_l^{(i)}(\phi, f_0; s)\, \Phi_l^{(i)}(f_\infty; \pi, \chi, s).$$

Remark. The coefficients $c_l^{(i)}(\phi; x)$ can be expressed as a linear combination of functions of x of the form $(\mathcal{M}_x^{(i)}\phi(\bar{y}_t), e_s)$, where $\{\bar{y}_t\}$ are a finite number of points in X_i, $\{e_s\}$ is an orthonormal basis of W and (,) is the inner product on W. Thus the coefficients of our zeta functions are, roughly speaking, mean values (or periods) of automorphic forms.

The simplest case where the assumption (A-4) is satisfied is the following:

The case of Größencharacters – ϕ is a unitary character of L_0^+.

It is known that (A-4) holds also in the following two cases:

Compact Case – L_0^+ is a compact Lie group (by the theorem of Peter-Weyl);

Symmetric Case – X_i $(1 \le i \le \nu)$ are reductive symmetric space (by a theorem of van den Ban, see [B1,Cor. 3.10], [B2, Lemma 2.1]).

In Compact case, the zeta functions $\zeta_l^{(i)}(\phi, f_0; s)$ have been studied in detail in [S6] and we obtained the functional equations satisfied by $\zeta_l^{(i)}(\phi, f_0; s)$ (for concrete examples, see also [S4] and [S7]). Therefore, in the subsequent sections, we consider exclusively Symmetric case.

§2 Functional equation of the zeta integral

Recall that, in the theory of p.v.'s developed in [SS] and [S1], the proof of the existence of analytic continuations and functional equations of zeta functions is based on the following three properties:

1. Analytic continuation and the functional equation of the zeta integral;

2. Functional equations satisfied by local zeta functions;

3. The existence of b-functions (the Bernstein-Sato polynomials), which controles the singularities of zeta functions and the gamma-factor of functional equations. Moreover, by using the b-functions, one can eliminate the troublesome contribution of rational points in the singular set to the zeta integral (cf. Lemma 2.2).

We must extend these three properties to our general situation. The easiest part is the functional equation of the zeta integral, which we describe here.

We keep the notation in §1 and assume the conditions (A-1), (A-2) and (A-3). It is not necessary in the present section to assume (A-4). Instead we assume that

(A-5) $(\mathbf{G},\rho,\mathbf{V})$ *is decomposed over* $\mathbb{Q}$ *into direct product as*

$$(\mathbf{G},\rho,\mathbf{V})=(\mathbf{G},\rho_1\oplus\rho_2,\mathbf{E}\oplus\mathbf{F})$$

and the invariant subspace $\mathbf{F}$ *is a regular subspace.*

For the definition and elementary properties of regular subspaces, we refer to [S1, §2]. Note that, in [S1], we have introduced the notion of k-regularity, where k is the field of definition. However the $\bar{k}$-regularity implies the k-regularity (cf. [S6, §2.1]). Hence in the assumption (A-5), $\mathbf{F}$ is necessarily a $\mathbb{Q}$-regular subspace.

Let $\mathbf{F}^*$ be the vector space dual to $\mathbf{F}$ and ρ_2^* the rational representation of $\mathbf{G}$ on $\mathbf{F}^*$ contragredient to ρ_2. Set $(\mathbf{G},\rho^*,\mathbf{V}^*)=(\mathbf{G},\rho_1\oplus\rho_2^*,\mathbf{E}\oplus\mathbf{F}^*)$. The assumption (A-5) implies that $(\mathbf{G},\rho^*,\mathbf{V}^*)$ is also a p.v. defined over $\mathbb{Q}$ and $\mathbf{F}^*$ is its regular subspace. By Lemma 2.4 in [S1], the assumption (A-1) holds also for $(\mathbf{G},\rho^*,\mathbf{V}^*)$. Let $\mathbf{S}^*$ be the singular set of $(\mathbf{G},\rho^*,\mathbf{V}^*)$. Let $P_1^*,\dots,P_n^*$ be the basic relative invariants of $(\mathbf{G},\rho^*,\mathbf{V}^*)$ over $\mathbb{Q}$. Note that the number of basic relative invariants of $(\mathbf{G},\rho^*,\mathbf{V}^*)$ is equal to n, the number of basic relative invariants of $(\mathbf{G},\rho,\mathbf{V})$. Let χ_i^* be the $\mathbb{Q}$-rational character of $\mathbf{G}$ corresponding to P_i^*:

$$P_i^*(\rho^*(g)x^*)=\chi_i^*(g)P_i^*(x^*)\qquad(g\in\mathbf{G},\ x^*\in\mathbf{V}^*).$$

Let $X_{\rho^*}(\mathbf{G})_{\mathbb{Q}}$ be the subgroup of $X(\mathbf{G})_{\mathbb{Q}}$ generated by $\chi_1^*,\dots,\chi_n^*$. Since $X_\rho(\mathbf{G})_{\mathbb{Q}}=X_{\rho^*}(\mathbf{G})_{\mathbb{Q}}$, there exists an n by n unimodular matrix $U=(u_{ij})_{i,j=1}^n$ such that

$$\chi_i=\prod_{i=1}^{n}\chi_j^{*u_{ij}}\qquad(1\le i\le n).\tag{2.1}$$

Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be an n-tuple of half-integers such that

$$(2.2) \qquad (\det \rho_2(g))^2 = \prod_{i=1}^{n} \chi_i(g)^{2\lambda_i}.$$

(for the existence of λ, see [S1, Lemma 2.5]).

Let the function $\phi : L_0^+/\Gamma_L \longrightarrow W$ be the same as in §1.3. Then, as in (1.2), we can define the zeta integral attached to ϕ also for $(\mathbf{G}, \rho^*, \mathbf{V}^*)$:

$$\begin{aligned} Z_\phi^*(s)(f_\infty^* \otimes f_0^*) &= Z_\phi^*(s_1, \ldots, s_n)(f_\infty^* \otimes f_0^*) \\ &= \frac{1}{v(\Gamma)} \int_{T^+} \prod_{i=1}^{n} \chi_i^*(t)^{s_i} \Delta(t)\, dt \int_{G_0^+/\Gamma} \phi(h) \sum_{x^* \in \mathbf{V}_{\mathbf{Q}}^* - \mathbf{S}_{\mathbf{Q}}^*} f_0^*(x^*) f_\infty^*(\rho(thu)x^*)\, dh\, du \end{aligned}$$
$(f_\infty^* \otimes f_0^* \in \mathcal{S}(\mathbf{V}_{\mathbf{R}}^*) \otimes \mathcal{S}(\mathbf{V}_{\mathbf{Q}}^*))$.

Now let us recall the Poisson summation formula for functions in $f_\infty \otimes f_0 \in \mathcal{S}(\mathbf{V}_{\mathbf{R}}) \otimes \mathcal{S}(\mathbf{V}_{\mathbf{Q}})$. For $f_0 \in \mathcal{S}(\mathbf{V}_{\mathbf{Q}})$ and $x_2^* \in \mathbf{F}_{\mathbf{Q}}^*$, take a lattice $\mathcal{L}$ in $\mathbf{F}_{\mathbf{Q}}^*$ for which the value of $f_0(x_1, x_2)$ $(x_1 \in \mathbf{E}_{\mathbf{Q}}, x_2 \in \mathbf{F}_{\mathbf{Q}})$ is determined by the coset of x_2 modulo $\mathcal{L}$ and x_2^* is in the dual lattice

$$\mathcal{L}^* = \{x_2^* \in \mathbf{F}_{\mathbf{Q}}^* \mid \langle x_2^*, \mathcal{L} \rangle \subset \mathbb{Z}\}.$$

Put

$$\widehat{f_0}(x_1, x_2^*) = v(\mathcal{L})^{-1} \sum_{x_2 \in \mathbf{F}_{\mathbf{Q}}/\mathcal{L}} f_0(x_1, x_2)\, e^{2\pi i \langle x_2, x_2^* \rangle},$$

where $v(\mathcal{L}) = \int_{\mathbf{F}_{\mathbf{R}}/\mathcal{L}} dx_2$. Then $\widehat{f_0}(x_1, x_2^*)$ is independent of the choice of $\mathcal{L}$ and defines a function in $\mathcal{S}(\mathbf{V}_{\mathbf{Q}}^*)$. The function $\widehat{f_0}$ is called the *partial Fourier transform of* f_0 *with respect to* $\mathbf{F}$.

We define the partial Fourier transform $\widehat{f_\infty} \in \mathcal{S}(\mathbf{V}_{\mathbf{Q}}^*)$ of $f_\infty \in \mathcal{S}(\mathbf{V}_{\mathbf{Q}})$ with respect to $\mathbf{F}$ by setting

$$\widehat{f_\infty}(x_1, x_2^*) = \int_{\mathbf{F}_{\mathbf{R}}} f_\infty(x_1, x_2)\, e^{-2\pi i \langle x_2, x_2^* \rangle}\, dx_2.$$

Then the partial Fourier transforms

$$\widehat{\ }: \mathcal{S}(\mathbf{V}_{\mathbf{Q}}) \longrightarrow \mathcal{S}(\mathbf{V}_{\mathbf{Q}}^*) \quad \text{and} \quad \widehat{\ }: \mathcal{S}(\mathbf{V}_{\mathbf{R}}) \longrightarrow \mathcal{S}(\mathbf{V}_{\mathbf{R}}^*)$$

are linear isomorphisms and the following Poisson summation formula holds:

$$(2.3) \qquad \begin{aligned} &\sum_{(x_1, x_2) \in \mathbf{V}_{\mathbf{Q}}} f_0(x_1, x_2)\, f_\infty(\rho(g)(x_1, x_2)) \\ &= \det \rho_2(g)^{-1} \sum_{(x_1, x_2^*) \in \mathbf{V}_{\mathbf{Q}}^*} \widehat{f_0}(x_1, x_2^*)\, \widehat{f_\infty}(\rho^*(g)(x_1, x_2^*)) \end{aligned}$$
$$(f_\infty \otimes f_0 \in \mathcal{S}(\mathbf{V}_{\mathbf{R}}) \otimes \mathcal{S}(\mathbf{V}_{\mathbf{Q}}), g \in G^+).$$

Let B (resp. B^*) be the domain in $\mathbb{C}^n$ on which $Z_\phi(s)(f_\infty \otimes f_0)$ (resp. $Z_\phi^*(s)(f_\infty^* \otimes f_0^*)$) converges absolutely. Denote by D (resp. D^*) be the convex hull of $(B^*U^{-1}+\lambda)\cup B$ (resp. $(B-\lambda)U \cup B^*$) in $\mathbb{C}^n$. Then it is clear that $(D-\lambda)U = D^*$.

Proposition 2.1 *Let $f_\infty \in \mathcal{S}(\mathbf{V}_\mathbb{R})$ be a function satisfying that f_∞ and $\widehat{f_\infty}$ vanish identically on $\mathbf{S}_\mathbb{R}$ and $\mathbf{S}_\mathbb{R}^*$, respectively. Then $Z_\phi(s)(f_\infty \otimes f_0)$ and $Z_\phi^*(s)(\widehat{f_\infty} \otimes \widehat{f_0})$ have analytic continuations to holomorphic functions on D and D^*, respectively, and satisfy the functional equation*

$$Z_\phi^*((s-\lambda)U)(\widehat{f_\infty} \otimes \widehat{f_0}) = Z_\phi(s)(f_\infty \otimes f_0) \qquad (s \in D).$$

The proof of Proposition 2.1, which is based on (2.3), is quite similar to that of [S1, Lemma 6.1] and we do not reproduce it here.

For the later use, we recall the construction of functions f_∞ satisfying the assumption in Proposition 2.1. Let $r = n - \operatorname{rank} X_{\rho_1}(\mathbf{G})_\mathbb{Q}$. Then, among the basic relative invariants $P_1, \ldots, P_n$ of $(\mathbf{G}, \rho, \mathbf{V})$ (resp. $P_1^*, \ldots, P_n^*$ of $(\mathbf{G}, \rho^*, \mathbf{V}^*)$) over $\mathbb{Q}$, there exist precisely $n-r$ relative invariants which are constant as functions of x_2 on $\mathbf{F}$ (resp. x_2^* on $\mathbf{F}^*$). Hence we may assume that

$$P_i(x_1, x_2) = P_i^*(x_1, x_2^*) = P_i(x_1) \qquad (i = r+1, \ldots, n).$$

These $P_i(x_1)$ $(r+1 \le i \le n)$ are the basic relative invariants of $(\mathbf{G}, \rho_1, \mathbf{E})$ over $\mathbb{Q}$. We put

$$P_F(x_1, x_2) = \prod_{i=1}^{r} P_i(x_1, x_2) \quad \text{and} \quad P_F^*(x_1, x_2^*) = \prod_{i=1}^{r} P_i^*(x_1, x_2^*).$$

Lemma 2.2 ([S1, Lemma 6.2]) (i) *For an $f_\infty'^* \in C_0^\infty(\mathbf{V}_\mathbb{R}^* - \mathbf{S}_\mathbb{R}^*)$, put*

$$f_\infty = P_F(x_1, x_2) \cdot \widehat{f_\infty'^*}(x_1, x_2).$$

Then f_∞ and $\widehat{f_\infty}$ vanish on $\mathbf{S}_\mathbb{R}$ and $\mathbf{S}_\mathbb{R}^*$, respectively.

(ii) *For an $f_\infty' \in C_0^\infty(\mathbf{V}_\mathbb{R} - \mathbf{S}_\mathbb{R})$, put*

$$f_\infty = P_F^*(x_1, \frac{\partial}{\partial x_2}) \cdot f_\infty'(x_1, x_2).$$

Then f_∞ and $\widehat{f_\infty}$ vanish on $\mathbf{S}_\mathbb{R}$ and $\mathbf{S}_\mathbb{R}^*$, respectively.

§3 Prehomogeneous vector spaces with symmetric structure

3.1 In this section, we keep the notation in §1 and assume the conditions (A-1) and (A-2). As in §1.3, let $p : \mathbf{G}_0 \longrightarrow \mathbf{L}_0$ be the canonical surjection and put $\mathbf{L}_{(x)} = p(\mathbf{G}_x \cap \mathbf{G}_0)$ for $x \in \mathbf{V} - \mathbf{S}$.

We call the semi-direct product decomposition $\mathbf{G} = \mathbf{L}\mathbf{U}$ determines a *symmetric structure* on $(\mathbf{G}, \rho, \mathbf{V})$ over $\mathbb{Q}$, if for any $x \in \mathbf{V}_{\mathbb{Q}} - \mathbf{S}_{\mathbb{Q}}$, there exists an involution (= an automorphism of order 2) $\sigma : \mathbf{L}_0 \longrightarrow \mathbf{L}_0$ defined over $\mathbb{Q}$ such that

$$\mathbf{L}_0^\sigma := \{h \in \mathbf{L}_0 \mid \sigma(h) = h\} \supset \mathbf{L}_{(x)} \supset (\mathbf{L}_0^\sigma)^\circ . \tag{3.1}$$

Then, for any $x \in \mathbf{V}_{\mathbb{R}} - \mathbf{S}_{\mathbb{R}}$, there exists an involution σ of $\mathbf{L}_0$ defined over $\mathbb{R}$ satisfying (3.1). The involution σ induces an involution of L_0^+, which we denote also by σ, satisfying

$$(L_0^+)^\sigma \supset L_0^+ \cap \mathbf{L}_{(x)} = L_{(x)}^+ \supset \left(L_0^{+\sigma}\right)^\circ .$$

Therefore the homogeneous spaces X_i $(1 \le i \le \nu)$ defined in § 1.4 are reductive symmetric spaces and the construction of zeta functions given in §1 can be applied to $(\mathbf{G}, \rho, \mathbf{V})$ with symmetric structure.

Lemma 3.1 *Suppose that* $(\mathbf{G}, \rho, \mathbf{V})$ *satisfies the condition* (A-5) *in* §2, *namely,* $\mathbf{V}$ *contains a regular subspace* $\mathbf{F}$. *Then the decomposition* $\mathbf{G} = \mathbf{L}\mathbf{U}$ *determines a symmetric structure also on* $(\mathbf{G}, \rho^*, \mathbf{V}^*)$, *the p.v. dual to* $(\mathbf{G}, \rho, \mathbf{V})$ *with respect to* $\mathbf{F}$.

Proof. By (A-5), one can find a relative invariant P of $(\mathbf{G}, \rho, \mathbf{V})$ with coefficients in $\mathbb{Q}$ for which the rational mapping $\phi_P : \mathbf{V} - \mathbf{S} \longrightarrow \mathbf{V}^*$ defined by

$$\phi_P(x_1, x_2) = \left(x_1, \operatorname{grad}_{x_2}(\log P(x_1, x_2))\right)$$

gives rise to a $\mathbf{G}$-equivariant biregular mapping of $\mathbf{V} - \mathbf{S}$ onto $\mathbf{V}^* - \mathbf{S}^*$ defined over $\mathbb{Q}$. For $x \in \mathbf{V}_{\mathbb{Q}} - \mathbf{S}_{\mathbb{Q}}$, put $x^* = \phi_P(x) \in \mathbf{V}^*_{\mathbb{Q}} - \mathbf{S}^*_{\mathbb{Q}}$. Then we have $\mathbf{G}_x = \mathbf{G}_{x^*}$ and $\mathbf{L}_{(x)} = \mathbf{L}_{(x^*)}$ (cf. [S1, Lemma 2.4]). Now the assertion is obvious. ∎

3.2 Let $\mathbf{P}_L$ be a parabolic subgroup of $\mathbf{L}$ and put $\mathbf{P} = \mathbf{P}_L\mathbf{U}$. We denote the restriction of the representation ρ to $\mathbf{P}$ by the same symbol ρ. We do not assume that $\mathbf{P}_L$ is defined over $\mathbb{Q}$. In fact, in §4, we need to consider a parabolic subgroup defined over $\mathbb{R}$.

Lemma 3.2 *Suppose that* $\mathbf{G} = \mathbf{L}\mathbf{U}$ *determines a symmetric structure of* $(\mathbf{G}, \rho, \mathbf{V})$. *Then*
(i) $(\mathbf{P}, \rho, \mathbf{V})$ *is also a p.v.*
(ii) *If* $(\mathbf{G}, \rho, \mathbf{V})$ *is regular, so is* $(\mathbf{P}, \rho, \mathbf{V})$.

Proof. Let $\mathbf{V}_1 = \{x \in \mathbf{V} \mid P_1(x) = \cdots = P_n(x) = 1\}$. Then $\mathbf{V}_1$ is a single $\rho(\mathbf{G}_0)$-orbit (cf. [S6, Lemma 1.1]). Fix a point $x_0 \in \mathbf{V}_1$. The mapping $\beta : \mathbf{V}_1 \longrightarrow \mathbf{L}_0/\mathbf{L}_{(x_0)}$ defined by $\beta(\rho(hu)x_0) = h \cdot \mathbf{L}_{(x_0)}$ $(h \in \mathbf{L}_0, u \in \mathbf{U})$ is clearly $\mathbf{L}_0$-equivaraint. Since $\mathbf{L}_0/\mathbf{L}_{(x_0)}$ is a symmetric space and $\mathbf{P}_{L_0} = \mathbf{P}_L \cap \mathbf{L}_0$ is a parabolic subgroup of $\mathbf{L}_0$, there exists a Zariski-open $\mathbf{P}_{L_0}$-orbit Ω_0 is $\mathbf{L}_0/\mathbf{L}_{(x_0)}$ (see [V, §1]). Then $\Omega = \rho(\mathbf{T})\beta^{-1}(\Omega_0)$ is a Zariski-open $\mathbf{P}$-orbit in $\mathbf{V}$. Hence $(\mathbf{P}, \rho, \mathbf{V})$ is a p.v. The second assertion is obvious. ∎

We denote by $\mathbf{S}_P$ the singular set of $(\mathbf{P}, \rho, \mathbf{V})$. It is obvious that $\mathbf{S}_P \supset \mathbf{S}$. Recall that the parabolic subgroup $\mathbf{P}_{L_0} = \mathbf{P} \cap \mathbf{L}_0$ of $\mathbf{L}_0$ is called *σ-anisotropic* for an involution σ of $\mathbf{L}_0$ if $\mathbf{P}_{L_0} \cap \sigma(\mathbf{P}_{L_0})$ is a Levi subgroup of $\mathbf{P}_{L_0}$ (cf. [V, §1]).

Lemma 3.3 *Suppose that $\mathbf{G} = \mathbf{LU}$ determines a symmetric structure of $(\mathbf{G}, \rho, \mathbf{V})$ and $\mathbf{P}_{L_0} = \mathbf{P}_L \cap \mathbf{L}_0$ is σ-anisotropic for the involution σ corresponding to some $x_0 \in \mathbf{V} - \mathbf{S}$. Then*

(i) *the point x_0 is in $\mathbf{V} - \mathbf{S}_P$.*

(ii) *For $x \in \mathbf{V} - \mathbf{S}_P$, the isotropy subgroup $\mathbf{P}_x = \{p \in \mathbf{P} \mid \rho(p)x = x\}$ is (not necessarily connected) reductive.*

(iii) *The singular set $\mathbf{S}_P$ is a hypersurface.*

Proof. We use the notation in the proof of Lemma 3.2. By replacing x_0 by $\rho(t)x_0$ $(t \in \mathbf{T})$ if necessary, we may assume that $x_0 \in \mathbf{V}_1$. By [V, Theorem 1] and the assumption that $\mathbf{P}_{L_0}$ is σ-anisotropic, we see that $\beta(x_0)$ is in Ω_0. This implies the first assertion. To prove the second assertion, it is sufficient to consider the case where $x = x_0$. Since the identity component of $\mathbf{P}_{x_0}$ coincides with that of $(\mathbf{P}_{L_0} \cdot \mathbf{U})_{x_0} = \mathbf{P}_{x_0} \cap (\mathbf{P}_{L_0} \cdot \mathbf{U})$, we prove that $(\mathbf{P}_{L_0} \cdot \mathbf{U})_{x_0}$ is reductive. It is obvious that $(\mathbf{P}_{L_0} \cdot \mathbf{U})_{x_0}$ is the semi-direct product of $\mathbf{P}_{L_0} \cap \mathbf{L}_{(x_0)}$ and $\mathbf{U}_{x_0} = \mathbf{U} \cap \mathbf{G}_{x_0}$. By (A-1), $\mathbf{G}_{x_0}$ is reductive; hence its normal subgroup $\mathbf{U}_{x_0}$ is reductive. Put $\mathbf{L'}_0 = \mathbf{P}_{L_0} \cap \sigma(\mathbf{P}_{L_0})$. By the assumption, $\mathbf{L'}_0$ is a Levi subgroup of $\mathbf{P}_{L_0}$. The group $\mathbf{P}_{L_0} \cap \mathbf{L}_{(x_0)}$ is reductive. This proves the second part. The third assertion is an immediate consequence of the second. ∎

3.3 Let the assumption be as in Lemma 3.3. Take a field k such that $x_0 \in \mathbf{V}_k - \mathbf{S}_k$, $\mathbf{P}_{L_0}$ and the involution σ are defined over k. We examine the group $X_\rho(\mathbf{P})_k$ of k-rational characters corresponding to relative invariants of $(\mathbf{P}, \rho, \mathbf{V})$.

For simplicity, we assume that

(A-6) *the basic relative invariants $P_1, \ldots, P_n$ of $(\mathbf{G}, \rho, \mathbf{V})$ over $\mathbf{Q}$ are absolutely irreducible.*

This is equivalent to the condition

$$X_\rho(\mathbf{G})_{\mathbf{Q}} = X_\rho(\mathbf{G})_{\mathbf{C}}.$$

Let $\mathbf{T}'_0$ be the identity component of the center of $\mathbf{L}'_0$. The central torus $\mathbf{T}'_0$ is σ-stable. Hence we get a separable isogeny $\mathbf{T}'_{0,+} \times \mathbf{T}'_{0-} \to \mathbf{T}'_0$, where

$$\mathbf{T}'_{0+} = \{t \in \mathbf{T}'_0 \mid \sigma(t) = t\}^\circ \qquad \text{and} \qquad \mathbf{T}'_{0-} = \left\{t \in \mathbf{T}'_0 \;\middle|\; \sigma(t) = t^{-1}\right\}^\circ .$$

We consider the following commutative diagram of the natural mappings:

$$\begin{array}{ccc} X(\mathbf{P})_k & \hookrightarrow & X(\mathbf{P})_k \oplus X(\mathbf{T}'_0)_k \\ \uparrow & & \big\downarrow \text{restriction to } \mathbf{T} \times \mathbf{T}'_{0-} \\ X_\rho(\mathbf{P})_k & \xrightarrow{\xi} & X(\mathbf{T})_k \oplus X(\mathbf{T}'_{0-})_k . \end{array}$$

Here note that $X(\mathbf{T})_k = X(\mathbf{T})_{\mathbb{Q}}$, since $\mathbf{T}$ is a $\mathbb{Q}$-split torus.

Lemma 3.4 *The homomorphism $\xi : X_\rho(\mathbf{P})_k \longrightarrow X(\mathbf{T})_k \oplus X(\mathbf{T}'_{0-})_k$ is injective and of finite cokernel.*

Proof. Any character χ in $X_\rho(\mathbf{P})_k$ is trivial on $\mathbf{P}_{x_0}\mathbf{U}$. The group $\mathbf{P}_{x_0}\mathbf{U}$ contains $(\mathbf{P}_{x_0} \cap \mathbf{L}_{(x_0)})\mathbf{U}$ and hence $((\mathbf{L}'_0)^\sigma)^\circ\mathbf{U}$. Since $\mathbf{T}'_{0+}$ is a subgroup of $((\mathbf{L}'_0)^\sigma)^\circ$, χ is trivial on $\mathbf{T}'_{0+}$. This implies that ξ is injective. As we have already seen in §1, $\xi(X_\rho(\mathbf{G})_k) = \xi(X_\rho(\mathbf{G})_{\mathbb{Q}})$ is of finite index in $X(\mathbf{T})_k$ $(= X(\mathbf{T})_{\mathbb{Q}})$. Let χ be a k-rational character of $\mathbf{T}'_{0-}$. Then, for some integer e_1, χ^{e_1} can be extended to a k-rational character of $\mathbf{P}$ such that $\ker \chi^{e_1}$ contains $\mathbf{T}\mathbf{T}'_{0+}D(\mathbf{L}'_0)\mathbf{U}'$, where $D(\mathbf{L}'_0)$ is the derived group of $\mathbf{L}'_0$. Since $(\mathbf{P}_x)^\circ$ is contained in $\mathbf{T}\mathbf{T}'_{0+}D(\mathbf{L}'_0)\mathbf{U}'$, there exists an integer e such that χ^e is trivial on $\mathbf{P}_x$. This implies that $\chi^e \in X_\rho(\mathbf{P})_k$. Therefore $\xi(X_\rho(\mathbf{P})_k)$ is of finite index in $X(\mathbf{T})_k \oplus X(\mathbf{T}'_{0-})_k$. ∎

Let $P_1, \ldots, P_n, P_{n+1}, \ldots, P_{n+l}$ be the basic relative invariants of $(\mathbf{P}, \rho, \mathbf{V})$ over k, where $P_1, \ldots, P_n$ are the basic relative invariants of $(\mathbf{G}, \rho, \mathbf{V})$. We have $l = \operatorname{rank} X(\mathbf{T}'_{0-})_k$ by Lemma 3.4.

Let $\chi_{n+1}, \ldots, \chi_{n+l}$ be the k-rational characters correponding to $P_{n+1}, \ldots, P_{n+l}$, respectively. Take a positive integer e such that $(\chi_i^e|_{\mathbf{T}})$ $(n+1 \le i \le n+l)$ are in $\xi(X_\rho(\mathbf{G})_k)$. Then one can find $m_{ij} \in e^{-1}\mathbb{Z}$ $(1 \le i \le n, 1 \le j \le l)$ such that

$$\chi_{n+j}^e \Big/ \prod_{i=1}^n \chi_i^{em_{ij}} \equiv 1 \qquad \text{on } \mathbf{T}. \tag{3.2}$$

These m_{ij} will play a role in the algebraic construction of the Poisson kernel in §4.

§4 Functional equations — The case of symmetric structure of K_ϵ-type

Let $(\mathbf{G}, \rho, \mathbf{V})$ be a p.v. with symmetric structure $\mathbf{G} = \mathbf{L} \cdot \mathbf{U}$ satisfying the assumptions (A-1), (A-3), (A-5) and (A-6). The assumption (A-2) is automatically satisfied. In this section we prove the functional equation satisfied by zeta functions attached to automorphic forms under the following assumption:

(A-7) $\mathbf{L}_0$ *is semisimple and the symmetric spaces* $X_i = L_0^+/L_{(x_i)}^+$ $(1 \leq i \leq \nu)$ *are* K_ϵ*-space in the sense of* [OS].

4.1 Let K be a maximal compact subgroup of L_0^+ and θ the corresponding Cartan involution. Let P_0 be a minimal parabolic subgroup of L_0^+ with Langlands decomposition $P_0 = MAN$ with respect to θ. Denote by $\mathfrak{L}_0, \mathfrak{m}$ and $\mathfrak{a}$ the Lie algebras of L_0^+, M and A, respectively. Let Σ $(\subset \mathfrak{a}^*)$ be the set of restricted roots and Σ^+ the set of positive restricted roots corresponding to P_0. Put $\mathfrak{L}_0^\alpha = \{X \in \mathfrak{L}_0 \mid [H, X] = \alpha(H)X\}$ for $\alpha \in \Sigma$.

Following [OS], we call a mapping $\epsilon : \Sigma \to \{\pm 1\}$ a *signature of roots*, if it satisfies the condition

$$\begin{aligned} &\epsilon(\alpha) = \epsilon(-\alpha) && (\alpha \in \Sigma), \\ &\epsilon(\alpha+\beta) = \epsilon(\alpha)\epsilon(\beta) && \text{if } \alpha, \beta \in \Sigma \text{ and } \alpha+\beta \in \Sigma. \end{aligned}$$

For a signature of roots ϵ, define an involution θ_ϵ of $\mathfrak{L}_0$ by

$$\begin{aligned} &\theta_\epsilon(X) = \epsilon(-\alpha)\theta(X) && X \in \mathfrak{L}_0^\alpha, \alpha \in \Sigma, \\ &\theta_\epsilon(X) = \theta(X) && X \in \mathfrak{m} + \mathfrak{a}. \end{aligned}$$

Then a precise formulation of the condition (A-7) is as follows:

(A-7)$'$ *for each* $i = 1, \ldots, \nu$, *there exists a representative* $x_i \in \mathbf{V}_i$ *and a signature of roots* ϵ_i *such that* $L_{(x_i)}^+ = M \cdot K_{\epsilon_i}^\circ$, *where* $K_{\epsilon_i}^\circ$ *is the analytic subgroup of* L_0^+ *with the Lie algebra*

$$\mathfrak{k}_{\epsilon_i} = \{X \in \mathfrak{L}_0 \mid \theta_{\epsilon_i}(X) = X\}.$$

In this case, one can apply the results in [OS] to the homogeneous spaces $X_i = L_0^+/L_{(x_i)}^+$.

Let $W = N_K(A)/Z_K(A)$ be the Weyl gorup. Note that $M = Z_K(A)$. Define a subgroup $W^{(i)}$ of W by $W^{(i)} = (L_{(x_i)}^+ \cap N_K(A))/M$. Put $r_i = [W : W^{(i)}]$ and fix a complete system $\{w_1^{(i)}, \ldots, w_{r_i}^{(i)}\}$ of representatives of $W/W^{(i)}$. Then, by [OS, Proposition 1.10] (or by [Mat]), the set

$$\bigcup_{j=1}^{r_i} ANw_j^{(i)} L_{(x_i)}^+ = \bigcup_{j=1}^{r_i} P_0 w_j^{(i)} L_{(x_i)}^+ \qquad \text{(disjoint union)} \tag{4.1}$$

is an open dense subset of L_0^+.

Let $\mathbf{P}_{L_0}$ be a minimal $\mathbb{R}$-parabolic subgroup of $\mathbf{L}_0$ such that $\mathbf{P}_{L_0,\mathbb{R}} \cap L_0^+ = P_0$. The parabolic subgroup $\mathbf{P}_{L_0}$ is θ_ϵ-anisotropic. Put

$$\mathbf{P} = \mathbf{T}\mathbf{P}_{L_0}\mathbf{U} \quad \text{and} \quad P^+ = T^+ P_0 U^+.$$

Note that P^+ is not necessarily connected. Using the notation in §3.3, we have $\mathbf{T}_0' = \mathbf{T}_{0,-}'$ and $\mathbf{T}_{0,+}' = \{1\}$.

As in §3.2, let $\mathbf{S}_P$ the singular set of $(\mathbf{P}, \rho, \mathbf{V})$. Then it follows from (4.1) that the P^+-orbit decomposition of $\mathbf{V}_{\mathbb{R}} - \mathbf{S}_{P,\mathbb{R}}$ is given by

$$\mathbf{V}_{\mathbb{R}} - \mathbf{S}_{P,\mathbb{R}} = \bigcup_{i=1}^{\nu} \bigcup_{j=1}^{r_i} V_{ij}, \quad V_{ij} = \rho(P^+)x_{ij}, \quad x_{ij} = \rho(w_j^{(i)})x_i.$$

Let $P_1, \ldots, P_n, P_{n+1}, \ldots, P_{n+l}$ be the basic relative invariants of $(\mathbf{P}, \rho, \mathbf{V})$ over $\mathbb{R}$. As in §3.3, $P_1, \ldots, P_n$ are the basic relative invariants of $(\mathbf{G}, \rho, \mathbf{V})$. We have $l = \dim A$ in the present case. Let m_{ij} $(1 \leq i \leq n, 1 \leq j \leq l)$ be the rational numbers given by (3.2). Then the function

$$|p_j(x)| = |P_{n+j}(x)| \Big/ \prod_{i=1}^{n} |P_i(x)|^{m_{ij}} \qquad (i \leq j \leq l)$$

on $\mathbf{V}_{\mathbb{R}} - \mathbf{S}_{\mathbb{R}}$ satisfies that

$$|p_j(\rho(tmanu)x)| = \chi_{n+j}(a)\,|p_j(x)| \quad (t \in T^+, m \in M, a \in A, n \in N, u \in U^+, x \in \mathbf{V}_{\mathbb{R}} - \mathbf{S}_{\mathbb{R}}).$$

This implies that $|p_j|$ defines a function $\left|\overline{p}_j\right|$ on X_i:

$$\begin{tikzcd} V_i \arrow[rr, "|p_j|"] \arrow[dr, "-"'] & & \mathbb{R}_+^\times \\ & X_i \arrow[ur, "|\overline{p}_j|"'] & \end{tikzcd}.$$

By Lemma 3.4, $\{\chi_{n+1}, \ldots, \chi_{n+l}\}$ gives a basis of $X(\mathbf{T}_0')_{\mathbb{R}} \otimes \mathbb{C}$. We can identify $X(\mathbf{T}_0')_{\mathbb{R}} \otimes \mathbb{C}$ with $\mathfrak{a}_{\mathbb{C}}^* = \mathfrak{a}^* \otimes_{\mathbb{R}} \mathbb{C}$ by $X(\mathbf{T}_0')_{\mathbb{R}} \ni \chi \longmapsto \log(\chi \circ \exp) \in \mathfrak{a}^*$. For $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$, write $\lambda = \sum_{j=1}^{l} \lambda_j \log(\chi_{n+j} \circ \exp)$ and put

$$|p(x)|^{\lambda} = \prod_{j=1}^{l} |p_j(x)|^{\lambda_j} \qquad (x \in \mathbf{V}_{\mathbb{R}} - \mathbf{S}_{\mathbb{R}})$$

and

$$|p(x)|_{ij}^{\lambda} = \begin{cases} |p(x)|^{\lambda} & \text{if } x \in V_{ij}, \\ 0 & \text{otherwise.} \end{cases}$$

The function $|p(x)|_{ij}^{\lambda}$ is well-defined for $\Re(\lambda_1), \ldots, \Re(\lambda_l) > 0$ and we define $|p(x)|_{ij}^{\lambda}$ for arbitrary $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ by analytic continuation. We denote by $|\overline{p}(x)|_{ij}^{\lambda}$ the function on X_i induced by $|p(x)|_{ij}^{\lambda}$. Then $|\overline{p}(x)|_{ij}^{\lambda}$ $(1 \leq j \leq r_i)$ coincide with the functions $\exp\left\{\lambda\left(H_{\epsilon_i}^{w_j^{(i)}}(g)\right)\right\}$

defined by [OS, (3.33)].

4.2 Now we examine the space $\mathcal{E}(X_i;\pi,\chi)$ of spherical functions of type (π,χ) introduced in §1.5. Let $\mathcal{D}(X_i)=\mathcal{D}(L_0^+/L_{(x_i)}^+)$ be the algebra of L_0^+-invariant differential operators on X_i. Denote by $\mathcal{Z}(X_i)$ the subring of $\mathcal{D}(X_i)$ consisting of the restrictions $\overline{D}$ of bi-invariant differential operators D in $\mathcal{Z}(L_0^+)$. It is known that $\mathcal{Z}(X_i)=\mathcal{D}(X_i)$ if L_0^+ is of classical type and $\mathcal{D}(X_i)$ is a finite $\mathcal{Z}(X_i)$-module in general ([Hel1, §7], [Hel4]).

Let

$$\gamma_i:\mathcal{D}(L_0^+/L_{(x_i)}^+)\ (\cong U(\mathfrak{L}_0)^{\mathfrak{k}_{\epsilon_i}}/\left(U(\mathfrak{L}_0)^{\mathfrak{k}_{\epsilon_i}}\cap U(\mathfrak{L}_0)(\mathfrak{k}_{\epsilon_i})\right))\xrightarrow{\cong}U(\mathfrak{a})^W$$

be the standard isomorphism of $\mathcal{D}(L_0^+/L_{(x_i)}^+)$ onto the ring $U(\mathfrak{a})^W$ of the Weyl group invariants (cf. [OS, §2.3], [Hel3, Chap. II, §4, §5]). For $\mu\in\mathfrak{a}_{\mathbf{C}}^*$, we obtain an algebra homomorphism of $U(\mathfrak{a})^W$ into $\mathbf{C}$ by extending it to $U(\mathfrak{a})^W$, which we denote by the same symbol. Put

$$\chi_\mu:=\mu\circ\gamma_i:\mathcal{D}(L_0^+/L_{(x_i)}^+)\longrightarrow\mathbf{C}.$$

Let π be an irreducible unitary representation of K on W_π and $\chi:\mathcal{Z}(L_0^+)\to\mathbf{C}$ be an infinitesimal character. It is obvious that $\mathcal{E}(X_i;\pi,\chi)=\{0\}$ unless $\chi:\mathcal{Z}(L_0^+)\to\mathbf{C}$ factors through $\mathcal{Z}(X_i)=\mathcal{Z}(L_0^+/L_{(x_i)}^+)$:

$$\begin{tikzcd}
\chi:\mathcal{Z}(L_0^+) \arrow[rr] \arrow[dr] & & \mathbf{C} \\
& \mathcal{Z}(L_0^+/L_{(x_i)}^+) \arrow[ur] &
\end{tikzcd}$$

Now assume that $\mathcal{E}(X_i;\pi,\chi)\neq\{0\}$ and denote the character of $\mathcal{Z}(L_0^+/L_{(x_i)}^+)$ induced by χ also by the same symbol.

For $\mu\in\mathfrak{a}_{\mathbf{C}}^*$, put

$$\mathcal{E}(X_i;\pi,\chi_\mu)=\left\{\psi:X_i\to W_\pi\ \middle|\ \begin{array}{ll}\psi(k\overline{x})=\pi(k)\psi(\overline{x}) & (k\in K,\overline{x}\in X_i)\\ D\psi=\chi_\mu(D)\psi & (D\in\mathcal{D}(X_i))\end{array}\right\}.$$

Since $\mathcal{D}(X_i)\supset\mathcal{Z}(X_i)$, we have

$$\mathcal{E}(X_i;\pi,\chi_\mu)\subseteq\mathcal{E}(X_i;\pi,\chi_\mu|_{\mathcal{Z}(X_i)}).$$

On the other hand, since $\mathcal{D}(X_i)$ is a commutative algebra, the ring $\mathcal{D}(X_i)$ acts on $\mathcal{E}(X_i;\pi,\chi)$ ($\chi\in\mathrm{Hom}(\mathcal{Z}(L_0^+),\mathbf{C})$). We assume that

(A-8) *There exists a finite number of* $\mu_1,\ldots,\mu_d\in\mathfrak{a}_{\mathbf{C}}^*$ *such that*

$$(4.2)\qquad \mathcal{E}(X_i;\pi,\chi)=\bigoplus_{j=1}^{d}\mathcal{E}(X_i;\pi,\chi_{\mu_j}).$$

Remark. The assumption always holds unless the symmetric space X_i contains an irreducible symmetric space of type EVII or EIX ([Och]). Even in this exceptional case, the assumption holds for generic χ. By [OS, Lemma 2.24], $\mu_1,\dots,\mu_d$ do not depend on $i=1,\dots,\nu$. If G is of classical type, then $d=1$ for any χ. In some exceptional cases, it may occur that $d\geq 2$; however, for a generic χ, we may take $d=1$ ([Hel4]).

Now we define $\mathrm{End}(W_\pi)$-valued spherical functions $\Psi_{i,j}^{\pi,\mu}(\overline{x})$ ($\overline{x}\in X_i$) by the analytic continuation (with respect to μ) of the integral

$$(4.3)\qquad \Psi_{i,j}^{\pi,\mu}(\overline{x}):=\int_K \left|\overline{p}(k^{-1}\cdot\overline{x})\right|_{ij}^{\mu+\rho}\pi(k)\,dk\quad(\mu\in\mathfrak{a}_{\mathbf{C}}^*,1\leq i\leq\nu,1\leq j\leq r_i),$$

where $\rho=\frac{1}{2}\sum_{\alpha\in\Sigma^+}\alpha$ and dk is the normalized Haar measure on K. Then, using the Poisson integral representation of eigenfunctions on X_i of invariant differential operators ([OS, Theorem 5.1]), we immediately obtain the following proposition:

Proposition 4.1 *If $\mu\in\mathfrak{a}_{\mathbf{C}}^*$ satisfies $\frac{2\langle\mu,\alpha\rangle}{\langle\alpha,\alpha\rangle}\notin\mathbb{Z}$ for all $\alpha\in\Sigma$, then the linear mapping*

$$\begin{array}{rcl}\mathcal{P}_{i,\mu}:\bigoplus^{r_i}W_\pi^M & \longrightarrow & \mathcal{E}(X_i;\pi,\chi_\mu)\\ (v_j)_{j=1}^{r_i} & \longmapsto & \sum_{j=1}^{r_i}\Psi_{i,j}^{\pi,\mu}(\overline{x})\cdot v_j\end{array}$$

is an isomorphism, where

$$W_\pi^M=\{v\in W_\pi\mid\pi(m)v=v\ (m\in M)\}.$$

Thus we have constructed a basis of $\mathcal{E}(X_i;\pi,\chi)$ for generic $\mu\in\mathfrak{a}_{\mathbf{C}}^*$.

Let $\mu_1,\dots,\mu_d$ be the elements in $\mathfrak{a}_{\mathbf{C}}^*$ appearing in the right hand side of the decomposition (4.2). We assume in the following that $\frac{2\langle\mu_l,\alpha\rangle}{\langle\alpha,\alpha\rangle}\notin\mathbb{Z}$ for all $\alpha\in\Sigma$ and $1\leq l\leq d$. Then, for $\phi\in\mathcal{A}(L_0^+/\Gamma;\pi,\chi)$ and $x\in\mathbf{V}_{\mathbf{Q}}\cap V_i$, one can find constants $v_{j,\mu_l}^{(i)}(\phi;x)$ such that

$$\mathcal{M}_x^{(i)}\phi(\overline{y})=\sum_{l=1}^{d}\sum_{j=1}^{r_i}v_{j,\mu_l}^{(i)}(\phi;x)\cdot\Psi_{i,j}^{\pi,\mu_l}(\overline{y}).$$

We put

$$\zeta_{j,l}^{(i)}(\phi,f_0;s)=\frac{1}{v(\Gamma)}\sum_{x\in\Gamma\backslash\mathbf{V}_{\mathbf{Q}}\cap V_i}\frac{\mu(x)f_0(x)v_{j,\mu_l}^{(i)}(\phi;x)}{\prod_{t=1}^{n}|P_t(x)|^{s_t}}$$

and

$$\Phi_j^{(i)}(f_\infty;\pi,\mu_l,s)=\int_{V_i}\prod_{t=1}^{n}|P_t(y)|^{s_t}\,\Psi_{i,j}^{\pi,\mu_l}(\overline{y})f_\infty(y)\,\Omega(y).$$

Here $\zeta_{j,l}^{(i)}(\phi,f_0;s)$ are Dirichlet series with values in W_π^M and $\Phi_j^{(i)}(f_\infty;\pi,\mu_l,s)$ are local zeta functions with values in $\mathrm{End}(W_\pi)$.

Now Proposition 1.4 can be formulated as follows:

Proposition 4.2 *We have*

$$Z_\phi(s)(f_\infty\otimes f_0)=\sum_{i=1}^{\nu}\sum_{j=1}^{r_i}\sum_{l=1}^{d}\Phi_j^{(i)}(f_\infty;\pi,\mu_l,s)\cdot\zeta_{j,l}^{(i)}(\phi,f_0;s).$$

4.3 Let $(\mathbf{G},\rho,\mathbf{V})=(\mathbf{G},\rho_1\oplus\rho_2,\mathbf{E}\oplus\mathbf{F})$ and assume that $\mathbf{F}$ is a regular subspace. Denote by $(\mathbf{G},\rho^*,\mathbf{V}^*)$ the p.v. dual to $(\mathbf{G},\rho,\mathbf{V})$ with respect to $\mathbf{F}$. In the following we indicate with the superscript * the notion for $(\mathbf{G},\rho^*,\mathbf{V}^*)$. For example, we denote by $P_1^*,\ldots,P_n^*$ the basic relative invariants of $(\mathbf{G},\rho^*,\mathbf{V}^*)$.

Take a relative invariant P of $(\mathbf{G},\rho,\mathbf{V})$ with coefficients in $\mathbb{Q}$ such that ϕ_P defined in the proof of Lemma 3.1 gives a biregular map of $\mathbf{V}-\mathbf{S}$ onto $\mathbf{V}^*-\mathbf{S}^*$. Since ϕ_P is defined over $\mathbb{Q}$ and $\mathbf{G}$-equivariant, we have a one to one correspondence of P^+-orbits in $\mathbf{V}_\mathbb{R}-\mathbf{S}_{P,\mathbb{R}}$ and those in $\mathbf{V}_\mathbb{R}^*-\mathbf{S}_{P,\mathbb{R}}^*$. Hence we have

$$\mathbf{V}_\mathbb{R}^*-\mathbf{S}_{P,\mathbb{R}}^*=\bigcup_{i=1}^{\nu}\bigcup_{j=1}^{r_i}V_{ij}^*,\quad V_{ij}^*=\phi_P(V_{ij})=\rho^*(P^+)x_{ij}^*,\quad x_{ij}^*=\phi_P(x_{ij})=\rho^*(w_j^{(i)})\phi_P(x_i).$$

Since $L_{(x_i)}^+=L_{(x_i^*)}^+$ for $x_i^*=\phi_P(x_i)$, we may identify $X_i=L_0^+/L_{(x_i)}^+$ with $X_i^*=L_0^+/L_{(x_i^*)}^+$ and the assumption (A-7) holds also for $(\mathbf{G},\rho^*,\mathbf{V}^*)$. Moreover we have the commutative diagram

$$\begin{tikzcd}
V_i \arrow[rr, "\phi_P"] \arrow[dr, "-"'] & & V_i^* \\
& X_i \arrow[ur, "-"'] &
\end{tikzcd}.$$

For $x^*=\phi_P(x)$ $(x\in V_i)$, it is easy to check the following identity:

$$|\overline{p}^*(x^*)|_{ij}^{\mu+\rho}=|\overline{p}(x)|_{ij}^{\mu+\rho}.$$

If x is in $V_i\cap\mathbf{V}_\mathbb{Q}$ (and hence x^* is in $V_i\cap\mathbf{V}_\mathbb{Q}^*$), then we have

$$\begin{aligned}
\mathcal{M}_{x^*}^{(i)}\phi(\overline{y}) &= \mathcal{M}_x^{(i)}\phi(\overline{y})\quad(\overline{y}\in X_i),\\
v_{j,\mu_l}^{(i)}(\phi;x^*) &= v_{j,\mu_l}^{(i)}(\phi;x)
\end{aligned}$$

The zeta functions and the local zeta functions associated with $(\mathbf{G}, \rho^*, \mathbf{V}^*)$ are defined as follows:

$$\zeta_{j,l}^{*(i)}(\phi, f_0^*; s) = \frac{1}{v(\Gamma)} \sum_{x^* \in \Gamma \backslash \mathbf{V}_{\mathbf{Q}}^* \cap V_i^*} \frac{\mu(x^*) f_0^*(x^*) v_{j,\mu_l}^{(i)}(\phi; x^*)}{\prod_{t=1}^{n} |P_t^*(x^*)|^{s_t}},$$

$$\Phi_j^{*(i)}(f_\infty^*; \pi, \mu_l, s) = \int_{V_i^*} \prod_{t=1}^{n} |P_t^*(y^*)|^{s_t} \, \Psi_{i,j}^{\pi,\mu_l}(\overline{y}^*) f_\infty^*(y^*) \, \Omega^*(y^*).$$

Theorem 4.3 *For any $f_\infty \in \mathcal{S}(\mathbf{V}_{\mathbf{R}})$ and $f_\infty^* \in \mathcal{S}(\mathbf{V}_{\mathbf{R}}^*)$, the integrals $\Phi_j^{(i)}(f_\infty; \pi, \mu, s)$, $\Phi_j^{*(i)}(f_\infty^*; \pi, \mu, s)$ $((\mu, s) \in \mathfrak{a}_{\mathbf{C}}^* \times \mathbf{C}^n)$ converge absolutely, when $\Re(s_1) > \delta_1, \ldots, \Re(s_n) > \delta_n$ and $\Re(\langle \mu, \alpha \rangle) > 0$ for all $\alpha \in \Delta$. Moreover they have analytic continuations to meromorphic functions of (μ, s) in $\mathfrak{a}_{\mathbf{C}}^* \times \mathbf{C}^n$ and satisfy the functional equation*

$$\Phi_j^{(i)}(f_\infty; \pi, \mu, s) = \sum_{i^*=1}^{\nu} \sum_{j^*=1}^{r_i} \Gamma_{j,j^*}^{(i,i^*)}(\mu, s) \Phi_{j^*}^{*(i^*)}(\widehat{f_\infty}; \pi, \mu, (s - \lambda)U),$$

where $\Gamma_{j,j^*}^{(i,i^*)}(\mu, s)$ are meromorphic functions independent of f_∞ and π having an elementary expression in terms of the gamma function and the exponential function.

Proof. From (4.3), we have

$$\Phi_j^{(i)}(f_\infty; \pi, \mu, s) = \int_{V_i} \prod_{t=1}^{n} |P_t(y)|^{s_t} \left\{ \int_K \left| \overline{p}(k^{-1}\overline{y}) \right|_{ij}^{\mu+\rho} \pi(k) \, dk \right\} f_\infty(y) \, \Omega(y).$$

Since P_t's are K-invariant, we obtain

$$(4.4) \qquad \Phi_j^{(i)}(f_\infty; \pi, \mu, s) = \int_{V_{ij}} \prod_{t=1}^{n} |P_t(y)|^{s_t} \cdot |p(y)|^{\mu+\rho} \left\{ \int_K f_\infty(\rho(k)y) \pi(k) \, dk \right\} \Omega(y).$$

Similarly we obtain

$$(4.5) \qquad \Phi_j^{*(i)}(f_\infty^*; \pi, \mu, s) = \int_{V_{ij}^*} \prod_{t=1}^{n} |P_t^*(y^*)|^{s_t} \cdot |p^*(y^*)|^{\mu+\rho} \left\{ \int_K f_\infty^*(\rho^*(k)y^*) \pi(k) \, dk \right\} \Omega^*(y^*).$$

From these expressions, the convergence of the integrals is obvious. Moreover, since any matrix coefficient of

$$\int_K f_\infty(\rho(k)y) \pi(k) \, dk \quad \text{(resp. } \int_K f_\infty^*(\rho^*(k)y) \pi(k) \, dk)$$

is a rapidly decreasing function on $V_{\mathbb{R}}$ (resp. $V_{\mathbb{R}}^*$), the integrals $\Phi_j^{(i)}$ (resp. $\Phi_j^{*(i)}$) have analytic continuations to meromorphic functions on $\mathfrak{a}_{\mathbb{C}}^* \times \mathbb{C}^n$. We note further that, for $u, v \in W_\pi$,

$$\left\langle \int_K \widehat{f_\infty}(\rho(k)y)\pi(k)\,dk \cdot u, v \right\rangle = \left(\left\langle \int_K f_\infty(\rho(k)y)\pi(k)\,dk \cdot u, v \right\rangle \right)^{\widehat{}} .$$

By [S1, Theorem 1], there exist meromorphic functions $\Gamma_{j,j^*}^{(i,i^*)}(\mu, s)$ on $\mathfrak{a}_{\mathbb{C}}^* \times \mathbb{C}^n$ such that the functional equation

$$\left\langle \Phi_j^{(i)}(f_\infty; \pi, \mu, s)u, v \right\rangle = \sum_{i^*=1}^{\nu} \sum_{j^*=1}^{r_i} \Gamma_{j,j^*}^{(i,i^*)}(\mu, s) \left\langle \Phi_{j^*}^{*(i^*)}(\widehat{f_\infty}; \pi, \mu, (s-\lambda)U)u, v \right\rangle$$

holds for all $u, v \in W_\pi$. This proves the theorem. ∎

For $(\mu, s) \in \mathfrak{a}_{\mathbb{C}}^* \times \mathbb{C}^n$, put

$$P_{\mu,s}(y) = \prod_{i=1}^{n} P_i(y)^{s_i - \delta_i - \sum_{j=1}^{l} m_{ij}\mu_j} \prod_{j=1}^{l} P_{n+j}(y)^{\mu_j}.$$

Let P_F^* be the relative invariant of (G, ρ^*, V^*) introduced just before Lemma 2.2. Then, by [S1, §3], there exists a polynomial $b_F(s, \mu)$, the b-function of $(\mathbf{G}, \rho, \mathbf{V})$ with respect to F, satisfying

(4.6) $$P_F^*(y_1, \tfrac{\partial}{\partial y_2})P_{\mu,s}(y) = b_F(s, \mu)P_{\mu,s+\alpha}(y),$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is defined by $\chi_F^* = \chi_1^{\alpha_1} \cdots \chi_n^{\alpha_n}$. We can similarly define the b-function $b_F^*(s, \mu)$ of $(\mathbf{G}, \rho^*, \mathbf{V}^*)$ with respect to F.

Now we are in a position to prove the functional equation of the zeta functions $\zeta_{j,l}^{(i)}(\phi, f_0; s)$ and $\zeta_{j,l}^{*(i)}(\phi, f_0^*; s)$.

Theorem 4.4 *Assume that* $\frac{2\langle \mu_l, \alpha \rangle}{\langle \alpha, \alpha \rangle} \notin \mathbb{Z}$ *for all* $\alpha \in \Sigma$ *and* $1 \leq l \leq d$. *Then*

(i) *the zeta functions* $\zeta_{j,l}^{(i)}(\phi, f_0; s)$ *and* $\zeta_{j,l}^{*(i)}(\phi, f_0^*; s)$ *can be extended to meromorphic functions of* s *in* D *and* D^*, *respectively (for the definition of* D *and* D^*, *see* §2).

(ii) *The functions* $b_F(s, \mu_l)\zeta_{j,l}^{(i)}(\phi, f_0; s)$ *and* $b_F^*(s, \mu_l)\zeta_{j,l}^{*(i)}(\phi, f_0^*; s)$ *are holomorphic functions of* s *in* D *and* D^*, *respectively.*

(iii) *The following functional equation holds for any* $f_0 \in \mathcal{S}(\mathbf{V}_{\mathbb{Q}})$:

$$\zeta_{j^*,l}^{*(i^*)}(\phi, \widehat{f_0}; (s-\lambda)U) = \sum_{i=1}^{\nu} \sum_{j=1}^{r_i} \Gamma_{j,j^*}^{(i,i^*)}(\mu_l, s)\zeta_{j,l}^{(i)}(\phi, f_0; s).$$

Proof. (i) and (ii): Let the notation be as in §2. For an $f'_\infty \in C_0^\infty(V_{ij})$, put $f_\infty = P_F^*(x_1, \frac{\partial}{\partial x_2})f'_\infty(x_1, x_2)$. Then, by Lemma 2.2, we can apply Proposition 2.1 to f_∞ and we see that the function

$$Z_\phi(s)(f_\infty \otimes f_0) = \sum_{l=1}^{d} \Phi_j^{(i)}(f_\infty; \pi, \mu_l, s)\zeta_{j,l}^{(i)}(\phi, f_0; s)$$

is a holomorphic function of s in D. On the other hand

$$\Phi_j^{(i)}(f_\infty; \pi, \mu_l, s) = \int_{V_{ij}} \prod_{t=1}^{n} |P_t(y)|^{s_t} \cdot |p(y)|^{\mu_l+\rho} \left\{ \int_K P_F^*(y_1, \frac{\partial}{\partial y_2}) f'_\infty(\rho(k)y)\pi(k)\, dk \right\} \Omega(y).$$

Since P_F^* is K-invariant, we have

$$P_F^*(y_1, \frac{\partial}{\partial y_2}) f'_\infty(\rho(k)y) = P_F^*(y_1, \frac{\partial}{\partial y_2}) \left({}^k f'_\infty\right)(y), \quad {}^k f'_\infty(y) = f'_\infty(\rho(k)y).$$

Hence, integrating by parts, we obtain

$$\Phi_j^{(i)}(f_\infty; \pi, \mu_l, s) = \pm b_F(s, \mu_l)\Phi_j^{(i)}(f'_\infty; \pi, \mu_l; s + \alpha),$$

where we use the identity (4.6). Thus we see that

$$Z_\phi(s)(f_\infty \otimes f_0) = \sum_{l=1}^{d} \pm b_F(s, \mu_l)\Phi_j^{(i)}(f'_\infty; \pi, \mu_l, s + \alpha)\zeta_{j,l}^{(i)}(\phi, f_0; s) \tag{4.7}$$

is a holomorphic function in D.

Now we need the following lemma, whose proof is not hard and is omitted.

Lemma 4.5 *Let $V = \mathbb{C}^m$ and $W = \mathbb{C}^n$. Let $\Psi : X \longrightarrow \mathrm{Hom}(V, W)$ be an $\mathrm{Hom}(V, W)$-valued function on a domain X in $\mathbb{R}^N$. We identify $\mathrm{Hom}(V, W)$ with $M(m, n; \mathbb{C})$ and denote by Ψ_{ij} the (i, j)-entry of Ψ. Put*

$$\Psi_j(x) = \begin{pmatrix} \Psi_{1j}(x) \\ \vdots \\ \Psi_{mj}(x) \end{pmatrix} : X \longrightarrow \mathbb{C}^m \quad (1 \leq j \leq n).$$

Assume that the functions $\Psi_1, \ldots, \Psi_n$ are linearly independent over $\mathbb{C}$. Then there exist $f_1, \ldots, f_n \in C_0^\infty(X)$ such that the rank of the matrix

$$\begin{pmatrix} \int_X \Psi(x) f_1(x)\, dx \\ \vdots \\ \int_X \Psi(x) f_n(x)\, dx \end{pmatrix} \begin{matrix} \}m \\ \vdots \\ \}m \end{matrix} \in M(mn, n; \mathbb{C})$$

is equal to n.

When $\frac{2\langle\mu_l,\alpha\rangle}{\langle\alpha,\alpha\rangle} \notin \mathbb{Z}$ $(1 \le l \le d)$, the lemma can be applied to the function

$$\Psi : V_{ij} \longrightarrow \mathrm{Hom}(\bigotimes^{d} W_\pi^M, W_\pi)$$

defined by

$$\Psi(x)(v_1, \dots, v_d) = |P(x)|^s \sum_{l=1}^{d} \Psi_{i,j}^{\pi,\mu_l}(\overline{x}) \cdot v_l.$$

Hence, by (4.7), we see that the functions $b_F(s,\mu_l)\zeta_{j,l}^{(i)}(\phi, f_0; s)$ are holomorphic in D. The holomorphy of $b_F^*(s,\mu_l)\zeta_{j,l}^{*(i)}(\phi, f_0^*; s)$ can be shown quite similarly.

(iii): Now we take $f'^*_\infty \in C_0^\infty(V^*_{i^*j^*})$ and put $f^*_\infty(x_1, x_2^*) = P_F(x_1, \frac{\partial}{\partial x_2^*}) f'^*_\infty(x_1, x_2^*)$ and $f_\infty = \widehat{f^*_\infty}$. The we can apply Proposition 2.1 to f_∞ and get the functional equation

$$Z_\phi^*((s-\lambda)U)(f^*_\infty \otimes \widehat{f_0}) = Z_\phi(s)(f_\infty \otimes f_0) \qquad (s \in D).$$

By proposition 4.2 and Theorem 4.3, we have

$$\begin{aligned}
&\sum_{l^*=1}^{d} \Phi_{j^*}^{*(i^*)}(\widehat{f_\infty}; \pi, \mu_{l^*}, (s-\lambda)U)\zeta_{j^*l^*}^{*(i^*)}(\phi, \widehat{f_0}; (s-\lambda)U) \\
&\quad = \sum_{i=1}^{\nu} \sum_{j=1}^{r_i} \sum_{l=1}^{d} \Phi_j^{(i)}(f_\infty; \pi, \mu_{l^*}, s)\zeta_{jl}^{(i)}(\phi, f_0; s) \\
&\quad = \sum_{i=1}^{\nu} \sum_{j=1}^{r_i} \sum_{l=1}^{d} \Gamma_{i^*,j^*}^{i,j}(\mu_{l^*}, s)\Phi_{j^*}^{*(i^*)}(\widehat{f_\infty}; \pi, \mu_{l^*}, (s-\lambda)U)\zeta_{jl}^{(i)}(\phi, f_0; s).
\end{aligned}$$

Therefore

$$\begin{aligned}
&\sum_{l=1}^{d} \Phi_{j^*}^{*(i^*)}(\widehat{f_\infty}; \pi, \mu_l, (s-\lambda)U) \\
&\quad \times \left(\zeta_{j^*l}^{*(i^*)}(\phi, \widehat{f_0}; (s-\lambda)U) - \sum_{i=1}^{\nu} \sum_{j=1}^{r_i} \Gamma_{i^*,j^*}^{i,j}(\mu_l, s)\zeta_{jl}^{(i)}(\phi, f_0; s) \right) = 0.
\end{aligned}$$

By the argument based upon Lemma 4.5, we see that the functional equation

$$\zeta_{j^*l}^{*(i^*)}(\phi, \widehat{f_0}; (s-\lambda)U) = \sum_{i=1}^{\nu} \sum_{j=1}^{r_i} \Gamma_{i^*,j^*}^{i,j}(\mu_l, s)\zeta_{jl}^{(i)}(\phi, f_0; s).$$

holds for any $s \in D$. ∎

Remarks. 1. As we mentioned at the beginning of §2, the functional equation of zeta functions are based on local functional equations, the existence of b-functions anbd the functional equations of the zeta integrals. In the case considered above, the local functional equation (Thoerem 4.3) and the b-function (4.6) are reduced to the usual local functional equations and the b-functions of the prehomogeneous vector space $(\mathbf{P}, \rho, \mathbf{V})$.

2. Even when the symmetric spaces $X_i = L_0^+/L_{(x_i)}^+$ is not of K_ϵ-type, we can argue quite similarly to prove the functional equations of zeta functions attached to automorphic forms on the basis of the results of Oshima [O1]. In the general case, $\mathbf{P}$ is not necessarily minimal parabolic, and the functional equations are reduced to the local functional equations discussed in [S6].

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