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保型形式の周期について

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F を n 次の総実代数体、B を F 上の quaternion algebra で

$$B \otimes_{\mathbb{Q}} \mathbb{R} \cong M_2(\mathbb{R})^r \times \mathbb{H}^{n-r}, \quad r > 0$$

をみたすものとする。$G = \mathrm{Res}_{F/\mathbb{Q}}(B^\times)$ とおく。B の split する F の archimedean places の集合を δ_1 とする。($|\delta_1| = r$.) F' は $\sum_{\tau \in \delta_1} \tau(x)$, $x \in F$ で生成される代数体とする. K を G_{A_f} の open compact subgroup, K_∞ を $G_{\infty +}$ の maximal compact subgroup とする。このとき、F' の上に定義される Shimura variety (non-connected canonical model) S_K が定まり、$S_K(\mathbb{C}) = G_{\mathbb{Q}} \backslash G_A / K K_\infty$ となる。S_K の F' 上の zeta 函数 $Z(s, S_K/F')$ は Langlands [8] により決定され、$\prod_{\pi} L(s, \pi, r_1)$ の形（一般には若干の修正項がつく）である。ここに π は K により決る G_A の有限個の automorphic representation を動く。r_1 は G の L-group の $2^r[F':\mathbb{Q}]$ 次元表現である.

この様な π (Hilbert modular form) には、$\mathrm{Gal}(\overline{\mathbb{Q}}/F)$ の λ-adic 表現 σ_λ が L 函数を保つ形で対応している。(cf. Taylor [18], 及びその引用文献。) 他方 $Z(s, S_K/F')$ は、S_K の cohomology 群における $\mathrm{Gal}(\overline{\mathbb{Q}}/F')$ の λ-adic 表現

から得られる。ゆえに、$\mathrm{Gal}(\overline{\mathbb{Q}}/F)$ の表現から $\mathrm{Gal}(\overline{\mathbb{Q}}/F')$ の表現を構成する自然な方法があるだろう、と推定される。誘導表現の概念を拡張した対応 $\sigma_\lambda \longrightarrow \otimes\mathrm{Ind}_H^{H'}\sigma_\lambda$, $H=\mathrm{Gal}(\overline{\mathbb{Q}}/F)$, $H'=\mathrm{Gal}(\overline{\mathbb{Q}}/F')$ を構成することにより、$L(s,\pi,r_1)$ に attach した λ-adic 表現が explicit に書けることを §1 で示した。

S_K の zeta 函数の特殊値が、Shimura [16] の周期不変量 $P(x,\delta,r)$, $Q(x,\delta)$ で表わされるであろう、という予想を志村先生が1989年に筆者に伝えられた。この志村予想が本稿を書く主要な motivation になった。

$Z(s, S_K/F')$ の特殊値を motive の見地から扱ったのが §2 ~ §5 である。対応 $\sigma_\lambda \longrightarrow \otimes\mathrm{Ind}_H^{H'}\sigma_\lambda$ は λ-adic 表現の compatible system を compatible system に写す。元の表現 σ_λ が motive から得られていれば、$\otimes\mathrm{Ind}_H^{H'}\sigma_\lambda$ も motive から得られるであろう、と予想できる。この様にして restriction of scalar を拡張した、F 上の motive を F' 上の motive に写す functor $M \longrightarrow \otimes\mathrm{Res}_{F/F'}M$ を得る。そこで Deligne の予想 [6] を適用すれば、$L(s, \otimes\mathrm{Res}_{F/F'}M)$ の特殊値が予想の level ではわかる。M が Hilbert modular form に対応する motive で、$|\delta_1| = r \geq 2$ のとき、実際幾つかの $Q(x,\delta)$ と π の積で特殊値の超越部分が表わさ

れる。

この計算で本質的なのは，MがF上のmotiveのとき，Deligneのperiods $c^{\pm}(Res_{F/\mathbb{Q}}(M))$を，$\tau \in Hom(F, \mathbb{C})$でindexづけられる$\tau$-periodsの積として表わしておくことである。Deligneは$L(s, M) = L(s, Res_{F/\mathbb{Q}}(M))$であるから，$c^{\pm}(Res_{F/\mathbb{Q}}(M))$を主に考えたが，tensor積等の操作によるperiodの変化を見るには，この分解が重要である。この様な分解はShimura [16]にある周期の定義からsuggestされたものである。

§6では志村不変量$Q(\alpha, \delta)$を予想は仮定せずに考察した。主要結果は$Q(\alpha, \delta)$がconsistentに定義できることである。

記号　1° Fを有限次代数体とする。Fから$\mathbb{C}$の中へのnon-trivial homomorphisms全体の集合をJ_Fにより表わす。I_Fにより、J_Fで生成されるfree $\mathbb{Z}$-moduleを表わす。

2° 体k上の有限次元vector spacesの成すcategoryを$V(k)$により表わす。

3° 環Rに係数をもつ、n行m列行列全体の集合を$M_{n,m}(R)$とかく。$M_{n,n}(R)$を$M_n(R)$と略す。

4° 群N, Hと準同型$H \longrightarrow Aut(N)$が与えられたときNとHの半直積を$N \times_s H$とかく。

5° 加群Mに群Gが作用しているとき、M^GによりMのG不変元の成す加群を表わす。

6° $\mathfrak{H}$は複素上半平面を表わす。

7° $a, b \in \mathbb{C}$とする。a, bの少くとも一方がnon-zeroであって、その商（$a \neq 0$のときはb/a）が$\mathbb{Q}$に入るとき、$a \sim b$とかく。

§1. New functors $\otimes_\Omega$ Ind, $\otimes_\Omega$ Res and zeta functions of Shimura varieties

Let G be a group and H be a subgroup of index $n < \infty$. Let k be a field and V be a vector space over k of dimension $d < \infty$. Let σ be a representation of H into $GL(V)$. Let $G = \cup_{i=1}^{n} s_i H$ be a coset decomposition and put $\Omega = \cup_{i=1}^{r} s_i H$, where r is any integer such that $1 \leq r \leq n$. Let H' be the stabilizer of Ω under the natural action of G on G/H:

$$(1.1) \qquad H' = \{g \in G \mid g\Omega = \Omega\}.$$

We can construct a representation τ of H' in the following way. For every i, $1 \leq i \leq r$, we prepare a vector space $s_i V$ over k which is isomorphic to V. Take $g \in H'$. Then we have

$$g s_i = s_{j(i)} h_i, \qquad 1 \leq i \leq r, \qquad h_i \in H.$$

Here $i \longrightarrow j(i)$ is a permutation on r-letters. Put $W = \bigotimes_{i=1}^{r} s_i V$ and set

$$(1.2) \qquad \tau(g)(\otimes_{i=1}^{r} s_i v_i) = \otimes_{i=1}^{r} s_{j(i)} \sigma(h_i) v_i, \qquad s_i v_i \in s_i V_i.$$

Extending (1.2) k-linearly to whole W, we can easily verify that τ defines a representation of H' on W.

The definition (1.2) is somewhat informal. We can rewrite it as follows. Let V_i, $1 \leq i \leq r$ be a vector space over k isomorphic to V. Put $W_1 = \bigotimes_{i=1}^{r} V_i$. For $g \in H'$, set

$$g^{-1} s_i = s_{k(i)} h_i^*, \qquad 1 \leq i \leq r, \qquad h_i^* \in H.$$

Then we find $i \longrightarrow k(i)$ is a permutation on r-letters and that

$$j(k(i)) = i, \qquad h_{k(i)} = (h_i^*)^{-1} = s_i^{-1} g s_{k(i)}.$$

Put

$$(1.3) \qquad \tau_1(g)(\otimes_{i=1}^{r} v_i) = \otimes_{i=1}^{r} \sigma(s_i^{-1} g s_{k(i)}) v_{k(i)}, \qquad v_i \in V_i.$$

This is merely a reformulation of (1.2) identifying $s_i V$ with V_i. Thus, by (1.3), τ_1 defines a representation of H' on W_1 which is equivalent to τ. We see easily that the equivalence class of τ does not depend on a choice of $\{s_i\}$. We denote τ_1 by $\bigotimes_\Omega \mathrm{Ind}_H^{H'} \sigma$ or $\bigotimes_\Omega \mathrm{Ind}(\sigma; H \longrightarrow H')$. We can perform similar construction replacing $\otimes$ by $\oplus$. The representation constructed using $\oplus$ instead of $\otimes$ in (1.3) shall be denoted by $\oplus_\Omega \mathrm{Ind}_H^{H'} \sigma$ or $\oplus_\Omega \mathrm{Ind}(\sigma; H \longrightarrow H')$. We have

$$(1.4) \qquad \dim(\bigotimes_\Omega \mathrm{Ind}_H^{H'} \sigma) = (\dim \sigma)^r, \qquad \dim(\bigoplus_\Omega \mathrm{Ind}_H^{H'} \sigma) = r(\dim \sigma).$$

Examples. (1) If $\Omega = G$, then $H' = G$. Clearly $\oplus_G \mathrm{Ind}_H^G\ \sigma$ is the usual induced representation.
(2) If $\Omega = H$, then $H' = H$. We have $\otimes_H \mathrm{Ind}_H^H\ \sigma \cong \oplus_H \mathrm{Ind}_H^H\ \sigma \cong \sigma$.
(3) Assume $\Omega = G$, $\dim \sigma = 1$. Then $\sigma \longrightarrow \bigotimes_G \mathrm{Ind}_H^G \sigma$ is the dual map of the transfer map $G/[G,G] \longrightarrow H/[H,H]$.

Let $\tau = \bigotimes_\Omega \mathrm{Ind}_H^{H'} \sigma$ be realized by (1.3). Let χ_σ and χ_τ denote characters of σ and τ respectively. We can express χ_τ in the following way. Let $\{e_1, \cdots, e_d\}$ be a basis of V over k. Put

$$\sigma(h)e_i = \sum_{j=1}^{d} \sigma_{ji}(h)e_j, \qquad h \in H, \qquad 1 \le i \le d.$$

Then $\{e_{j_1} \otimes e_{j_2} \otimes \cdots \otimes e_{j_r}\}$ make a basis of $\otimes_{i=1}^r V_i$ when $j_1, \cdots, j_r$ run over $[1,d]^r$. We have

$$\tau(g)(\otimes_{i=1}^r e_{j_i}) = \otimes_{i=1}^r \sigma(s_i^{-1} g s_{k(i)}) e_{j_{k(i)}},$$
$$\sigma(s_i^{-1} g s_{k(i)}) e_{j_{k(i)}} = \sum_{l=1}^{d} \sigma_{l j_{k(i)}}(s_i^{-1} g s_{k(i)}) e_l.$$

Hence $\otimes_{i=1}^r e_{j_i}$ contributes

$$\prod_{i=1}^{r} \sigma_{j_i j_{k(i)}}(s_i^{-1} g s_{k(i)})$$

to the trace. Therefore we obtain

$$\chi_\tau(g) = \sum_{j_1, \cdots, j_r \in [1,d]^r} \prod_{i=1}^{r} \sigma_{j_i j_{k(i)}}(s_i^{-1} g s_{k(i)}), \qquad g \in H'. \tag{1.5}$$

If $g \in \cap_{i=1}^r s_i H s_i^{-1} \subseteq H'$, then (1.5) simplifies to

$$\chi_\tau(g) = \prod_{i=1}^{r} \chi_\sigma(s_i^{-1} g s_i), \qquad g \in \cap_{i=1}^r s_i H s_i^{-1}. \tag{1.6}$$

The above construction $\bigotimes_\Omega \mathrm{Ind}_H^{H'} \sigma$ applies also to the case where σ is a λ-adic representation of a Galois group or σ is a representation of a Weil group. In other words, the continuity condition of $\bigotimes_\Omega \mathrm{Ind}_H^{H'} \sigma$ can easily be derived from that of σ.

We shall consider the case of λ-adic representation in more detail. Let F and E be algebraic number fields of finite degree. Let λ be a finite place of E and let

$$\sigma_\lambda : \mathrm{Gal}(\overline{\mathbf{Q}}/F) \longrightarrow GL(V)$$

be a λ-adic representation . Here V is a $d < \infty$ dimensional vector space over E_λ. Take $G = \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$, $H = \mathrm{Gal}(\overline{\mathbf{Q}}/F)$, $G = \cup_{i=1}^n s_i H$, $\Omega = \cup_{i=1}^r s_i H$ and apply the above consruction. Let F' be the fixed field of H' and $\tilde{F}$ be the normal closure of F. Put $K = \mathrm{Gal}(\overline{\mathbf{Q}}/\tilde{F})$. We have $H' = \mathrm{Gal}(\overline{\mathbf{Q}}/F') \supseteq K$. Then $\tau_\lambda = \bigotimes_\Omega \mathrm{Ind}_H^{H'} \sigma_\lambda$ defines a λ-adic representation

$$\tau_\lambda : \mathrm{Gal}(\overline{\mathbf{Q}}/F') \longrightarrow GL(W)$$

where W is a d^r-dimensional vector space over E_λ.

Let S be a finite set of prime ideals of F. We assume that σ_λ is unramified outside of S. If $\mathfrak{p}$ is a prime ideal of F such that $\mathfrak{p} \notin S$, we set

$$(1.7) \qquad f_\mathfrak{p}(\sigma_\lambda, X) = \det(1 - \sigma_\lambda(F_\mathfrak{p})X) \in E_\lambda[X],$$

where $F_\mathfrak{p}$ denotes a representative from the Frobenius conjugacy class of $\mathfrak{p}$ in $\mathrm{Gal}(\overline{\mathbf{Q}}/F)$.

Let $\mathfrak{p}'$ be a prime ideal of F' and let $\mathfrak{P}'$ be a prime divisor of $\mathfrak{p}'$ in $\tilde{F}$. For $1 \leq i \leq r$, let $\mathfrak{p}_i$ be the restriction of $s_i^{-1}\mathfrak{P}'$ to F. Then the set of prime ideals $\{ \mathfrak{p}_1, \mathfrak{p}_2, \cdots, \mathfrak{p}_r \}$ does not depend on the choice of $\mathfrak{P}'$ and $\{ s_i \}$. Let S' be the set of $\mathfrak{p}'$ such that either one of $\mathfrak{p}_i$ ramifies in $F/\mathbf{Q}$ or that σ_λ ramifies at one of $\mathfrak{p}_i$.

THEOREM 1.1. *Let the notation be the same as above. Then τ_λ is unramified outside of S'. If $f_\mathfrak{p}(\sigma_\lambda, X) \in E[X]$ whenever $\mathfrak{p} \notin S$, then for any prime ideal $\mathfrak{p}' \notin S'$ of F', we have $f_\mathfrak{p}(\tau_\lambda, X) \in E[X]$. Furtheremore if λ' is another finite place of E and $\sigma_{\lambda'}$ is a λ'-representation of Gal($\overline{\mathbf{Q}}/F$) unramified outside of S such that $f_\mathfrak{p}(\sigma_\lambda, X) = f_\mathfrak{p}(\sigma_{\lambda'}, X)$ if $\mathfrak{p} \notin S$, then we have $f_\mathfrak{p}(\tau_\lambda, X) = f_\mathfrak{p}(\tau_{\lambda'}, X)$ for $\mathfrak{p}' \notin S'$.*

PROOF: From the realization of τ_λ by (1.3), we have

$$(1.8) \qquad \mathrm{Ker}(\tau_\lambda) \supseteq \cap_{i=1}^r s_i \mathrm{Ker}(\sigma_\lambda) s_i^{-1}.$$

Assume $\mathfrak{p}' \notin S'$. First we shall show that τ_λ is unramified at $\mathfrak{p}'$. Let $\tilde{\mathfrak{P}}'$ be a prime divisor of $\mathfrak{p}'$ in $\overline{\mathbf{Q}}$ and let $I_{\tilde{\mathfrak{P}}'}$ be the inertia group of $\tilde{\mathfrak{P}}'$. It suffices to show $\tau_\lambda(I_{\tilde{\mathfrak{P}}'}) = \{1\}$. By (1.8), this assertion follows if we could show

$$(1.9) \qquad s_i^{-1} I_{\tilde{\mathfrak{P}}'} s_i \subseteq H,$$

$$(1.10) \qquad \sigma_\lambda(s_i^{-1} I_{\tilde{\mathfrak{P}}'} s_i) = \{1\}$$

for every $1 \leq i \leq r$. Let $\mathfrak{P}'$ be a prime ideal of $\tilde{F}$ which lies under $\tilde{\mathfrak{P}}'$. Since $s_i^{-1} I_{\tilde{\mathfrak{P}}'} s_i = I_{s_i^{-1}\tilde{\mathfrak{P}}'}$, (1.9) is equivalent to $I_{s_i^{-1}\mathfrak{P}'} \subseteq \mathrm{Gal}(\tilde{F}/F)$, where $I_{s_i^{-1}\mathfrak{P}'}$ is the inertia group of $s_i^{-1}\mathfrak{P}'$ in $\mathrm{Gal}(\tilde{F}/\mathbf{Q})$. This condition is equivalent to that $s_i^{-1}\mathfrak{P}'$ is unramified in $F/\mathbf{Q}$, i.e., $\mathfrak{p}_i$ is unramified in $F/\mathbf{Q}$. Hence (1.9) is verified. Since $s_i^{-1}\tilde{\mathfrak{P}}'$ is a prime divisor of $\mathfrak{p}_i$ in $\overline{\mathbf{Q}}$, (1.10) follows from the assumption that σ_λ is unramified at $\mathfrak{p}_i$.

Next we shall show E-rationality for $\mathfrak{p}' \notin S'$. Put

$$K = \mathrm{Gal}(\overline{\mathbf{Q}}/\tilde{F}), \qquad \overline{G} = G/K, \qquad \overline{H} = H/K,$$
$$\overline{\Omega} = \Omega/K, \qquad \overline{F}_{\mathfrak{p}'} = F_{\mathfrak{p}'} \mod K \in \overline{G}.$$

Let U be the cyclic subgroup of $\overline{G}$ generated by $\overline{F}_{\mathfrak{p}'}$. Let

$$\overline{\Omega} = \cup_{j=1}^{m} U\bar{t}_j\overline{H}, \qquad \bar{t}_j \in \overline{G} \tag{1.11}$$

be a double coset decomposition of $\overline{\Omega}$. For every j, let n_j be the minimal positive integer a such that $\overline{F}^{a}_{\mathfrak{p}'} \in \bar{t}_j\overline{H}\bar{t}_j^{-1}$. Then

$$\overline{\Omega} = \cup_{j=1}^{m} \cup_{i=0}^{n_j-1} \overline{F}^{i}_{\mathfrak{p}'}\bar{t}_j\overline{H} \tag{1.12}$$

is a coset decomposition of $\overline{\Omega}$. Take $t_j \in G$ so that $t_j \mod K = \bar{t}_j$. Then

$$\Omega = \cup_{j=1}^{m} \cup_{i=0}^{n_j-1} s_{ij}H, \qquad s_{ij} = F^{i}_{\mathfrak{p}'}t_j \tag{1.13}$$

is a coset decomposition of Ω. We may realize τ_λ using (1.13) and (1.3) on

$$W = \bigotimes_{j=1}^{m} \bigotimes_{i=0}^{n_j-1} V_{ij}, \qquad V_{ij} \cong V.$$

Since

$$F_{\mathfrak{p}'}^{-1}s_{ij}H = \begin{cases} s_{n_j-1,j}H & \text{if} \quad i = 0, \\ s_{i-1,j}H & \text{if} \quad 1 \le i \le n_j - 1, \end{cases}$$

we have

$$\tau_\lambda(F_{\mathfrak{p}'})(\otimes_{j=1}^{m} \otimes_{i=1}^{n_j-1} v_{ij}) = \otimes_{j=1}^{m}(\sigma_\lambda(t_j^{-1}F_{\mathfrak{p}'}^{n_j}t_j)v_{n_j-1,j} \otimes (\otimes_{i=1}^{n_j-1} v_{i-1,j})).$$

For $1 \le j \le m$, let A_j be the linear operator on $\bigotimes_{i=0}^{n_j-1} V_{ij}$ defined by

$$A_j(\otimes_{i=1}^{n_j-1} v_{ij}) = \sigma_\lambda(t_j^{-1}F_{\mathfrak{p}'}^{n_j}t_j)v_{n_j-1,j} \otimes (\otimes_{i=1}^{n_j-1} v_{i-1,j}).$$

Then we have $\tau_\lambda(F_{\mathfrak{p}'}) = \otimes_{j=1}^{m} A_j$. Therefore it suffices to show

$$\det(1 - A_jX) \in E[X], \qquad 1 \le j \le m. \tag{1.14}$$

Let $\tilde{\mathfrak{P}}'$ be a prime divisor of $\mathfrak{p}'$ in $\overline{\mathbf{Q}}$ and let $F_{\tilde{\mathfrak{P}}'} \in \mathrm{Gal}(\overline{\mathbf{Q}}/F')$ be a Frobenius element of $\tilde{\mathfrak{P}}'$. We may take $F_{\mathfrak{p}'} = F_{\tilde{\mathfrak{P}}'}$. We have $t_j^{-1}F_{\mathfrak{p}'}^{n_j}t_j = F^{n_j}_{t_j^{-1}\tilde{\mathfrak{P}}'}$. Let $\mathfrak{p}_j$ (resp. $\mathfrak{p}'_j$) be the restriction of $t_j^{-1}\tilde{\mathfrak{P}}'$ to F (resp. F'). Let f_j (resp. f'_j) be the degree of

$\mathfrak{p}_j$ (resp. $\mathfrak{p}'_j$) in $F/\mathbf{Q}$ (resp. $F'/\mathbf{Q}$). Let $\tilde{F}_{t_j^{-1}\tilde{\mathfrak{P}}'}$ be a Frobenius element of $t_j^{-1}\tilde{\mathfrak{P}}'$ in $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$. We may assume

$$(\tilde{F}_{t_j^{-1}\tilde{\mathfrak{P}}'})^{f_j} = F_{\mathfrak{p}_j}, \qquad (\tilde{F}_{t_j^{-1}\tilde{\mathfrak{P}}'})^{f'_j} = F_{\mathfrak{p}'_j} = F_{t_j^{-1}\tilde{\mathfrak{P}}'}.$$

Hence we have

$$t_j^{-1}F_{\mathfrak{p}'}^{n_j}t_j = (\tilde{F}_{t_j^{-1}\tilde{\mathfrak{P}}'})^{n_j f'_j} \in H.$$

Therefore f_j must divide $n_j f'_j$ and we obtain

$$t_j^{-1}F_{\mathfrak{p}'}^{n_j}t_j = F_{\mathfrak{p}_j}^{n_j f'_j/f_j}.$$

Now the assertion (1.14) and also the last assertion of Theorem 1.1 follows from the next Lemma.

LEMMA 1.2. *Let V be a finite dimensional vector space over a field k and let $A \in End(V)$. Let $W = \bigotimes_{i=0}^{n-1} V_i$, $V_i \cong V$. Define $A_1 \in End(W)$ by*

$$A_1(\otimes_{i=0}^{n-1} v_i) = Av_{n-1} \otimes (\otimes_{i=1}^{n-1} v_{i-1}).$$

Put $f_A(X) = \det(1 - AX)$, $f_{A_1}(X) = \det(1 - A_1X)$. Then $f_{A_1}(X)$ depends only on $f_A(X)$. Furthermore if k_0 is a subfield of k such that $f_A(X) \in k_0[X]$, then we have $f_{A_1}(X) \in k_0[X]$.

The proof is omitted since it is easy. This completes the proof of Theorem 1.1.

Let $\sigma_\lambda : \mathrm{Gal}(\overline{\mathbf{Q}}/F) \longrightarrow GL(V)$ and $\tau_\lambda : \mathrm{Gal}(\overline{\mathbf{Q}}/F') \longrightarrow GL(W)$ be as above. If σ_λ is of Hodge-Tate type, then we can show that τ_λ is also of Hodge-Tate type and its type can be determined.

Let p be a rational prime which lies under λ. Set

$$\mu_{p^\infty} = \{ z \in \mathbf{Q} \mid z^{p^a} = 1 \quad \text{for some} \quad 1 \le a \in \mathbf{Z} \}.$$

Define a homomorphism χ of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ into $\mathbf{Z}_p^\times$ by

$$g(z) = z^{\chi(g)}, \qquad g \in \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}), \qquad z \in \mu_{p^\infty}.$$

Let $\mathfrak{p}$ be a prime factor of p in F and take a prime divisor $\tilde{\mathfrak{P}}$ of $\mathfrak{p}$ in $\overline{\mathbf{Q}}$. We identify $\overline{\mathbf{Q}}_p$ with $\overline{\mathbf{Q}}_{\tilde{\mathfrak{P}}}$ and consider μ_{p^∞} as a subgroup of $\overline{\mathbf{Q}}_p^\times$. We regard E_λ as a subfield of $\overline{\mathbf{Q}}_p$. Let $\mathbf{C}_p = \hat{\overline{\mathbf{Q}}}_p$ be the completion of $\overline{\mathbf{Q}}_p$. Put $V_{\mathbf{C}_p} = \mathbf{C}_p \otimes_{E_\lambda} V$. Then $\mathrm{Gal}(\overline{\mathbf{Q}}_{\tilde{\mathfrak{P}}}/F_{\mathfrak{p}} \vee E_\lambda) \ni g$ acts on $V_{\mathbf{C}_p}$ by

$$g(c \otimes v) = g(c) \otimes \sigma_\lambda(g)v, \qquad c \in \mathbf{C}_p, \quad v \in V.$$

For every $i \in \mathbf{Z}$, let

$$V^i = \{ v \in V_{\mathbf{C}_p} \mid g(v) = \chi(g)^i v \quad \text{for every} \quad g \in \mathrm{Gal}(\overline{\mathbf{Q}}_{\tilde{\mathfrak{P}}}/F_\mathfrak{p} \vee E_\lambda) \}$$

and put $V(i) = \mathbf{C}_p \otimes_{F_\mathfrak{p} \vee E_\lambda} V^i$. Then $\oplus_{i \in \mathbf{Z}} V(i)$ can be considered as a sub $\mathbf{C}_p$-vector space of $V_{\mathbf{C}_p}$ (cf. Serre [10], III-6). We call σ_λ is of Hodge-Tate type at $\mathfrak{p}$ if

$$V_{\mathbf{C}_p} = \oplus_{i_\mathfrak{p} \in \mathbf{Z}} V(i_\mathfrak{p}). \tag{1.15}$$

PROPOSITION 1.3. *Assume that σ_λ is of Hodge-Tate type (1.15) at every prime factor $\mathfrak{p}$ of p in F. Let $\mathfrak{p}'$ be any prime factor of p in F'. Define prime ideals $\mathfrak{p}_1, \cdots, \mathfrak{p}_r$ of F as above. Then τ_λ is of Hodge-Tate type at $\mathfrak{p}'$ such that*

$$\mathbf{C}_p \otimes_{E_\lambda} W = \oplus_{i_{\mathfrak{p}_1}, \cdots, i_{\mathfrak{p}_r} \in \mathbf{Z}^r} W(i_{\mathfrak{p}_1} + i_{\mathfrak{p}_2} + \cdots + i_{\mathfrak{p}_r}).$$

PROOF: Let $\tilde{\mathfrak{P}}'$ be a prime factor of $\mathfrak{p}'$ in $\overline{\mathbf{Q}}$. Let $\tilde{F}$ be the normal closure of F in $\overline{\mathbf{Q}}$ and $\mathfrak{P}'$ be the restriction of $\tilde{\mathfrak{P}}'$ to $\tilde{F}$. For every $1 \leq i \leq r$, take $v_i \in V^{i_{\mathfrak{p}_i}}$ so that

$$gv_i = \chi(g)^{i_{\mathfrak{p}_i}} v_i, \qquad g \in \mathrm{Gal}(\overline{\mathbf{Q}}_{\tilde{\mathfrak{P}}'}/F_{\mathfrak{p}_i} \vee E_\lambda).$$

Then $v_1 \otimes \cdots \otimes v_r \in \mathbf{C}_p \otimes_{E_\lambda} W$ and we can easily verify that

$$g(\otimes_{i=1}^r v_i) = \chi(g)^{i_{\mathfrak{p}_1} + \cdots + i_{\mathfrak{p}_r}} (\otimes_{i=1}^r v_i) \quad \text{if} \quad g \in \mathrm{Gal}(\overline{\mathbf{Q}}_{\tilde{\mathfrak{P}}'}/\tilde{F}_{\mathfrak{P}'} \vee E_\lambda).$$

In view of the injectivity result of [10], III-6 and III-31, Theorem 1, the assertion follows immmediately.

Let $\sigma_\lambda : \mathrm{Gal}(\overline{\mathbf{Q}}/F) \longrightarrow GL(V)$ be as before. We define L-series $L(s, \sigma_\lambda)$ attached to σ_λ by

$$L(s, \sigma_\lambda) = \prod_{\mathfrak{p} \notin S} \det(1 - \sigma_\lambda(F_\mathfrak{p}) N(\mathfrak{p})^{-s})^{-1},$$

a formal Dirichlet series with coefficients in E_λ. We assume that F is normal over $\mathbf{Q}$ until (1.17). For $1 \leq i \leq r$, put

$$\sigma_\lambda^i(h) = \sigma_\lambda(s_i^{-1} h s_i), \qquad h \in H = \mathrm{Gal}(\overline{\mathbf{Q}}/F).$$

Put $\tau_\lambda = \bigotimes_\Omega \mathrm{Ind}_H^{H'} \sigma_\lambda$. Then we have

$$\tau_\lambda | H \cong \otimes_{i=1}^r \sigma_\lambda^i. \tag{1.16}$$

Therefore we obtain

$$\mathrm{Ind}_H^G(\otimes_{i=1}^r \sigma_\lambda^i) \cong \oplus_{\chi \in \widehat{G/H}} (\tau_\lambda \otimes \chi).$$

Here χ extends over irreducible representations of G/H and we have assumed that E_λ is sufficiently large so that every χ is realized over E_λ. Now the well known property of L-series yields

$$L(s, \otimes_{i=1}^{r} \sigma_\lambda^i) = \prod_{\chi \in \widehat{G/H}} L(s, \tau_\lambda \otimes \chi) \tag{1.17}$$

up to finitely many Euler factors.

We are going to consider a relation between $\bigotimes_\Omega \mathrm{Ind}_H^{H'} \sigma_\lambda$ and the Langlands L-function used to express the zeta functions of certain Shimura varieties (cf. Langlands [8]). Let F be a totally real algebraic number field and B be a quaternion algebra over F. Set $H = \mathrm{Gal}(\overline{\mathbf{Q}}/F)$. Fix an embedding of $\overline{\mathbf{Q}}$ into $\mathbf{C}$. Then J_F can be identified with $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})/H$. Let $G = \mathrm{Res}_{F/\mathbf{Q}}(B^\times)$. Then the L-group ${}^LG$ of G is given by

$${}^LG = GL_2(\mathbf{C})^{J_F} \times_s \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$$

where the multiplication is defined by

$$(g_1, \sigma_1)(g_2, \sigma_2) = (g_1\sigma_1(g_2), \sigma_1\sigma_2), \quad g_1, g_2 \in GL_2(\mathbf{C})^{J_F}, \quad \sigma_1, \sigma_2 \in \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}).$$

Here we take the action of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ on $GL_2(\mathbf{C})^{J_F}$ by

$$\sigma(g) = (g_{\sigma^{-1}\tau})_{\tau \in J_F} \quad \text{for} \quad g = (g_\tau)_{\tau \in J_F}, \quad g_\tau \in GL_2(\mathbf{C}).$$

Put ${}^LG^0 = GL_2(\mathbf{C})^{J_F}$. We shall define two representations of ${}^LG$. Let $V = \oplus_{\tau \in J_F} V_\tau$, $V_\tau \cong \mathbf{C}^2$. Let r_0^* be the standard representation of ${}^LG^0$ on V, i.e.,

$$g(\oplus_{\tau \in J_F} v_\tau) = \oplus_{\tau \in J_F}\, g_\tau v_\tau, \qquad g = (g_\tau) \in {}^LG^0.$$

For $\sigma \in \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$, define $I_\sigma \in GL(V)$ by

$$I_\sigma(\oplus_{\tau \in J_F} v_\tau) = \oplus_{\tau \in J_F}\, v_{\sigma^{-1}\tau}.$$

Then we can verify

$$I_{\sigma_1\sigma_2} = I_{\sigma_1} I_{\sigma_2}, \qquad \sigma_1, \sigma_2 \in \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}), \tag{1.18}$$

$$I_\sigma r_0^*(g) = r_0^*(\sigma(g)) I_\sigma, \qquad \sigma \in \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}), \quad g \in {}^LG^0. \tag{1.19}$$

Put

$$r_0((g, \sigma)) = r_0^*(g) I_\sigma, \qquad (g, \sigma) \in {}^LG.$$

By (1.18) and (1.19), we see easily that r_0 defines a representation of ${}^LG$ on V.

Let δ be a subset of J_F at which B splits and B ramifies at $J_F \backslash \delta$. We assume that δ is not empty. Since J_F is identified with $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})/H$, δ can be identified with a subset Ω of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})/H$. Let

$$H' = \{g \in \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \mid g\Omega = \Omega\}$$

and let F' be the subfield of $\overline{\mathbf{Q}}$ which corresponds to H'. Let $W = \bigotimes_{\tau \in \delta} V_\tau$, $V_\tau \cong \mathbf{C}^2$. Let r_1^* be the representation of ${}^LG^0$ on W defined by

$$g(\otimes_{\tau\in\delta} v_\tau) = \otimes_{\tau\in\delta}\, g_\tau v_\tau, \qquad g = (g_\tau) \in {}^LG^0.$$

For $\sigma \in \mathrm{Gal}(\overline{\mathbf{Q}}/F')$, define $J_\sigma \in GL(W)$ by

$$J_\sigma(\otimes_{\tau\in\delta} v_\tau) = \otimes_{\tau\in\delta}\, v_{\sigma^{-1}\tau}.$$

Then we have

$$J_{\sigma_1\sigma_2} = J_{\sigma_1} J_{\sigma_2}, \qquad J_\sigma r_1^*(g) = r_1^*(\sigma(g)) J_\sigma$$

for σ_1, σ_2, $\sigma \in \mathrm{Gal}(\overline{\mathbf{Q}}/F')$, $g \in {}^LG^0$. Therefore we can define a representation $r_1^{(0)}$ of $GL_2(\mathbf{C})^{J_F} \times_s \mathrm{Gal}(\overline{\mathbf{Q}}/F')$ by

$$r_1^{(0)}((g,\sigma)) = r_1^*(g) J_\sigma, \qquad g \in {}^LG^0, \quad \sigma \in \mathrm{Gal}(\overline{\mathbf{Q}}/F').$$

Then we let

$$r_1 = \mathrm{Ind}(r_1^{(0)}; {}^LG^0 \times_s \mathrm{Gal}(\overline{\mathbf{Q}}/F') \longrightarrow {}^LG).$$

THEOREM 1.4. *Let π be an automorphic representation of G_A. Let E be an algebraic number field of finite degree and λ be a finite place of E. Let σ_λ : $Gal(\overline{\mathbf{Q}}/F) \longrightarrow GL_2(E_\lambda)$ be a λ-adic representation. We assume that $L(s,\pi,r_0) = L(s,\sigma_\lambda)$ holds up to finitely many Euler factors, when we fix an embedding of E_λ into $\mathbf{C}$ and consider two L-series as Euler products over rational primes. Then we have*

$$L(s,\pi,r_1) = L(s, \bigotimes_\Omega Ind_H^{H'} \sigma_\lambda)$$

up to finitely many Euler factors.

PROOF: Let $\iota : E_\lambda \subset \mathbf{C}$ be the fixed embedding. Then $\iota\circ\sigma_\lambda$ defines a homorphism of $\mathrm{Gal}(\overline{\mathbf{Q}}/F)$ into $GL_2(\mathbf{C})$. Put $\rho = \iota\circ\sigma_\lambda$. Let $G = \cup_{i=1}^n s_i H$, $\Omega = \cup_{i=1}^r s_i H$. For $g \in \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$, set

$$\tilde{\rho}(g) = ((\rho(s_i^{-1} g s_{k(i)})), g) \in {}^LG. \tag{1.20}$$

Here the meaning of $(\rho(s_i^{-1} g s_{k(i)})) \in {}^LG^0$ is as follows. We identify J_F with $\{\, s_i|F; 1 \le i \le n\}$. Then the s_i-component of $(\rho(s_i^{-1} g s_{k(i)}))$ is $\rho(s_i^{-1} g s_{k(i)}) \in GL_2(\mathbf{C})$. We have set

$$g^{-1} s_i = s_{k(i)} h_i^*, \qquad 1 \le i \le n, \quad h_i^* \in H.$$

It can be verified that $\tilde{\rho}$ defines a homomorphism of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ into ${}^LG$. By the definition of $r_1^{(0)}$, we obtain

$$(1.21)\qquad (r_1^{(0)}\circ\tilde{\rho})(g))(\otimes_{i=1}^r v_{s_i}) = \otimes_{i=1}^r \rho(s_i^{-1}gs_{k(i)})v_{k(i)}, \qquad g\in \mathrm{Gal}(\overline{\mathbf{Q}}/F').$$

Comparing (1.21) with (1.3), we get

$$(1.22)\qquad \iota\circ\tau_\lambda \cong r_1^{(0)}\circ(\tilde{\rho}|\mathrm{Gal}(\overline{\mathbf{Q}}/F')),$$

where $\tau_\lambda = \bigotimes_\Omega \mathrm{Ind}_H^{H'}\sigma_\lambda$. Now let us show

$$(1.23)\qquad \iota\circ \mathrm{Ind}_{H'}^G\tau_\lambda \cong r_1\circ\tilde{\rho}.$$

Let $G = \cup_{i=1}^m t_i H'$. We realize $\mathrm{Ind}_{H'}^G\tau_\lambda$ by the similar formula to (1.3). Thus $\mathrm{Ind}_{H'}^G\tau_\lambda$ is realized on $\oplus_{i=1}^m W_i$, $W_i\cong W$, where W is the representation space of τ_λ. Put, for $g\in G$,

$$g^{-1}t_i = t_{l(i)}h_i', \qquad 1\le i\le m,\quad h_i'\in H'.$$

Then we have

$$(\mathrm{Ind}_{H'}^G\tau_\lambda)(g)(\oplus_{i=1}^m w_i) = \oplus_{i=1}^m \tau_\lambda(t_i^{-1}gt_{l(i)})w_{l(i)}, \qquad g\in G.$$

On the other hand, take a coset decomposition

$${}^LG = \cup_{i=1}^m \tilde{\rho}(t_i)({}^LG^0\times_s H')$$

and realize r_1 on $\oplus_{i=1}^m W_i'$, $W_i'\cong W'$ where W' is the representation space of $r_1^{(0)}$. Then we have

$$r_1(\tilde{\rho}(g))(\oplus_{i=1}^m w_i') = \oplus_{i=1}^m r_1^{(0)}(\tilde{\rho}(t_i^{-1}gt_{l(i)}))w_{l(i)}', \qquad w_i'\in W_i',\quad g\in G.$$

Since we may take $W_i' = W_i\otimes_{E_\lambda}\mathbf{C}$, (1.23) follows from (1.22). Let p be a rational prime at which π, $\mathrm{Ind}_H^G\sigma_\lambda$ and $\mathrm{Ind}_{H'}^G(\bigotimes_\Omega \mathrm{Ind}_H^{H'}\sigma_\lambda)$ are unramified and also

$$(1.24)\qquad L(s,\pi,r_0) = L(s,\sigma_\lambda)$$

holds at Euler p-factors. Fix a Frobenius element $F_p\in\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ of p. Let $\pi = \otimes_p\pi_p\otimes\pi_\infty$ and let $(g_p,F_p)\in {}^LG$ be the Langlands class of π_p. By (1.24), we have

$$(1.25)\qquad \tilde{\rho}(F_p) = (g_p,F_p).$$

Therefore we obtain

$$\begin{aligned}\det(1-Xr_1((g_p,F_p)) &= \det(1-Xr_1\circ\tilde{\rho}(F_p))\\ = \det(1-X(\iota\circ\mathrm{Ind}_{H'}^G\tau_\lambda)(F_p)) &= \iota(\det(1-X(\mathrm{Ind}_{H'}^G\tau_\lambda)(F_p)))\end{aligned}$$

by (1.23). This completes the proof.

Let M be a motive over F with coefficients in E. For every finite place λ of E, the λ-adic realization $H_\lambda(M)$ of M determines a λ-adic representation $\sigma_\lambda : \mathrm{Gal}(\overline{\mathbf{Q}}/F) \longrightarrow GL(H_\lambda(M))$; $\{\sigma_\lambda\}$ makes a compatible system of λ-adic representations. Let Ω be any non-empty subset of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})/\mathrm{Gal}(\overline{\mathbf{Q}}/F)$ and define F' as before. By Theorem 1.1, we have a compatible system of λ-adic representations $\{\otimes_\Omega \mathrm{Ind}_{\mathrm{Gal}(\overline{\mathbf{Q}}/F)}^{\mathrm{Gal}(\overline{\mathbf{Q}}/F')}\sigma_\lambda\}$ of $\mathrm{Gal}(\overline{\mathbf{Q}}/F')$. We conjecture that this system of representations is realized by a motive.

CONJECTURE 1.5. *There exists a motive M' over F' with coefficients in E such that the λ-adic representation of $Gal(\overline{\mathbf{Q}}/F')$ obtained from M' coincides with $\otimes_\Omega Ind_{\mathrm{Gal}(\overline{\mathbf{Q}}/F)}^{\mathrm{Gal}(\overline{\mathbf{Q}}/F')}\sigma_\lambda$ for every finite place λ of E.*

The rank of M' is $(\mathrm{rank}\ M)^r$ where $r = |\Omega|$. In analogy with the case of induced representations, we denote the above M' by $\otimes_\Omega \mathrm{Res}_{F/F'} M$. (Of course F' is not a subfield of F in general.) The computation of special values of the L-function attached to M' based on Deligne's conjecture shall be performed in §5 and shall be shown to be consistent with a conjecture and certain results of Shimura.

§2. Factorization of Deligne's period $c^\pm(M)$ of a motive M

Let E and F be algebraic number fields of finite degree. Let M be a motive over F with coefficients in E. Let λ be a finite place of E and consider the λ-adic realization $H_\lambda(M) \in V(E_\lambda)$ of M. For a prime ideal $\mathfrak{p}$ of F such that $(\lambda, \mathfrak{p}) = 1$, put

$$Z_\mathfrak{p}(M, X) = \det(1 - F_\mathfrak{p} X, H_\lambda(M)^{I_\mathfrak{p}})^{-1}, \tag{2.1}$$

where $F_\mathfrak{p}$ denotes a geometric Frobenius of $\mathfrak{p}$. It is conjectured that $Z_\mathfrak{p}(M, X) \in E[X]$ independently of λ. We shall assume this conjecture. For $\sigma \in J_E$, put

$$L_\mathfrak{p}(\sigma, M, s) = \sigma Z_\mathfrak{p}(M, N(\mathfrak{p})^{-s}), \tag{2.2}$$

$$L(\sigma, M, s) = \prod_\mathfrak{p} L_\mathfrak{p}(\sigma, M, s). \tag{2.3}$$

Let $\mathrm{Res}_{F/\mathbf{Q}}(M) = R_{F/\mathbf{Q}}(M)$ denote the motive over $\mathbf{Q}$ with coefficients in E obtained from M by the restriction of scalar. Then we have

$$L(\sigma, M, s) = L(\sigma, R_{F/\mathbf{Q}}(M), s) \tag{2.4}$$

for every $\sigma \in J_E$. Since $E \otimes_\mathbf{Q} \mathbf{C} \cong \mathbf{C}^{J_E}$, we can define a function $L^*(M, s)$ taking values in $E \otimes_\mathbf{Q} \mathbf{C}$ by arranging $L(\sigma, M, s)$. Deligne's conjecture predicts

$$L^*(M, 0)/c^+(R_{F/\mathbf{Q}}(M)) \in E \tag{2.5}$$

if 0 is critical for $R_{F/\mathbf{Q}}(M)$ (which is assumed to be homogeneous) with $E \subset E \otimes_{\mathbf{Q}} \mathbf{C}$ canonically. Here the period $c^+(R_{F/\mathbf{Q}}(M)) \in (E \otimes_{\mathbf{Q}} \mathbf{C})^\times$ is defined as follows.

Let M be a motive over $\mathbf{Q}$ with coefficients in E. Let $H_B(M) \in V(E)$ denote the Betti realization of M. Then the complex conjugation F_∞ acts on $H_B(M)$. We have

$$H_B(M) = H_B^+(M) \oplus H_B^-(M), \tag{2.6}$$

where $H_B^{\pm}(M)$ denotes the eigenspaces of $H_B(M)$ with eigenvalues ± 1. We assume that M is homogeneous of weight w. Then we have

$$H_B(M) \otimes_{E,\sigma} \mathbf{C} = \oplus_{p+q=w} H^{pq}(\sigma, M), \qquad \sigma \in J_E.$$

In view of the Gamma factor of the conjectural functional equation of $L^*(M,s)$, we find that if 0 is critical for M, then:

(2.7) Whenever $H^{pq}(\sigma, M) \neq \{0\}$ and $p < q$, $p < 0, q > -1$ must hold.

If w is odd, (2.7) is sufficient for 0 to be critical. If w is even, F_∞ must act on $\oplus_{\sigma \in J_E} H^{pp}(\sigma, M)$, $p = w/2$ by scalar. Put

$$F_\infty = (-1)^{p+\epsilon}, \ \epsilon = 0 \quad \text{or} \quad 1 \quad \text{on} \quad H^{pp}(\sigma, M).$$

Then

$$\begin{cases} p < \epsilon & \text{if} \quad p + \epsilon \quad \text{is even}, \\ -\epsilon - 1 < p & \text{if} \quad p + \epsilon \quad \text{is odd}, \end{cases} \tag{2.8}$$

must be satisfied; (2.7) and (2.8) are sufficient for 0 to be critical.

Remark. We see that $n \in \mathbf{Z}$ is critical for M hence the transcendental part of $L^*(M,n)$ is predictable by Deligne's conjecture if and only if the following conditions are satisfied. (Of course, we admit the conjectural functional equation for $L^*(M,s)$.)

$$p < n \leq q \qquad \text{if} \qquad H^{pq}(\sigma, M) \neq \{0\}, \quad p < q. \tag{2.9}$$

$$\begin{cases} n > p - \epsilon & \text{if} \quad p + \epsilon + n \quad \text{is even}, \\ n < p + \epsilon + 1 & \text{if} \quad p + \epsilon + n \quad \text{is odd}, \end{cases} \tag{2.10}$$

if $F_\infty = (-1)^{p+\epsilon}$, $\epsilon = 0$ or 1 on $H^{pp}(\sigma, M) \neq \{0\}$.

Let $H_{DR}(M) \in V(E)$ be the de Rham realization of M. We have the canonical isomorphism

$$I : H_B(M) \otimes_{\mathbf{Q}} \mathbf{C} \cong H_{DR}(M) \otimes_{\mathbf{Q}} \mathbf{C}$$

as $E \otimes_{\mathbf{Q}} \mathbf{C}$-modules. We choose $F^{\pm}(M) \in V(E)$ as certain subspaces of $H_{DR}(M)$ obtained from the Hodge filtration; explicitly we have

$$I^{-1}(F^{+}(M) \otimes_{E,\sigma} \mathbf{C}) = \begin{cases} \oplus_{p \geq q} H^{pq}(\sigma, M) & \text{if} \quad F_{\infty} = 1 \quad \text{on} \quad H^{pp}(\sigma, M), \\ \oplus_{p > q} H^{pq}(\sigma, M) & \text{if} \quad F_{\infty} = -1 \quad \text{on} \quad H^{pp}(\sigma, M). \end{cases}$$

$$I^{-1}(F^{-}(M) \otimes_{E,\sigma} \mathbf{C}) = \begin{cases} \oplus_{p > q} H^{pq}(\sigma, M) & \text{if} \quad F_{\infty} = 1 \quad \text{on} \quad H^{pp}(\sigma, M), \\ \oplus_{p \geq q} H^{pq}(\sigma, M) & \text{if} \quad F_{\infty} = -1 \quad \text{on} \quad H^{pp}(\sigma, M). \end{cases}$$

Put $H^{\pm}_{DR}(M) = H_{DR}(M)/F^{\mp}(M)$. We have the canonical isomorphisms

$$I^{\pm} : H^{\pm}_{B}(M) \otimes_{\mathbf{Q}} \mathbf{C} \cong H^{\pm}_{DR}(M) \otimes_{\mathbf{Q}} \mathbf{C}.$$

Let $\delta(M) = \det(I)$, $c^{\pm}(M) = \det(I^{\pm})$ be the determinants calculated by E-rational basis. Then $\delta(M)$, $c^{\pm}(M) \in (E \otimes_{\mathbf{Q}} \mathbf{C})^{\times}$ are determined up to multiplications by elements of E.

Now going back to the general case, let M be a motive over F with coefficients in E. We assume that F is totally real. For every $\tau \in J_F$, we have the Betti realization $H_{\tau,B}(M) \in V(E)$ of M and the complex conjugation F_{∞_τ} associated with τ acts on $H_{\tau,B}(M)$. Similarly to (2.6), we have

$$H_{\tau,B}(M) = H^{+}_{\tau,B}(M) \oplus H^{-}_{\tau,B}(M), \tag{2.11}$$

$H^{\pm}_{\tau,B}(M) \in V(E)$. We assume that $R_{F/\mathbf{Q}}(M)$ is homogeneous of weight w. Then we have

$$H_{\tau,B}(M) \otimes_{E,\sigma} \mathbf{C} = \oplus_{p+q=w} H^{pq}(\tau, \sigma, M), \qquad \sigma \in J_E.$$

If w is even, we assume that $\oplus_{\tau \in J_F} F_{\infty_\tau}$ acts on $\oplus_\tau \oplus_\sigma H^{pp}(\tau, \sigma, M)$, $p = w/2$ by scalar. The de Rham realization $H_{DR}(M) \in V(E)$ has the structure of a free $E \otimes_{\mathbf{Q}} F$-module. We have the canonical isomorphism

$$I_\tau : H_{\tau,B}(M) \otimes_{\mathbf{Q}} \mathbf{C} \cong H_{DR}(M) \otimes_{F,\tau} \mathbf{C}$$

as $(E \otimes_{\mathbf{Q}} \mathbf{C})$-modules. By the Hodge filtration obtained from the convergence of the spectral sequence

$$E_1^{pq} = H^q(M, \Omega^p) \Longrightarrow H^{p+q}_{DR}(M),$$

we can define subspaces $F^{\pm}(M) \in V(E)$ of $H_{DR}(M)$ as in the case $F = \mathbf{Q}$; $F^{\pm}(M)$ has the structure of a vector space over F. We have

$$I_\tau^{-1}(F^{+}(M) \otimes_{F,\tau} \mathbf{C}) = \begin{cases} \oplus_{\sigma \in J_E} \oplus_{p \geq q} H^{pq}(\tau, \sigma, M) & \text{if} \quad F_{\infty_\tau} = 1 \quad \text{on} \quad H^{pp}(\tau, \sigma, M), \\ \oplus_{\sigma \in J_E} \oplus_{p > q} H^{pq}(\tau, \sigma, M) & \text{if} \quad F_{\infty_\tau} = -1 \quad \text{on} \quad H^{pp}(\tau, \sigma, M). \end{cases}$$

$$I_\tau^{-1}(F^-(M)\otimes_{F,\tau}\mathbf{C})$$
$$=\begin{cases}\oplus_{\sigma\in J_E}\oplus_{p>q}H^{pq}(\tau,\sigma,M) & \text{if } F_{\infty_\tau}=1 \text{ on } H^{pp}(\tau,\sigma,M),\\ \oplus_{\sigma\in J_E}\oplus_{p\geq q}H^{pq}(\tau,\sigma,M) & \text{if } F_{\infty_\tau}=-1 \text{ on } H^{pp}(\tau,\sigma,M).\end{cases}$$

Put $H_{DR}^{\pm}(M)=H_{DR}(M)/F^{\mp}(M)$. We have the canonical isomorphisms

$$I_\tau^{\pm}: H_{\tau,B}^{\pm}(M)\otimes_{\mathbf{Q}}\mathbf{C}\cong H_{DR}^{\pm}(M)\otimes_{F,\tau}\mathbf{C}$$

as $(E\otimes_{\mathbf{Q}}\mathbf{C})$-modules. Let $\delta_\tau(M)=\det(I_\tau)$, $c_\tau^{\pm}(M)=\det(I_\tau^{\pm})$ be the determinant calculated by E-rational basis of the left hand side and by $E\otimes_{\mathbf{Q}}F$-basis (since they are free $E\otimes_{\mathbf{Q}}F$-modules) of the right hand side modules. Then $\delta_\tau(M)$, $c_\tau^{\pm}(M)\in(E\otimes_{\mathbf{Q}}\mathbf{C})^\times$ are determined up to multiplications by elements of $(E\vee\tilde{F})$. Here $\tilde{F}$ denotes the normal closure of F in $\mathbf{Q}$.

PROPOSITION 2.1. *Let the notation be the same as above. We have*

$$c^+(R_{F/\mathbf{Q}}(M))=\prod_{\tau\in J_F}c_\tau^+(M),\qquad c^-(R_{F/\mathbf{Q}}(M))=\prod_{\tau\in J_F}c_\tau^-(M),$$
$$\delta(R_{F/\mathbf{Q}}(M))=\prod_{\tau\in J_F}\delta_\tau(M),$$

up to multiplications by elements of $E\vee\tilde{F}$.

PROOF: It is known (cf. Deligne [6]) that $H_B(R_{F/\mathbf{Q}}(M))=\oplus_{\tau\in J_F}H_{\tau,B}(M)$ as vector spaces over E and that $H_{DR}(R_{F/\mathbf{Q}}(M))$ can be identified with $H_{DR}(M)$ forgetting its structure as a vector space over F. We see that

$$H_B^+(R_{F/\mathbf{Q}}(M))=\oplus_{\tau\in J_F}H_{\tau,B}^+(M),\qquad H_B^-(R_{F/\mathbf{Q}}(M))=\oplus_{\tau\in J_F}H_{\tau,B}^-(M),$$

and that $H_{DR}^{\pm}(R_{F/\mathbf{Q}}(M))$ is identified with $H_{DR}^{\pm}(M)$ forgetting the structure of a vector space over F. We have the isomorphism of $(E\otimes_{\mathbf{Q}}\mathbf{C})$-modules

$$I^+: H_B^+(R_{F/\mathbf{Q}}(M))\otimes_{\mathbf{Q}}\mathbf{C}\cong H_{DR}^+(R_{F/\mathbf{Q}}(M))\otimes_{\mathbf{Q}}\mathbf{C}.$$

Since

$$\text{(2.12)}\qquad \begin{aligned}H_{DR}^+(R_{F/\mathbf{Q}}(M))\otimes_{\mathbf{Q}}\mathbf{C}&\cong H_{DR}^+(R_{F/\mathbf{Q}}(M))\otimes_F F\otimes_{\mathbf{Q}}\mathbf{C}\\&\cong\oplus_{\tau\in J_F}(H_{DR}^+(M)\otimes_{F,\tau}\mathbf{C}),\end{aligned}$$

I^+ may be written as

$$I^+:\oplus_{\tau\in J_F}(H_{\tau,B}^+(M)\otimes_{\mathbf{Q}}\mathbf{C})\cong\oplus_{\tau\in J_F}(H_{DR}^+(M)\otimes_{F,\tau}\mathbf{C}).$$

Restricting I^+ to a direct factor, we obtain

$$I_\tau^+: H_{\tau,B}^+(M)\otimes_{\mathbf{Q}}\mathbf{C}\cong H_{DR}^+(M)\otimes_{F,\tau}\mathbf{C}.$$

The isomorphism (2.12) does not preserve E-structure but preserve $E\vee\tilde{F}$-structure on both sides. Hence we obtain the first assertion. The second and the last assertions can be proved in similar way. This completes the proof.

§3. Variations of periods $c_\tau^\pm(M)$ under standard operations

(I) Let M and N be motives over F with coefficients in E. Let $d(M)$ and $d(N)$ be the ranks of M and N respectively. For example, we have $d(M) = \dim_E H_{\tau,B}(M)$ for every $\tau \in J_F$. We assume that $R_{F/\mathbf{Q}}(M)$ and $R_{F/\mathbf{Q}}(N)$ are homogeneous of weights w and w' respectively. For $\tau \in J_F$, we obviously have

$$(3.1)\quad \begin{aligned} &H_{\tau,B}(M\otimes N) = H_{\tau,B}(M)\otimes_E H_{\tau,B}(N),\\ &H^+_{\tau,B}(M\otimes N) = (H^+_{\tau,B}(M)\otimes_E H^+_{\tau,B}(N))\oplus(H^-_{\tau,B}(M)\otimes_E H^-_{\tau,B}(N)),\\ &H^-_{\tau,B}(M\otimes N) = (H^+_{\tau,B}(M)\otimes_E H^-_{\tau,B}(N))\oplus(H^-_{\tau,B}(M)\otimes_E H^+_{\tau,B}(N)),\\ &H_{DR}(M\otimes N) = H_{DR}(M)\otimes_{(E\otimes_{\mathbf{Q}}F)} H_{DR}(N).\end{aligned}$$

Since

$$\begin{aligned} H_{\tau,B}(M)\otimes_E H_{\tau,B}(N)\otimes_{\mathbf{Q}}\mathbf{C} &\cong (H_{\tau,B}(M)\otimes_{\mathbf{Q}}\mathbf{C})\otimes_{E\otimes\mathbf{C}}(H_{\tau,B}(N)\otimes_{\mathbf{Q}}\mathbf{C}),\\ H_{DR}(M)\otimes_{(E\otimes_{\mathbf{Q}}F)} H_{DR}(N)\otimes_{F,\tau}\mathbf{C} &\\ \cong (H_{DR}(M)\otimes_{F,\tau}\mathbf{C})&\otimes_{E\otimes\mathbf{C}}(H_{DR}(N)\otimes_{F,\tau}\mathbf{C}),\end{aligned}$$

we have

$$(3.2)\qquad \delta_\tau(M\otimes N) = \delta_\tau(M)^{d(N)}\delta_\tau(N)^{d(M)}.$$

Assume $d(N)=1$, w' is even and put $p' = w'/2$. Assume further that $H_B(R_{F/\mathbf{Q}}(N))\otimes_{\mathbf{Q}}\mathbf{C}$ is of Hodge type (p',p'). If $H_B(R_{F/\mathbf{Q}}(N))\otimes_{\mathbf{Q}}\mathbf{C}$ does not have a component of Hodge type (p,p), we have

$$F^\pm(M\otimes N) = F^\pm(M)\otimes_{E\otimes F} H_{DR}(N).$$

In view of (3.1), we immediately obtain

$$(3.3)\qquad c_\tau^\pm(M\otimes N) = c_\tau^{\pm\epsilon_\tau}(M)\delta_\tau(N)^{d_\tau^\pm(M)},$$

where $F_{\infty_\tau} = (-1)^{\epsilon_\tau}$ on $H_{\tau,B}(N)$ and $d_\tau^\pm(M) = \dim_E H^\pm_{\tau,B}(M)$. If $H_B(R_{F/\mathbf{Q}}(M))\otimes_{\mathbf{Q}}\mathbf{C}$ has a component of type (p,p), we assume that F_∞ acts on both of $\oplus_{\sigma\in J_E}H^{pp}(\sigma,R_{F/\mathbf{Q}}(M))$ and $\oplus_{\sigma\in J_E}H^{p'p'}(\sigma,R_{F/\mathbf{Q}}(N))$ by scalar. Then we obtain

$$(3.4)\quad c_\tau^\pm(M\otimes N) = \begin{cases} c_\tau^\pm(M)\delta_\tau(N)^{d_\tau^\pm(M)} & \text{if } F_\infty = 1 \text{ on } H_B(R_{F/\mathbf{Q}}(N)),\\ c_\tau^\mp(M)\delta_\tau(N)^{d_\tau^\mp(M)} & \text{if } F_\infty = -1 \text{ on } H_B(R_{F/\mathbf{Q}}(N)).\end{cases}$$

For $n \in \mathbf{Z}$, let $T(n)$ denote the Tate motive over F. We have

$$L^*(M \otimes T(n), s) = L^*(M, s+n), \tag{3.5}$$

$$\delta_\tau(T(n)) = (2\pi\sqrt{-1})^n, \tag{3.6}$$

$$F_\infty \quad \text{acts on} \quad H_B(R_{F/\mathbf{Q}}(T(n))) \quad \text{by} \quad (-1)^n. \tag{3.7}$$

Hence, if F_∞ acts on $\oplus_{\sigma \in J_E} H^{pp}(\sigma, R_{F/\mathbf{Q}}(M))$ by scalar, we obtain

$$\begin{aligned} c_\tau^\pm(M(n)) &= \begin{cases} (2\pi\sqrt{-1})^{nd_\tau^\pm(M)} c_\tau^\pm(M) & \text{if } n \text{ is even,} \\ (2\pi\sqrt{-1})^{nd_\tau^\mp(M)} c_\tau^\mp(M) & \text{if } n \text{ is odd,} \end{cases} \\ \delta_\tau(M(n)) &= (2\pi\sqrt{-1})^{nd(M)} \delta_\tau(M), \end{aligned} \tag{3.8}$$

where $M(n) = M \otimes T(n)$.

(II) Let M, N and related notations be the same as in the beginning of (I).

PROPOSITION 3.1. *We assume* $\oplus_{\sigma \in J_E} H^{pp}(\sigma, R_{F/\mathbf{Q}}(M)) = \{0\}$. *We further assume that if*

$$H^{pq}(\tau, \sigma, M) \neq \{0\}, \quad p > q \quad \text{and} \quad H^{p'q'}(\tau, \sigma, M) \neq \{0\}, \quad p' \geq q'$$

for $\tau \in J_F$, $\sigma \in J_E$, *then* $p - q > p' - q'$ *holds. Then we have*

$$c_\tau^+(M \otimes N) = c_\tau^+(M)^{d_\tau^+(N)} c_\tau^-(M)^{d_\tau^-(N)} \delta_\tau(N)^{d_\tau^+(M)},$$
$$c_\tau^-(M \otimes N) = c_\tau^+(M)^{d_\tau^-(N)} c_\tau^-(M)^{d_\tau^+(N)} \delta_\tau(N)^{d_\tau^-(M)}.$$

PROOF: By the assumption, we immediately obtain

$$F^+(M) = F^-(M), \quad F^\pm(M \otimes N) = F^\pm(M) \otimes_{E \otimes F} H_{DR}(N).$$

Let

$$I_\tau^M : H_{\tau,B}(M) \otimes_{\mathbf{Q}} \mathbf{C} \cong H_{DR}(M) \otimes_{F,\tau} \mathbf{C}$$
$$I_\tau^N : H_{\tau,B}(N) \otimes_{\mathbf{Q}} \mathbf{C} \cong H_{DR}(N) \otimes_{F,\tau} \mathbf{C}$$

be canonical isomorphisms. Let $u_1^\pm, \cdots, u_n^\pm$ (resp. $v_1, \cdots, v_m$) be a basis of $H_{\tau,B}^\pm(M)$ (resp. $H_{\tau,B}(N)$) over E where $n = d_\tau^\pm(M)$, $m = d_\tau(N)$. Let $e_1^-, \cdots, e_n^-$ be a basis of $F^-(M)$ as free $E \otimes F$-module. Take $e_1^+, \cdots, e_n^+$ so that $e_1^+, \cdots, e_n^+$, $e_1^-, \cdots, e_n^-$ becomes a basis of $H_{DR}(M)$ as free $E \otimes F$-module. Let $d_1, \cdots, d_m$ be a basis of $H_{DR}(N)$ as free $E \otimes F$-module. Put

$$I_\tau^M(u_i^\pm) = \sum_{j=1}^n x_{ij}^{+,\pm} e_j^+ + \sum_{j=1}^n x_{ij}^{-,\pm} e_j^-,$$
$$I_\tau^N(v_i) = \sum_{j=1}^m y_{ij} d_j$$

with $x_{ij}^{\pm,\pm}$, $y_{ij} \in E \otimes_{\mathbf{Q}} \mathbf{C}$. Put

$$X_{11} = (x_{ij}^{+,+}),\quad X_{12} = (x_{ij}^{-,+}),\quad X_{21} = (x_{ij}^{+,-}),\quad X_{22} = (x_{ij}^{-,-}) \in M_n(E \otimes_{\mathbf{Q}} \mathbf{C}),$$
$$Y = (y_{ij}) \in M_m(E \otimes_{\mathbf{Q}} \mathbf{C}).$$

Then we have

$$c_\tau^+(M) = \det(X_{11}),\quad c_\tau^-(M) = \det(X_{21}),\quad \delta_\tau(N) = \det(Y).$$

We may assume that $v_1, \cdots, v_t$ (resp. $v_{t+1}, \cdots, v_m$) is a basis of $H_{\tau,B}^+(N)$ (resp. $H_{\tau,B}^-(N)$) where $t = d_\tau^+(N)$. We have

$$(I_\tau^M \otimes I_\tau^N)(u_i^+ \otimes v_j) = (\sum_{k=1}^n x_{ik}^{+,+} e_k^+) \otimes (\sum_{l=1}^m y_{jl} d_l)$$
$$(I_\tau^M \otimes I_\tau^N)(u_i^- \otimes v_j) = (\sum_{k=1}^n x_{ik}^{+,-} e_k^+) \otimes (\sum_{l=1}^m y_{jl} d_l)$$

modulo $F^-(M \otimes N)$. Therefore we have

$$c_\tau^+(M) = \det\begin{pmatrix} X_{11} \otimes Y_1 \\ X_{21} \otimes Y_2 \end{pmatrix},\qquad c_\tau^-(M) = \det\begin{pmatrix} X_{11} \otimes Y_2 \\ X_{21} \otimes Y_1 \end{pmatrix},$$

where $Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$ with $Y_1 \in M_{t,m}(E \otimes_{\mathbf{Q}} \mathbf{C})$, $Y_2 \in M_{m-t,m}(E \otimes_{\mathbf{Q}} \mathbf{C})$. Hence we obtain

$$c_\tau^+(M) = \det(X_{11})^t \det(X_{21})^{m-t} (\det Y)^n,$$
$$c_\tau^-(M) = \det(X_{11})^{m-t} \det(X_{21})^t (\det Y)^n,$$

and the assertion follows.

(III) Let $n \geq 2$ and suppose that we are given motives M_i over F with coefficients in E for $1 \leq i \leq n$. We assume that M_i is of rank 2 for every i and let

$$H_{\tau,B}(M_i) \otimes_{E,\sigma} \mathbf{C} = H^{a_i(\tau,+),a_i(\tau,-)}(\tau, \sigma, M_i) \oplus H^{a_i(\tau,-),a_i(\tau,+)}(\tau, \sigma, M_i),$$

$1 \leq i \leq n$, $\tau \in J_F$. We assume that $a_i(\tau,+) > a_i(\tau,-)$ for every $\tau \in J_F$ and i. We shall give a formula for $c_\tau^\pm(M_1 \otimes M_2 \otimes \cdots \otimes M_n)$, which is suggested by Blasius [2]. Let Λ be the set of all maps from $\{1, 2, \cdots, n\}$ to $\{\pm 1\}$. Set

$$\Lambda_\pm = \{\lambda \in \Lambda \mid \prod_{i=1}^n \lambda(i) = \pm 1\},$$
$$\Lambda^+ = \{\lambda \in \Lambda \mid \sum_{i=1}^n a_i(\tau, \lambda(i)) > \sum_{i=1}^n a_i(\tau, -\lambda(i))\}.$$

We have $|\Lambda_\pm| = 2^{n-1}$. We assume that

$$\sum_{i=1}^{n} a_i(\tau, \lambda(i)) \neq \sum_{i=1}^{n} a_i(\tau, -\lambda(i)) \quad \text{for every} \quad \lambda \in \Lambda. \tag{3.9}$$

We note that if (3.9) is not satisfied, then the action of F_{∞_τ} on $H^{pp}(\tau, \sigma, M_1 \otimes \cdots \otimes M_n)$ is not a scalar. By (3.9), we have $|\Lambda^+| = 2^{n-1}$ since $\lambda \in \Lambda^+$ is equivalent to $-\lambda \notin \Lambda^+$. Let n_i (resp. m_i) be the number of $\lambda \in \Lambda^+$ such that $\lambda(i) = 1$ (resp. $\lambda(i) = -1$). We have

$$n_i + m_i = 2^{n-1}. \tag{3.10}$$

PROPOSITION 3.2. *We assume that (3.9) holds for every $\tau \in J_F$. Then we have*

$$c_\tau^\pm(M_1 \otimes M_2 \otimes \cdots \otimes M_n) = \prod_{i=1}^{n} (c_\tau^+(M_i) c_\tau^-(M_i))^{(n_i - m_i)/2} \delta_\tau(M_i)^{m_i}.$$

PROOF: Take $u_i^\pm$ so that

$$E u_i^\pm = H_{\tau,B}^\pm(M_i), \qquad 1 \le i \le n.$$

Choose d_i^- so that

$$(E \otimes F) d_i^- = F^-(M_i) = F^+(M_i)$$

and choose d_i^+ so that

$$H_{DR}(M_i) = (E \otimes F) d_i^+ + (E \otimes F) d_i^-, \qquad 1 \le i \le n.$$

Let

$$I_\tau^{M_i} : H_{\tau,B}(M_i) \otimes_{\mathbf{Q}} \mathbf{C} \cong H_{DR}(M_i) \otimes_{F,\tau} \mathbf{C}$$

be the canonical isomorphism and put

$$I_\tau^{M_i}(u_i^\pm) = x_i^{+,\pm} d_i^+ + x_i^{-,\pm} d_i^-, \qquad 1 \le i \le n$$

with $x_i^{\pm,\pm} \in E \otimes_{\mathbf{Q}} \mathbf{C}$. Then we have

$$c_\tau^\pm(M_i) = x_i^{+,\pm}, \qquad \delta_\tau(M_i) = \det \begin{pmatrix} x_i^{+,+} & x_i^{-,+} \\ x_i^{+,-} & x_i^{-,-} \end{pmatrix}.$$

A basis of $H_{\tau,B}^\pm(M_1 \otimes \cdots \otimes M_n)$ over E is given by $\otimes_{i=1}^n u_i^{\epsilon(i)}$ when ϵ extends over $\Lambda_\pm$. Also we see easily that a basis of $H_{DR}^\pm(M_1 \otimes \cdots \otimes M_n)$ is given by $\otimes_{i=1}^n d_i^{\lambda(i)}$ mod $F^-(M_1 \otimes \cdots \otimes M_n)$ when λ extends over Λ^+. Since

$$\otimes_i I_\tau^{M_i}(\otimes_i u_i^{\epsilon(i)}) = \otimes_i (x_i^{+,\epsilon(i)} d_i^+ + x_i^{-,\epsilon(i)} d_i^-) = \sum_{\lambda \in \Lambda^+} \prod_{i=1}^{n} x_i^{\lambda(i),\epsilon(i)} \otimes_i d_i^{\lambda(i)}$$

mod $F^-(M_1 \otimes M_2 \otimes \cdots \otimes M_n)$, we have

$$c_\tau^{\pm}(M_1 \otimes M_2 \otimes \cdots \otimes M_n) = \det(X^{\pm}),$$

where $X^{\pm}$ is the $2^{n-1} \times 2^{n-1}$-matrix whose (λ, ϵ)-entry for $\lambda \in \Lambda^+$, $\epsilon \in \Lambda_{\pm}$ is given by $\prod_{i=1}^n x_i^{\lambda(i),\epsilon(i)}$. We shall prove the formula for c_τ^+ since the other case can be shown similarly.

It suffices to show

$$(3.11)\quad \det(X^+) = c\prod_{i=1}^n (x_i^{+,+}x_i^{+,-})^{(n_i-m_i)/2}(x_i^{+,+}x_i^{-,-} - x_i^{-,+}x_i^{+,-})^{m_i}, \qquad c \in \mathbf{Q},$$

regarding $x_i^{\pm,\pm}$, $1 \le i \le n$ as indeterminates. It is obvious that $\det(X^+)$ is a homogeneous polynomial of degree $2^{n-1}n$ with $\mathbf{Z}$-coefficients of $4n$-variables $x_i^{\pm,\pm}$. Fix i, $1 \le i \le n$. If we change variables $x_i^{+,\pm} \longrightarrow \mu x_i^{+,\pm}$ with $\mu \in \mathbf{C}$, then every (λ, ϵ)-entry of X^+ with $\lambda(i) = 1$ is multiplied by μ. Hence $\det(X^+)$ is multiplied by μ^{n_i}. Therefore we have

$$(3.12)\qquad \det(X^+) = \sum_{a+b=n_i} (x_i^{+,+})^a(x_i^{+,-})^b Q_{a,b}$$

where $Q_{a,b}$ is a polynomial which does not contain the variables $x_i^{+,\pm}$. Suppose $\lambda \in \Lambda^+$, $\lambda(i) = -1$. Put $\lambda'(j) = \lambda(j)$, $j \ne i$, $\lambda'(i) = 1$. Then $\lambda' \in \Lambda^+$ since $a_i(\tau, +) > a_i(\tau, -)$. Thus we may set

$$X^+ = \begin{pmatrix} x_i^{+,+}A & x_i^{+,-}B \\ x_i^{+,+}C & x_i^{+,-}D \\ x_i^{-,+}C & x_i^{-,-}D \end{pmatrix}$$

where A, B, C and D are $(n_i - m_i) \times 2^{n-2}$, $(n_i - m_i) \times 2^{n-2}$, $m_i \times 2^{n-2}$ and $m_i \times 2^{n-2}$ matrices respectively which does not contain the variables $x_i^{\pm,\pm}$. By standard operations on matrices, we have

$$\begin{aligned} \det(X^+) &= \det\begin{pmatrix} x_i^{+,+}A & x_i^{+,-}B \\ x_i^{+,+}C & x_i^{+,-}D \\ 0 & x_i^{-,-}D - (x_i^{-,+}x_i^{+,-}/x_i^{+,+})D \end{pmatrix} \\ &= \det\begin{pmatrix} A & x_i^{+,+}x_i^{+,-}B \\ C & 0 \\ 0 & (x_i^{+,+}x_i^{-,-} - x_i^{-,+}x_i^{+,-})D \end{pmatrix} \\ &= (x_i^{+,+}x_i^{-,-} - x_i^{-,+}x_i^{+,-})^{m_i}\det\begin{pmatrix} A & x_i^{+,+}x_i^{+,-}B \\ C & 0 \\ 0 & D \end{pmatrix}. \end{aligned}$$

Hence we have

$$\det(X^+) = (x_i^{+,+}x_i^{-,-} - x_i^{-,+}x_i^{+,-})^{m_i} \sum_j (x_i^{+,+}x_i^{+,-})^j P_j$$

where P_j is a polynomial which does not contain the variables $x_i^{\pm,\pm}$. By (3.12), $P_j = 0$ except for $m_i + 2j = n_i$. Therefore we have

$$\det(X^+) = (x_i^{+,+}x_i^{-,-} - x_i^{-,+}x_i^{+,-})^{m_i} (x_i^{+,+}x_i^{+,-})^{(n_i-m_i)/2} Q$$

where Q is a polynomial with **Q**-coefficients which does not contain the variables $x_i^{\pm,\pm}$. Since this expression holds for arbitrary i, we obtain (3.11). This completes the proof.

§4. On motives attached to Hilbert modular forms and Shimura's invariants

Let F be a totally real algebraic number field of degree n over $\mathbf{Q}$. Let $k = (k(\tau)) \in \mathbf{Z}^{J(F)}$ be a weight. By the Hilbert modular cusp form of weight k, we understand an element of $S_k(\mathfrak{c}, \psi)$ in the notation of Shimura [12], p. 649. Assume that f is a non-zero common eigenfunction of all Hecke operators. We attach Dirichlet series $D(s, f)$ to f by (2.25) of [12]. Now the form of the Gamma factor and the functional equation of $D(s, f)$ (cf. (2.47), (2.48) of [12]) suggest the following conjecture.

CONJECTURE 4.1. *Assume $k(\tau)$ mod 2 is independent of τ and put $k_0 = \max_{\tau \in J_F} k(\tau)$. Let E be the algebraic number field of finite degree generated by eigenvalues of Hecke operators of f (cf. [12], Prop. 2.8.). Then there exists a motive M_f over F with coefficients in E which satisfies the following conditions.*

(1) $L(\sigma, M_f, s) = D(s, f^\sigma)$ for every $\sigma \in J_E$.

(2) $H_{\tau,B}(M_f) \otimes_{E,\sigma} \mathbf{C} \cong H^{(k_0+k(\tau))/2-1,\,(k_0-k(\tau))/2}(\tau, \sigma, M_f) \oplus H^{(k_0-k(\tau))/2,\,(k_0+k(\tau))/2-1}(\tau, \sigma, M_f)$, $\sigma \in J_E$, $\tau \in J_F$.

(3) $\bigwedge^2 M_f \cong \mathrm{Art}_{\psi^{-1}}(1 - k_0)$ where $\mathrm{Art}_{\psi^{-1}}$ denotes the Artin motive attached to ψ.

Let χ be a Hecke character of F of finite order. Let $\mathfrak{c}$ be the conductor of χ and $\mathbf{Q}(\chi)$ be the field generated over $\mathbf{Q}$ by values of χ. As in [6], §6, we can attach a motive $\mathrm{Art}_\chi = N_\chi$ over F with coefficients in $\mathbf{Q}(\chi)$ such that $L(s, \chi^\sigma) = L(\sigma, N_\chi, s)$ for every $\sigma \in J_{\mathbf{Q}(\chi)}$. The rank of N_χ is 1 and the Hodge type of $H_{\tau,B}(N_\chi) \otimes_{\mathbf{Q}} \mathbf{C}$ is $(0, 0)$ for every $\tau \in J_F$. For the real archimedean place ∞_τ corresponding to $\tau \in J_F$, we have

$$\chi_{\infty_\tau}(x) = \mathrm{sgn}(x)^{m_\tau}, \qquad x \in k_{\infty_\tau}^\times \cong \mathbf{R}^\times, \quad m_\tau = 0 \quad \text{or} \quad 1. \tag{4.1}$$

If $m_\tau = 0$ (resp. $m_\tau = 1$), then F_{∞_τ} acts on $H_{\tau,B}(N_\chi)$ by 1 (resp. -1). We are going to calculate $\delta(R_{F/\mathbf{Q}}(N_\chi))$.

For this purpose, let us recall the following facts concerning an Artin motive M over $\mathbf{Q}$. Let ρ be a representation of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ into $GL(V)$ where V is a vector space over E of finite dimension m. Then there exists an Artin motive $M = \mathrm{Art}_\rho$ over $\mathbf{Q}$ with coefficients in E such that (cf. [6])

$$L(s, \rho^\sigma) = L(\sigma, M, s) \quad \text{for every} \quad \sigma \in J_E, \tag{4.2}$$

$$H_B(M) = V, \qquad H_{DR}(M) = (V \otimes_{\mathbf{Q}} \overline{\mathbf{Q}})^{\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})}. \tag{4.3}$$

Obviously $\delta(M) = \delta(\bigwedge^m M)$ and $\bigwedge^m M$ is the Artin motive attached to the representation $\det \rho$ of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$. For a Dirichlet character η of $\mathbf{Q}$ of conductor (f), $f > 0$, put

$$g_0(\eta) = \sum_{u=1}^{f} \eta(u) \exp(2\pi\sqrt{-1}u/f). \tag{4.4}$$

Then, as is shown in [6], §6, we have

$$\delta(\bigwedge^m M) = g_0((\det \rho)_*)^{-1} \tag{4.5}$$

where $(\det \rho)_*$ denotes the Dirichlet character associated to $\det \rho$.

We may regard χ as a character of $\mathrm{Gal}(K/F)$ where K is a finite Galois extension of $\mathbf{Q}$. Put $\rho = \mathrm{Ind}(\chi; \mathrm{Gal}(K/F) \longrightarrow \mathrm{Gal}(K/\mathbf{Q}))$. Then $R_{F/\mathbf{Q}}(N_\chi)$ is the Artin motive associated with ρ. We have (cf. [5], Prop. 1.2)

$$(\det \rho)(\sigma) = \chi(t(\sigma))\epsilon(\sigma), \qquad \sigma \in \mathrm{Gal}(K/\mathbf{Q})$$

where t denotes the transfer map from $\mathrm{Gal}(K/\mathbf{Q})^{ab}$ to $\mathrm{Gal}(K/F)^{ab}$ and ϵ denotes the determinant of the left regular representation of $\mathrm{Gal}(K/\mathbf{Q})$ on $\mathrm{Gal}(K/\mathbf{Q})/\mathrm{Gal}(K/F)$. Let χ_* denote the character of ideal class group of conductor $\mathfrak{c}$ of F associated with χ and let ϵ_* denote the Dirichlet character associated with ϵ. We have

$$(\det \rho)_*(n) = \chi_*(n)\epsilon_*(n), \qquad n \in \mathbf{Z}, \quad n > 0.$$

Define a Gauss sum by

$$g(\chi) = \sum_{x \in \mathfrak{c}^{-1}\mathfrak{d}_F^{-1}/\mathfrak{d}_F^{-1},\, x >> 0} \chi_*(x\mathfrak{c}\mathfrak{d}_F) \exp(2\pi\sqrt{-1}\mathrm{Tr}_{F/\mathbf{Q}}(x)) \tag{4.6}$$

where $\mathfrak{d}_F$ denotes the different of F.

LEMMA 4.2. *Put* $E = \mathbf{Q}(\chi)$. *We have* $g(\chi)/g_0(\det\rho) \in E \vee \tilde{F}$.

We omit the proof which is not difficult.

Let $M = M_f$ and $N = N_\chi$ be as above. Define $m_\tau = 0, 1$ by (4.1) and let $\epsilon_\tau = +$ (resp. $-$) if $m_\tau = 0$ (resp. 1) for $\tau \in J_F$. We assume that $k(\tau) \geq 2$ for every τ. Let E denote the number field generated by the eigenvalues of f under Hecke operators and the values of χ. By (3.4), we have

$$c_\tau^+(M \otimes N) = c_\tau^{\epsilon_\tau}(M)\delta_\tau(N), \qquad c_\tau^-(M \otimes N) = c_\tau^{-\epsilon_\tau}(M)\delta_\tau(N).$$

By Proposition 2.1, we have

$$c^+(R_{F/\mathbf{Q}}(M \otimes N)) = \prod_\tau c_\tau^{\epsilon_\tau}(M)\delta_\tau(N), \qquad c^-(R_{F/\mathbf{Q}}(M \otimes N)) = \prod_\tau c_\tau^{-\epsilon_\tau}(M)\delta_\tau(N)$$

modulo $(E \vee \tilde{F})^\times$. We have, by Lemma 4.2,

$$\prod_\tau \delta_\tau(N) = \delta(R_{F/\mathbf{Q}}(N)) = g_0(\det\rho)^{-1} = g(\chi)^{-1}$$

modulo $(E \vee \tilde{F})^\times$. By (3.8) we obtain

$$c^+((R_{F/\mathbf{Q}}(M \otimes N))(m)) = \begin{cases} (2\pi\sqrt{-1})^{nm} \prod c_\tau^{\epsilon_\tau}(M) g(\chi)^{-1} & \text{if } m \text{ is even} \\ (2\pi\sqrt{-1})^{nm} \prod c_\tau^{-\epsilon_\tau}(M) g(\chi)^{-1} & \text{if } m \text{ is odd} \end{cases}$$

modulo $(E \vee \tilde{F})^\times$. Put

$$D(s, f, \chi^{-1}) = \sum_{\mathfrak{n}} c(\mathfrak{n}, f)\chi(n)^{-1} N(\mathfrak{n})^{-s}.$$

Then Deligne's conjecture predicts

$$D(m, f, \chi^{-1})/((2\pi\sqrt{-1})^{nm} \prod_{\tau \in J_F} c_\tau^{(-1)^m \epsilon_\tau}(M) g(\chi)^{-1}) \in E \vee \tilde{F} \tag{4.7}$$

if $m \in \mathbf{Z}$ is critical for $R_{F/\mathbf{Q}}(M \otimes N)$, that is

$$(k_0 - \min_{\tau \in J_F} k(\tau))/2 < m \leq (k_0 + \min_{\tau \in J_F} k(\tau))/2 - 1.$$

(cf. (2.9) in §2 .)

We see easily that (4.7) is consistent with Theorem 4.3, (I) of [12] by putting

$$u(r, f) = \prod_\tau c_\tau^{\epsilon_\tau}(M_f), \qquad r = (m_\tau). \tag{4.8}$$

However Shimura's result is more precise in two points. First it is shown that the quantity on the left of (4.7) belongs to E. Secondly it transforms covariantly under

$\sigma \in J_E$. We note one more important fact which cannot be derived from Deligne's conjecture. Define

$$I(f^\sigma) = (2\pi\sqrt{-1})^{n(1-k_0)}\pi^{\sum_{\tau\in J_F} k(\tau)} g(\psi)^{-1}\langle f^\sigma, f^\sigma\rangle, \qquad \sigma \in J_E$$

where E denotes the field generated by eigenvalus of Hecke operators of f. Consider $\{I(f^\sigma)\}$ as an element of $(E \otimes_{\mathbf{Q}} \mathbf{C})^\times$. Then Theorem 4.3, (II) of [12] suggests

$$\begin{aligned} &c^+(R_{F/\mathbf{Q}}(M_f))c^-(R_{F/\mathbf{Q}}(M_f)) \\ &= \prod_{\tau\in J_F} c_\tau^+(M_f)c_\tau^-(M_f) = \{I(f^\sigma)\} \qquad \bmod E \vee \tilde{F}. \end{aligned} \tag{4.9}$$

Now let $f \in S_k(\mathfrak{c}, \psi)$, $g \in S_l(\mathfrak{c}, \varphi)$ which are common eigenfunctions of all Hecke operators. Let

$$\begin{aligned} D(s,f) = \sum_{\mathfrak{n}} c(\mathfrak{n}, f)N(\mathfrak{n})^{-s}, &\qquad D(s,g) = \sum_{\mathfrak{n}} c(\mathfrak{n}, g)N(\mathfrak{n})^{-s}, \\ k_0 = \max\nolimits_{\tau\in J_F} k(\tau), &\qquad l_0 = \max\nolimits_{\tau\in J_F} l(\tau). \end{aligned}$$

Put

$$\begin{aligned} D(s,f,g) &= \sum_{\mathfrak{n}} c(\mathfrak{n}, f)c(\mathfrak{n}, g)N(\mathfrak{n})^{-s}, \\ \mathfrak{D}_{\mathfrak{c}}(s,f,g) &= L_{\mathfrak{c}}(2s+2-k_0-l_0, \psi\varphi)D(s,f,g). \end{aligned}$$

Here $L_{\mathfrak{c}}$ denotes the L-function whose Euler $\mathfrak{p}$-factors are dropped for $\mathfrak{p}|\mathfrak{c}$. Then $\mathfrak{D}_{\mathfrak{c}}(s,f,g)$ coincides with the L-function $L(id., M_f \otimes M_g, s)$ up to finitely many Euler $\mathfrak{p}$-factors. By Proposition 3.2, we have

$$\begin{aligned} &c_\tau^+(M_f \otimes M_g) = c_\tau^-(M_f \otimes M_g) \\ &= \begin{cases} c_\tau^+(M_f)c_\tau^-(M_f)\delta_\tau(M_g) & \text{if} \quad k(\tau) > l(\tau), \\ c_\tau^+(M_g)c_\tau^-(M_g)\delta_\tau(M_f) & \text{if} \quad k(\tau) < l(\tau). \end{cases} \end{aligned} \tag{4.10}$$

Hence, by (3.8), we obtain

$$\begin{aligned} &c_\tau^+((M_f \otimes M_g)(m)) = c_\tau^-(M_f \otimes M_g)(m)) \\ &= \begin{cases} (2\pi\sqrt{-1})^{2m}c_\tau^+(M_f)c_\tau^-(M_f)\delta_\tau(M_g) & \text{if} \quad k(\tau) > l(\tau), \\ (2\pi\sqrt{-1})^{2m}c_\tau^+(M_g)c_\tau^-(M_g)\delta_\tau(M_f) & \text{if} \quad k(\tau) < l(\tau). \end{cases} \end{aligned} \tag{4.11}$$

Let E be the field generated by eigenvalues of Hecke operators of f and g.

First we assume that

$$k(\tau) > l(\tau) \qquad \text{for every} \quad \tau \in J_F.$$

By Proposition 3.2 and (4.11), we obtain

$$\begin{aligned} & c^+(R_{F/\mathbf{Q}}((M_f \otimes M_g)(m)) \\ & = (2\pi\sqrt{-1})^{2mn} c^+(R_{F/\mathbf{Q}}(M_f)) c^-(R_{F/\mathbf{Q}}(M_f)) \delta(R_{F/\mathbf{Q}}(M_g)) \quad \mod (E \vee \tilde{F})^\times. \end{aligned}$$

Since $\bigwedge^2 M_g \cong \mathrm{Art}_{\varphi^{-1}}(1-l_0)$, we have $\delta(R_{F/\mathbf{Q}}(M_g)) = (2\pi\sqrt{-1})^{n(1-l_0)} g(\varphi)$. Thus we have shown

$$\begin{aligned} & c^+(R_{F/\mathbf{Q}}((M_f \otimes M_g))(m)) \\ & = (2\pi\sqrt{-1})^{n(2m+1-l_0)} g(\varphi)\{I(f^\sigma)\} \quad \mod (E \vee \tilde{F})^\times. \end{aligned} \tag{4.12}$$

From (4.12), we see easily that Deligne's conjecture is consistent with Shimura [12], Theorem 4.2. However Shimura's result is more precise in two points mentioned above and also in that the condition on weights is less restrictive.

Next assume that

$$k(\tau) > l(\tau) \quad \text{for} \quad \tau \in \delta, \quad k(\tau) < l(\tau) \quad \text{for} \quad \tau \in \delta'$$

where δ and δ' are subsets of J_F such that $\delta \cup \delta' = J_F$, $\delta \cap \delta' = \emptyset$. By Proposition 3.2, we have

$$\begin{aligned} & c^+(R_{F/\mathbf{Q}}((M_f \otimes M_g)) = c^-(R_{F/\mathbf{Q}}((M_f \otimes M_g)) = \prod_{\tau\in\delta} c_\tau^+(M_f) c_\tau^-(M_f) \\ & \times \prod_{\tau\in\delta'} c_\tau^+(M_g) c_\tau^-(M_g) \prod_{\tau\in\delta} \delta_\tau(M_f) \prod_{\tau\in\delta'} \delta_\tau(M_g) \quad \mod (E \vee \tilde{F})^\times. \end{aligned}$$

Since

$$\bigwedge^2 M_f \cong \mathrm{Art}_{\psi^{-1}}(1-k_0), \qquad \bigwedge^2 M_g \cong \mathrm{Art}_{\varphi^{-1}}(1-l_0),$$

we obtain

$$\begin{aligned} & c^+(R_{F/\mathbf{Q}}((M_f \otimes M_g)(m))) = (2\pi\sqrt{-1})^{2mn+(1-k_0)|\delta'|+(1-l_0)|\delta|} \\ & \times \prod_{\tau\in\delta} c_\tau^+(M_f) c_\tau^-(M_f) \prod_{\tau\in\delta'} c_\tau^+(M_g) c_\tau^-(M_g) \prod_{\tau\in\delta} \delta_\tau(\mathrm{Art}_{\varphi^{-1}}) \prod_{\tau\in\delta'} \delta_\tau(\mathrm{Art}_{\psi^{-1}}) \end{aligned} \tag{4.13}$$

by (3.8). Since $\delta_\tau(\mathrm{Art}_{\varphi^{-1}}) \sim 1$, $\delta_\tau(\mathrm{Art}_{\psi^{-1}}) \sim 1$, we have

$$\begin{aligned} & \mathfrak{D}_\epsilon(m,f,g) \sim \pi^{2mn+(1-k_0)|\delta'|+(1-l_0)|\delta|} \\ & \qquad \prod_{\tau\in\delta} c_\tau^+(M_f) c_\tau^-(M_f) \prod_{\tau\in\delta'} c_\tau^+(M_g) c_\tau^-(M_g) \end{aligned} \tag{4.14}$$

if m is critical for $\mathrm{Res}_{F/\mathbf{Q}}(M_f \otimes M_g)$, that is

$$\frac{k_0+l_0}{2} - \frac{|k(\tau)-l(\tau)|}{2} - 1 < m < \frac{k_0+l_0}{2} + \frac{|k(\tau)-l(\tau)|}{2}$$

for all $\tau \in J_F$.

We shall show that (4.14) is consistent with Shimura's results and a part of his conjectures. (The part involving modular forms of half integral weight shall remain mysterious.)

Let χ (resp. χ') be the system of eigenvalues of Hecke operators attached to f (resp. g). Set

$$Q(\chi,\delta) = \pi^{(k_0-1)|\delta|-\sum_{\tau\in\delta}k(\tau)} \prod_{\tau\in\delta} c_\tau^+(M_f)c_\tau^-(M_f). \tag{4.15}$$

We note that

$$D(s,\chi,\eta) = \sum_{\mathfrak{A}} \eta(\mathfrak{A})\chi(\mathfrak{A})N(\mathfrak{A})^{-s-1} = D(s+\frac{k_0}{2},f,\eta)$$

when η is a Hecke chracter of finite order of F,

$$D(s,\chi,\chi') = \sum_{\mathfrak{A}} \chi(\mathfrak{A})\chi'(\mathfrak{A})N(\mathfrak{A})^{-s} = D(s+\frac{k_0+l_0}{2}-2,f,g)$$

in the normalization of Shimura [15], [16]. By (4.14) and (4.15), we obtain

$$\begin{aligned} D(\frac{t}{2},\chi,\chi') &= D(\frac{t}{2}+\frac{k_0+l_0}{2}-2,f,g) \\ &\sim \pi^{tn+\sum_{\tau\in\delta}k(\tau)+\sum_{\tau\in\delta'}l(\tau)}\pi^{-2n}Q(\chi,\delta)Q(\chi',\delta')/L_\epsilon(t-2,\psi\varphi) \end{aligned}$$

if $t \in \mathbf{Z}$ satisfies

$$\begin{aligned} &t \equiv k_0+l_0 \mod 2, \\ &-\frac{|k(\tau)-l(\tau)|}{2}+1 < \frac{t}{2} \le \frac{|k(\tau)-l(\tau)|}{2}+1 \qquad \text{for every} \quad \tau\in J_F. \end{aligned}$$

Since $L_\epsilon(t-2,\psi\varphi) \sim \pi^{(t-2)n}$, we obtain

$$D(\frac{t}{2},\chi,\chi') \sim \pi^{\sum_{\tau\in\delta}k(\tau)+\sum_{\tau\in\delta'}l(\tau)}Q(\chi,\delta)Q(\chi',\delta'). \tag{4.16}$$

This is consistent with Theorem 5.3 of [15] and the definition of $Q(\chi,\delta)$ in [16].

Next we assume that $k(\tau)$ is even for every $\tau\in J_F$ and put, for $r\in(\mathbf{Z}/2\mathbf{Z})^{J_F}$, $\delta\subseteq J_F$,

$$V(\chi,r) = \pi^{nk_0/2}\prod_{\tau\in J_F} c_\tau^{(-1)^{k_0/2}r_\tau}(M_f), \tag{4.17}$$

$$P(\chi,\delta,r) = \pi^{-\sum_{\tau\in\delta}k(\tau)/2}\pi^{k_0|\delta|/2}\pi^{-|\delta|}\prod_{\tau\in\delta} c_\tau^{(-1)^{k_0/2}r_\tau}(M_f). \tag{4.18}$$

If t is an integer such that $|t| < k(\tau)/2$ for every τ and η is a Hecke character of finite order of F such that

$$\eta(x) = \mathrm{sgn}(x)^{t+r}, \qquad x \in F_\infty^\times,$$

then we have

$$D(t,\chi,\eta) = D(t+\frac{k_0}{2}, f, \eta) \sim \pi^{n(t+k_0/2)} \prod_{\tau \in J_F} c_\tau^{(-1)^{k_0/2} r_\tau}(M_f) \sim \pi^{nt} V(\chi, r).$$

This is consistent with (8.2a) of [16].

We see easily that $\bar{\chi}$ attaches to $f \otimes \psi^{-1}$. Since we have assumed that $k(\tau)$ is even for every τ, $\psi(x) = 1$, $x \in F_\infty^\times$ holds. Therefore $c_\tau^\pm(M_f) \sim c_\tau^\pm(M_f \otimes \mathrm{Art}_\psi)$ for every τ. Thus we may take $P(\bar{\chi}, \delta, r) \sim P(\chi, \delta, r)$ for every $\delta \subseteq J_F$, $r \in (\mathbf{Z}/2\mathbf{Z})^{J_F}$. Now (C3) and (C4) of [16], p. 293 state that

$$V(\chi, r) \sim \pi^{(\sum_{\tau \in J_F} k(\tau)/2)+n} P(\chi, \delta, r) P(\bar{\chi}, \iota - \delta, r - r\delta) \tag{4.19}$$

$$Q(\chi, \delta) \sim \pi^{|\delta|} P(\chi, \delta, r) P(\bar{\chi}, \delta, \delta - r) \tag{4.20}$$

for every $r \in (\mathbf{Z}/2\mathbf{Z})^{J_F}$, $\delta \subseteq J_F$. It is obvious that (4.19) and (4.20) follow from (4.15), (4.17) and (4.18) in view of $P(\chi, \delta, r) \sim P(\bar{\chi}, \delta, r)$.

§5. Special values of zeta functions of Shimura varieties

Let M be a motive over F with coefficients in E. Let $M' = \otimes_\Omega \mathrm{Res}_{F/F'} M$. Our first task is to compute τ'-periods, $\tau' \in J_{F'}$ of M' from τ-periods, $\tau \in J_F$ of M. For this purpose, we supplement Conjecture 1.5 with specifying Betti and de Rham realizations of M'.

Let $\tau' \in J_{F'}$ and take any $\tilde{\tau}' \in G$ such that $\tilde{\tau}'|F' = \tau'$. Then $\{\tilde{\tau}' s_i|F \mid 1 \le i \le r\}$ defines a set of r-distinct elements of J_F which does not depend on the choice of $\tilde{\tau}'$ and s_i. We have

$$H_{\tau',B}(M') = \otimes_{i=1}^r H_{\tilde{\tau}' s_i|F,B}(M) \tag{5.1}$$

where $\otimes_{i=1}^r$ denotes the tensor product as E-modules. If

$$H_{\tau,B}(M) \otimes_{E,\sigma} \mathbf{C} = \oplus H^{pq}(\tau, \sigma, M), \qquad \tau \in J_F, \quad \sigma \in J_E$$

is the Hodge decomposition,

$$H_{\tau',B}(M') \otimes_{E,\sigma} \mathbf{C} = \otimes_{i=1}^r H^{pq}(\tilde{\tau}' s_i|F, \sigma, M) \tag{5.2}$$

is the Hodge decomposition of $H_{\tau',B}(M')$.

The de Rham realization of M' is

$$H_{DR}(M') = (\otimes_{i=1}^{r}(H_{DR}(M) \otimes_{F,s_i} \overline{\mathbf{Q}}))^{\mathrm{Gal}(\overline{\mathbf{Q}}/F')}, \tag{5.3}$$

where $\otimes_{i=1}^{r}$ denotes the tensor product as $E \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}$-modules and $\sigma \in \mathrm{Gal}(\overline{\mathbf{Q}}/F')$ acts as

$$\sigma(\otimes_{i=1}^{r}(v_i \otimes_{F,s_i} a_i)) = \otimes_{i=1}^{r}(v_i \otimes_{F,\sigma s_i} \sigma(a_i)), \qquad v_i \in H_{DR}(M), \quad a_i \in \overline{\mathbf{Q}}.$$

Since $H_{DR}(M)$ has a structure of free $E \otimes_{\mathbf{Q}} F$-module, we can verify that the right hand side of (5.3) has a natural structure of free $E \otimes_{\mathbf{Q}} F'$-module of rank = $\mathrm{rank}(M)^r$. From (5.3), we obtain an isomorphism as $E \otimes_{\mathbf{Q}} \mathbf{C}$-modules

$$H_{DR}(M') \otimes_{F',\tau'} \mathbf{C} \cong \otimes_{i=1}^{r}(H_{DR}(M) \otimes_{F,\tilde{\tau}'s_i|F} \mathbf{C}), \qquad \tau' \in J_{F'}. \tag{5.4}$$

Here the isomorphism is given by

$$(\otimes_{i=1}^{r}(v_i \otimes_{F,s_i} a_i)) \otimes_{F',\tau'} 1 \longrightarrow (\otimes_{i=1}^{r}(v_i \otimes_{F,\tilde{\tau}'s_i|F} \tilde{\tau}'(a_i)), \quad v_i \in H_{DR}(M), \quad a_i \in \overline{\mathbf{Q}}.$$

For $\tau \in J_F$, let

$$I_\tau : H_{\tau,B}(M) \otimes_{\mathbf{Q}} \mathbf{C} \cong H_{DR}(M) \otimes_{F,\tau} \mathbf{C}$$

be the canonical isomorphism of $E \otimes_{\mathbf{Q}} \mathbf{C}$-modules. Let $\tau' \in J_{F'}$. In view of (5.1) and (5.4), we can take the canonical isomorphism

$$I_{\tau'} : H_{\tau',B}(M') \otimes_{\mathbf{Q}} \mathbf{C} \cong H_{DR}(M) \otimes_{F',\tau'} \mathbf{C}$$

as

$$I_{\tau'} = \otimes_{i=1}^{r} I_{\tilde{\tau}'s_i|F}. \tag{5.5}$$

Proposition 5.1. *Let M be a motive over F with coefficients in E. We assume that M is of rank 2 and let*

$$H_{\tau,B}(M) \otimes_{E,\sigma} \mathbf{C} = H^{a(\tau,+),a(\tau,-)}(\tau,\sigma,M) \oplus H^{a(\tau,-),a(\tau,+)}(\tau,\sigma,M),$$

$\tau \in J_F$, $\sigma \in J_E$ be the Hodge decomposition. We assume

$$a(\tau,+) > a(\tau,-). \tag{5.6}$$

Let Ω be any subset of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})/\mathrm{Gal}(\overline{\mathbf{Q}}/F)$ such that $|\Omega| = r \geq 2$. Define F' and $M' = \bigotimes_\Omega \mathrm{Res}_{F/F'} M$ as above. Let Λ be the set of all maps from $\{1, 2, \cdots, r\}$ to $\{\pm 1\}$. We assume that

$$\sum_{i=1}^{r} a(\tilde{\tau}'s_i|F, \lambda(i)) \neq \sum_{i=1}^{r} a(\tilde{\tau}'s_i|F, -\lambda(i)) \tag{5.7}$$

holds for every $\lambda \in \Lambda$. Set

$$\Lambda^+ = \{\lambda \in \Lambda \mid \sum_{i=1}^{r} a(\tilde{\tau}' s_i|F, \lambda(i)) > \sum_{i=1}^{r} a(\tilde{\tau}' s_i|F, -\lambda(i))\}.$$

and let n_i (resp. m_i) be the number of $\lambda \in \Lambda^+$ such that $\lambda(i) = 1$ (resp. $\lambda(i) = -1$), $1 \le i \le r$. (n_i and m_i depend on τ'.) Then we have, modulo $E \vee \tilde{F}$,

$$c^{\pm}_{\tau'}(M') = \prod_{i=1}^{r} (c^{+}_{\tilde{\tau}' s_i|F}(M) c^{-}_{\tilde{\tau}' s_i|F}(M))^{(n_i - m_i)/2} \delta_{\tilde{\tau}' s_i|F}(M)^{m_i} \tag{5.8}$$

for every $\tau' \in J_{F'}$.

PROOF: For every $\tau \in J_F$, take $u^{\pm}_{\tau}$ so that

$$Eu^{\pm}_{\tau} = H^{\pm}_{\tau,B}(M).$$

Take d^- so that

$$(E \otimes F)d^- = F^-(M) = F^+(M),$$

and then take d^+ so that

$$H_{DR}(M) = (E \otimes F)d^+ + (E \otimes F)d^-.$$

Put

$$I_{\tilde{\tau}' s_i|F}(u^{\pm}_{\tilde{\tau}' s_i|F}) = x_i^{+,\pm} d^+ + x_i^{-,\pm} d^-,$$

with $x_i^{\pm,\pm} \in E \otimes_{\mathbf{Q}} \mathbf{C}$. Then we have

$$c^{\pm}_{\tilde{\tau}' s_i|F}(M) = x_i^{+,\pm}, \qquad \delta_{\tilde{\tau}' s_i|F}(M) = \det \begin{pmatrix} x_i^{+,+} & x_i^{-,+} \\ x_i^{+,-} & x_i^{-,-} \end{pmatrix}.$$

These quantities are elements of $(E \otimes_{\mathbf{Q}} \mathbf{C})^{\times}$ determined modulo multiplications by $E \vee \tilde{F}$. Set

$$\Lambda_{\pm} = \{\lambda \in \Lambda \mid \prod_{i=1}^{r} \lambda(i) = \pm 1\}.$$

By (5.1), a basis of $H^{\pm}_{\tau',B}(M')$ over E is given by $\otimes_{i=1}^{n} u^{\epsilon(i)}_{\tilde{\tau}' s_i|F}$ when ϵ extends over $\Lambda_{\pm}$. By (5.2) and (5.7), we have $F^+(M') = F^-(M')$. A basis of $H_{DR}(M') \otimes_{F',\tau'} \tilde{F}$ modulo $F^-(M') \otimes_{F',\tau'} \tilde{F}$ over $E \otimes_{\mathbf{Q}} \tilde{F}$ is given by $\otimes_{i=1}^{r} d^{\lambda(i)}$ where λ extends over Λ^+. We have

$$\otimes_{i=1}^{r} I_{\tilde{\tau}' s_i|F}(\otimes_{i=1}^{r} u^{\epsilon(i)}_{\tilde{\tau}' s_i|F}) = \otimes_{i=1}^{r}(x_i^{+,\epsilon(i)} d^+ + x_i^{-,\epsilon(i)} d^-) = \sum_{\lambda \in \Lambda^+} \prod_{i=1}^{r} x_i^{\lambda(i),\epsilon(i)} \otimes_{i=1}^{r} d_i^{\lambda(i)}$$

mod $F^-(M') \otimes_{F',\tau'} \tilde{F}$. Therefore we obtain

$$c^{\pm}_{\tau'}(M') = \det(X^{\pm})$$

where $X^{\pm}$ is the $2^{r-1} \times 2^{r-1}$-matrix whose (λ, ϵ)-entry for $\lambda \in \Lambda^+$, $\epsilon \in \Lambda_{\pm}$ is given by $\prod_{i=1}^{r} x_i^{\lambda(i),\epsilon(i)}$. Now (5.8) follows from (3.10) (and from the same formula for $\det(X^-)$). This completes the proof.

By (3.8), we obtain

$$\begin{aligned} &c^{\pm}_{\tau'}(M'(m)) \\ (5.9) \qquad &= (2\pi\sqrt{-1})^{2^{r-1}m} \prod_{i=1}^{r} (c^{+}_{\tilde{\tau}'s_i|F}(M) c^{-}_{\tilde{\tau}'s_i|F}(M))^{(n_i-m_i)/2} \delta_{\tilde{\tau}'s_i|F}(M)^{m_i}, \end{aligned}$$

and by Proposition 2.1, we have

$$\begin{aligned} (5.10) \qquad &c^{\pm}(\mathrm{Res}_{F'/\mathbf{Q}}(M')(m)) = (2\pi\sqrt{-1})^{2^{r-1}mn'} \\ &\times \prod_{\tau' \in J_{F'}} \prod_{i=1}^{r} (c^{+}_{\tilde{\tau}'s_i|F}(M) c^{-}_{\tilde{\tau}'s_i|F}(M))^{(n_i-m_i)/2} \delta_{\tilde{\tau}'s_i|F}(M)^{m_i}, \end{aligned}$$

mod $E \vee \tilde{F}$, where $n' = [F' : \mathbf{Q}]$.

Now let $f \in S_k(\mathfrak{c}, \psi)$ be a new form and let M_f be the motive over F with coefficients in E attached to f as is given in Conjecture 4.1. We set $M'_f = \bigotimes_{\Omega} \mathrm{Res}_{F'/F}(M_f)$. We regard $c^+(\mathrm{Res}_{F'/\mathbf{Q}}(M'_f)(m))$ as a complex number fixing an embedding $E \subset \mathbf{C}$. Since $\bigwedge^2 M_f \cong \mathrm{Art}_{\psi^{-1}}(1-k_0)$, we have $\delta_\tau(M_f) \sim \pi^{1-k_0}$ for every $\tau \in J_F$. Then (5.10) yields

$$\begin{aligned} &c^{\pm}(\mathrm{Res}_{F'/\mathbf{Q}}(M'_f)(m)) \\ (5.11) \qquad &\sim \pi^{2^{r-1}mn'} \prod_{\tau' \in J_{F'}} (\pi^{1-k_0 \sum_{i=1}^{r} m_i} \prod_{i=1}^{r} (c^{+}_{\tilde{\tau}'s_i|F}(M_f) c^{-}_{\tilde{\tau}'s_i|F}(M_f))^{(n_i-m_i)/2}), \end{aligned}$$

Deligne's conjecture predicts

$$(5.12) \qquad L(m, M_f) \sim c^{\pm}(\mathrm{Res}_{F'/\mathbf{Q}}(M'_f)(m))$$

if $m \in \mathbf{Z}$ is critical for M'_f. In view of the relation (4.15) of $Q(\chi, \delta)$ with $c^{\pm}_\tau(M_f)$, $c^{\pm}(\mathrm{Res}_{F'/\mathbf{Q}}(M'_f)(m))$ can be expressed as a product of π and $Q(\chi, \delta)$. (Note that we have assumed $r \geq 2$). Here χ denotes the system of eigenvalues of Hecke operators attached to f.

We shall show that (5.11) and (5.12) are in perfect accordance with a certain result of Shimura [14], II. Assume that F contains a subfield F_0 such that $[F : F_0] = 2$. Set $\Omega = \mathrm{Gal}(\overline{\mathbf{Q}}/F_0)/\mathrm{Gal}(\overline{\mathbf{Q}}/F)$. Then we have $H' = \mathrm{Gal}(\overline{\mathbf{Q}}/F_0)$, $F' = F_0$.

Apply the above construction. We obtain a motive M_f' over F' with coefficients in E. Set

$$J_{F'} = \{\tau_1, \tau_2, \cdots, \tau_{n/2}\}$$

and choose $\tilde{\tau}_i \in J_F$ so that $\tilde{\tau}_i|F' = \tau_i$, $1 \leq i \leq n/2$. Let σ be the generator of $\mathrm{Gal}(F/F')$. We assume that the weight of f satisfies

$$k(\tilde{\tau}_i|F) > k(\tilde{\tau}_i\sigma|F), \qquad 1 \leq i \leq n/2. \tag{5.13}$$

Then we have $n_1 = 2$, $m_1 = 0$, $n_2 = 1$, $m_2 = 1$ for every $\tau' \in J_{F'}$. Hence we get

$$c^{\pm}(\mathrm{Res}_{F'/\mathbf{Q}}(M_f')(m)) \sim \pi^{mn}\pi^{(1-k_0)n/2}\prod_{i=1}^{n/2}(c^+_{\tilde{\tau}_i}(M_f)c^-_{\tilde{\tau}_i}(M_f)) \tag{5.14}$$

by (5.11). Set

$$\delta = \{\tilde{\tau}_1, \tilde{\tau}_2, \cdots, \tilde{\tau}_{n/2}\} \subset J_F.$$

Since

$$Q(\chi, \delta) \sim \pi^{(k_0-1)n/2-\sum_{\tau\in\delta}k(\tau)}\prod_{i=1}^{n}(c^+_{\tilde{\tau}_i|F}(M_f)c^-_{\tilde{\tau}_i|F}(M_f)),$$

we obtain

$$c^{\pm}(\mathrm{Res}_{F'/\mathbf{Q}}(M_f')(m)) \sim \pi^{mn}\pi^{(1-k_0)n}\pi^{\sum_{\tau\in\delta}k(\tau)}Q(\chi,\delta).$$

By (2.9), Conjecture 4.1 and (5.2), we see easily that $m \in \mathbf{Z}$ is critical for M_f' if and only if

$$k_0 - \frac{k(\tilde{\tau}_i|F) - k(\tilde{\tau}_i\sigma|F)}{2} - 1 < m \leq k_0 + \frac{k(\tilde{\tau}_i|F) - k(\tilde{\tau}_i\sigma|F)}{2} - 1$$

holds for every $1 \leq i \leq n/2$. Put

$$L^*(s,\chi) = L(s + k_0 - 2, M_f').$$

Then if $m \in \mathbf{Z}$ satisfies

$$1 - \frac{k(\tilde{\tau}_i|F) - k(\tilde{\tau}_i\sigma|F)}{2} < m \leq 1 + \frac{k(\tilde{\tau}_i|F) - k(\tilde{\tau}_i\sigma|F)}{2} \tag{5.15}$$

for every $1 \leq i \leq n/2$,

$$L^*(m,\chi) \sim \pi^{(m-1)n}\pi^{\sum_{\tau\in\delta}k(\tau)}Q(\chi,\delta) \tag{5.16}$$

is predicted.

We see easily that (5.16) is in accordance with [14], II, Theorem 3.11. ($C(s)$ there essentially coincides with $L^*(s,\chi)$.) Furthermore Shimura obtained the result without assuming that $k(\tau)$ mod 2 is independent of $\tau \in J_F$. In this sense, Shimura's result is more general than (5.16).

§6. 志村の不変量 $Q(x, \delta)$ について

F は $\mathbb{Q}$ 上 n 次の総実体、B は F 上の quaternion algebra とする。B は $\delta \subset J_F$ で split し、$\delta' = J_F \setminus \delta$ で split しないとする。従って

(6.1) $\quad B \otimes_{\mathbb{Q}} \mathbb{R} \cong M_2(\mathbb{R})^r \times \mathbb{H}^{n-r}, \quad r = |\delta|.$

ここに $\mathbb{H}$ は Hamilton quaternion algebra を表わす。このとき B を signature (δ, δ') の quaternion algebra という。以下、$r \geqq 1$ と仮定する。

$G = Res_{F/\mathbb{Q}}(B^\times)$ とおく。G は $\mathbb{Q}$ 上の代数群であり、$G_K = (B \otimes_{\mathbb{Q}} K)^\times$ が任意の $\mathbb{Q}$-algebra K に対して成立つ。G_A により G の adelization, G_{A_f}, G_∞ はそれぞれ G_A の finite part, archimedean part を表わす。

$$G_\infty \cong GL(2, \mathbb{R})^r \times (\mathbb{H}^\times)^{n-r}$$

である。$G_{\infty+}$ により G_∞ の単位元の連結成分を表わす。Z を G の center とする。

Shimura [14], II に従い、保型因子、保型形式、arithmeticity を定義する。一部を復習しておこう。

表現 $\sigma_m : \mathbb{H}^\times \longrightarrow GL_{m+1}(\mathbb{C})$ を、埋込み $\mathbb{H}^\times \subset GL_2(\mathbb{C})$ と、m 次の symmetric tensor 表現 $GL_2(\mathbb{C}) \longrightarrow GL_{m+1}(\mathbb{C})$ をつないで定義する。(arithmeticity の定義には、$\overline{\mathbb{Q}}$-structure をこめたとり方が必要である。cf. [14])

$k = \sum_{\tau \in \delta} k(\tau)\tau$, $\kappa = \sum_{\tau \in \delta'} \kappa(\tau)\tau \in I_F$, $k(\tau), \kappa(\tau) \geqq 0$ を weight とし、$d = \prod_{\tau \in \delta'} (\kappa(\tau)+1)$ とおく。$\mathbb{C}^d$ を $\bigotimes_{\tau \in \delta'} \sigma_{\kappa(\tau)}$ の表現空間とみる。$x \in G_{A+} = G_{A_f} \times G_{\infty+}$, $z = (z_\tau)_{\tau \in \delta} \in \mathfrak{H}^r$ とする。

(16.2)
$$J(x, z) = \prod_{\tau \in \delta} (c_\tau z_\tau + d_\tau)^{k(\tau)} N(x_\tau)^{-k(\tau)/2}$$
$$\bigotimes_{\tau \in \delta'} \sigma_{\kappa(\tau)}(x_\tau) \prod_{\tau \in \delta'} N(x_\tau)^{-\kappa(\tau)/2},$$
$$x_\infty = (x_\tau)_{\tau \in J_F}, \quad x_\tau = \begin{pmatrix} * & * \\ c_\tau & d_\tau \end{pmatrix}, \ \tau \in \delta$$

とおく。ここに N は reduced norm を表わす。$\mathfrak{H}^r$ 上の $\mathbb{C}^d$-valued function f に対して

(16.3) $(f\|_{k,\kappa}(x))(z) = f(x(z))J(x, z)^{-1}$,

$z \in \mathfrak{H}^r$ とおく。Γ を $G_{Q+} = G_Q \cap G_{A+}$ の congruence subgroup とする.

$S_{k,\kappa}(\Gamma) = \{ f \mid f$ は $\mathfrak{H}^r$ 上の $\mathbb{C}^d$-valued holomorphic function で $f\|_{k,\kappa}\gamma = f$, $\forall\gamma \in \Gamma$ をみたす $\}$

を、weight が (k, κ) の Γ に関する cusp form の空間とする。但し、$B = M_2(F)$ のときは、更に周知の cusp についての条件が課せられているものとする。

$$S_{k,\kappa} = \bigcup_{\Gamma} S_{k,\kappa}(\Gamma)$$

とおく。ここに Γ は G_{Q+} の全ての合同部分群を走る。

W_0 を G_{A_f} の open compact subgroup とし $W = W_0 G_{\infty+} \subset G_A$ とおく。$\mathfrak{S}_{k,\kappa}(W)$ により、G_A 上の $\mathbb{C}^d$-valued function f で次の条件 (6.4), (6.5) をみたすものの成す vector space とする。

(6.4) $f(\alpha x w) = f(x)$, $\alpha \in G_{\mathbb{Q}}$, $x \in G_A$, $w \in W_0$.

(6.5) $\forall x \in G_{A_f}$, $\exists g_x \in S_{k,\kappa}$ such that

$f(xy) = (g_x \|_{k,\kappa}\, y)(i)$ for $\forall y \in G_{\infty+}$.

ここに $i = (\sqrt{-1}, \cdots, \sqrt{-1}) \in \mathfrak{H}^r$.

$$G_A = \bigcup_{i=1}^{h} G_{\mathbb{Q}} x_\lambda W, \quad \Gamma_\lambda = x_\lambda W x_\lambda^{-1} \cap G_{\mathbb{Q}}$$

とおくと、

$$(6.6) \quad \begin{array}{ccc} \mathfrak{S}_{k,\kappa}(W) & \cong & \prod_{\lambda=1}^{h} S_{k,\kappa}(\Gamma_\lambda) \\ \Psi & & \Psi \\ f & \longrightarrow & (f_1, \cdots, f_h) \end{array}$$

が canonical に成立つ。$f, g \in \mathfrak{S}_{k,\kappa}(W)$ に対して

$$(6.7) \quad \langle f, g \rangle = \int_{Z_{\infty+} G_{\mathbb{Q}} \backslash G_A} {}^t\overline{f(x)}\, g(x)\, dx$$

とおく。ここに $Z_{\infty+}$ は Z_∞ における 1 の連結成分を表わし、$\mathrm{vol}(Z_{\infty+} G_{\mathbb{Q}} \backslash G_A) = 1$ となるように invariant measure を normalize しておく。この内積は [14], II, (1.12) で与えられているものと一致する。

B の maximal order R を 1 つとる。$W_0 = \prod_v (W_0)_v$, $(W_0)_v$ は $B_v^\times$ の open compact subgroup で、$(W_0)_v = R_v^\times$ が殆んど全ての v に対して成立つ、という形の W_0 を全て考え

$$\mathcal{S}_{k,\kappa} = \bigcup_{W_0} \mathcal{S}_{k,\kappa}(W), \quad W = W_0 G_{\infty +}$$

とおく。

$S_{k,\kappa}(\overline{\mathbb{Q}})$ は $\overline{\mathbb{Q}}$-rational elements の成す $S_{k,\kappa}$ の subset とする。($B = M_2(F)$ のときは cusp $(i\infty)^r$ の回りでの Fourier 展開の係数が全て $\overline{\mathbb{Q}}$ に入ることと同値である。)

$\mathcal{S}_{k,\kappa} \ni f$ が $\overline{\mathbb{Q}}$-rational $\Longleftrightarrow f_\lambda \in S_{k,\kappa}(\overline{\mathbb{Q}}),\ 1 \leq {}^\forall\lambda \leq h$

と定義する。(cf. (6.6))

(6.8) $\quad S_{k,\kappa} = S_{k,\kappa} \otimes_{\overline{\mathbb{Q}}} \mathbb{C}, \quad \mathcal{S}_{k,\kappa} = \mathcal{S}_{k,\kappa}(\overline{\mathbb{Q}}) \otimes_{\overline{\mathbb{Q}}} \mathbb{C}$

が成立つ。 B による依存性を強調する時は、上の記号に B を挿入して($\mathcal{S}_{k,\kappa}(B, \overline{\mathbb{Q}})$ 等と)表わすことにする。

L を $\mathbb{C}$ の subfield とする。$\mathcal{H}_L(G_{A_f})$ は G_{A_f} 上の L-valued locally constant compactly supported functions が convolution について成す L 上の algebra とする。但し G_{A_f} の measure は、$\mathrm{vol}(R_v^\times) = 1$ をみたす $B_v^\times$ の measure の product measure から得られているとする。G_{A_f} の open compact subgroup K に対し、$\mathcal{H}_L(G_{A_f}, K)$ は、K 両側不変な函数の成す $\mathcal{H}_L(G_{A_f})$ の subalgebra とする。

$f \in \mathcal{S}_{k,\kappa}$, $\varphi \in \mathcal{H}_L(G_{A_f})$ に対して

(6.9) $\quad \rho(\varphi) f = f * \check{\varphi}, \quad \check{\varphi}(x) = \varphi(x^{-1}),\ x \in G_{A_f}$

とおくと、$\rho(\varphi) f \in \mathcal{S}_{k,\kappa}$ で ρ は $\mathcal{H}_L$ の表現を与える。さらに、$L \subset \overline{\mathbb{Q}}$ で f が $\overline{\mathbb{Q}}$-rational ならば、$\rho(\varphi) f$ も

$\overline{\mathbb{Q}}$-rational である。([14], II, p.577, l.26~28 による。)

(6.10) $\langle \rho(\varphi)f, g\rangle = \langle f, \rho(\tilde{\varphi})g\rangle$,

$\tilde{\varphi}(x) = \overline{\varphi(x^{-1})}$ が成立つ。v を F の finite place,

$f \in \mathcal{S}_{k,\kappa}(W)$, $W = W_0 G_{\infty+}$, $W_0 = \prod_w (W_0)_w$, $(W_0)_v = R_v^\times$

B は v で split すると仮定し, 同型 $B_v \cong M_2(F_v)$, $R_v \cong M_2(\mathcal{O}_v)$ を固定する。φ を

$$\prod_{w \neq v} (W_0)_w \times R_v^\times \begin{pmatrix} 1 & 0 \\ 0 & \varpi_v \end{pmatrix} R_v^\times$$

の characteristic function $\times \mathrm{vol}(W_0)^{-1}$, ととることにより Hecke operator $T(v)$ の f への作用 $T(v)f$ を定める。ここに, $\mathcal{O}_v$, ϖ_v は F_v の整数環, 素元である。

F の各 finite place v に $\chi(v) \in \mathbb{C}$ を対応させる写像 χ がある $f(\neq 0) \in \mathcal{S}_{k,\kappa}(B)$ について

$$T(v)f = \chi(v)f \quad \text{for almost all } v$$

をみたすとき, Hecke operator の eigenvalue の system χ は $S_{k,\kappa}(B)$ に occur するといい, f は χ に属するという. χ に属する $f \in \mathcal{S}_{k,\kappa}(B)$ 全体の vector space を $W(\chi, B)$ とかき,

$$W(\chi, B, \overline{\mathbb{Q}}) = W(\chi, B) \cap \mathcal{S}_{k,\kappa}(B, \overline{\mathbb{Q}})$$

とおく.

Theorem 6.1. χ を $S_{k,\kappa}(B)$ に occur する Hecke operator の eigenvalue の system, $f, g, h \in W(B, \chi, \overline{\mathbb{Q}})$, $f \neq 0$ とする。このとき，$\langle g, h\rangle / \langle f, f\rangle \in \overline{\mathbb{Q}}$.

Proof. G_{A_f} の open compact subgroup W_0 を十分小さくとり，$f, g, h \in S_{k,\kappa}(W)$, $W = W_0 G_{\infty +}$ とできる。$H_{\mathbb{C}}(G_A)$ を Jacquet - Langlands[7] の意味での G_A の Hecke 環とし，U を f, g, h により，$H_{\mathbb{C}}(G_A)$ の作用で生成される $G_{\mathbb{Q}} \backslash G_A$ 上の $\mathbb{C}^d$ - valued functions の空間とする。このとき，Shimizu - Jacquet - Langlands 対応と，$GL(2)$ における strong multiplicity 1 theorem により，U は $H_{\mathbb{C}}(G_A)$ の既約表現空間になることがわかる。$U_1 = U \cap S_{k,\kappa}(B)$ とおくと，U_1 は $H_{\mathbb{C}}(G_{A_f})$ 不変であり，U_1 における $H_{\mathbb{C}}(G_{A_f})$ の表現は既約である。さらに，$U_1^{W_0} = S_{k,\kappa}(W)$ であり，$U_1^{W_0}$ は $H_{\mathbb{C}}(G_{A_f}, W_0)$ - 不変であるが，この表現は既約である。(cf. Bernstein - Zelevinski [1], p. 17.) よって次の Lemma が証明できればよい。

Lemma 6.2. k は complex conjugation で stable な $\overline{\mathbb{Q}}$ の subfield, V は k 上の $d < \infty$ 次元 vector space とし，$V = V_0 \otimes_k \mathbb{C}$ とおく。H は k 上の associative algebra, ρ は H の V における既約表現で $\rho(H) V_0 \subseteq V_0$ をみたす

と仮定する。$\langle\ ,\ \rangle$ は V 上の positive definite hermitian form で、$\forall u \in \mathfrak{A}$, $\exists \tilde{u} \in \mathfrak{A}$ such that

(6.10) $\langle \rho(u)x, y \rangle = \langle x, \rho(\tilde{u})y \rangle$, $\forall x, \forall y \in V$

をみたすと仮定する。このとき、$x_1, \cdots, x_d$ が V_0 の k 上の(任意の) basis で、$X = (\langle x_i, x_j \rangle) \in M_d(\mathbb{C})$ とおくと、$c \in \mathbb{C}^\times$ があり、$c^{-1}X \in M_d(k)$ が成立つ。

<u>Proof.</u> basis $x_1, \cdots, x_d$ により V を k^d と同一視し、ρ を $\mathfrak{A} \longrightarrow M_d(k)$ とみる。(6.10) から

(*) $\rho(u)X = X\overline{\rho(\tilde{u})}$, $\forall u \in \mathfrak{A}$

を得る。任意の $\sigma \in Aut(\mathbb{C}/k)$ に対して、(*) から

(**) $\rho(u)X^\sigma = X^\sigma\overline{\rho(\tilde{u})}$.

$\det X \neq 0$ ゆえ、(*), (**) から

$$\rho(u)X^\sigma X^{-1} = X^\sigma X^{-1}\rho(u), \quad \forall u \in \mathfrak{A}.$$

ゆえに $X^\sigma X^{-1}$ は scalar で

(***) $X^\sigma = d_\sigma X$, $d_\sigma \in \mathbb{C}^\times$

を得る。$c = \langle x_1, x_1 \rangle$ ととる。(***) から (1,1)-成分をみて、$d_\sigma = c^\sigma / c$. ゆえに $X = cX_0$ とおくと、$X_0^\sigma = X_0$. これが $\forall \sigma \in Aut(\mathbb{C}/k_0)$ に対して成立つから、$X_0 \in M_d(k)$. QED

$f \in W(B, \alpha, \overline{\mathbb{Q}})$, $f \neq 0$ をとり

$$Q(\chi, B) = \langle \mathfrak{f}, \mathfrak{f} \rangle \mod \overline{\mathbb{Q}}^\times$$

とおく。$Q(\chi, B) \in \mathbb{C}^\times / \overline{\mathbb{Q}}^\times$ は、Theorem 6.1 により $\mathfrak{f}$ のとり方に依らないで定まる。同様の証明法は高次元の場合にも使える。(例えば [11] で扱われている case。) Shimura [15], Theorem 5.6 は次の様に幾分精密化できる。

<u>Theorem 6.3.</u> B_1, B_2 は F 上の quaternion algebra で共に signature は (δ, δ') とする。Hecke operator の eigenvalue の system χ が $\mathcal{S}_{k,\kappa}(B_1)$, $\mathcal{S}_{k,\kappa}(B_2)$ に共に occur すると仮定し、$\mathfrak{g} \in W(\chi, B_1, \overline{\mathbb{Q}})$, $\mathfrak{h} \in W(\chi, B_2, \overline{\mathbb{Q}})$, $\mathfrak{g} \neq 0$ をとる。このとき、$\forall \tau \in \delta$ に対して $k(\tau) \geqq 2$ ならば、$\langle \mathfrak{g}, \mathfrak{g} \rangle \sim \langle \mathfrak{h}, \mathfrak{h} \rangle$.

<u>Proof.</u> $\mathfrak{f} \in \mathcal{S}_{m,0}(M_2(F), \overline{\mathbb{Q}})$,

$$m_\tau = \begin{cases} k(\tau), & \tau \in \delta \\ \kappa(\tau) + 2, & \tau \in \delta' \end{cases}$$

を χ に属する non-zero form とし、$\pi = \bigotimes_v \pi_v$ を $\mathfrak{f}$ が生成する $H_{\mathbb{C}}(GL_2(F_A))$ の automorphic representation とする。Theorem 6.1 により、B_1, B_2 は F 上の quaternion algebra として同型ではない、と仮定してよい。ゆえに B_1, B_2 の少くとも一方は、F のある finite place v_0 で分岐する。Jacquet-Langlands [7], Theorems 14.4, 15.1 により、

π_{v_0} は special 又は absolutely cuspidal. F 上の quaternion algebra B' で $\delta \cup \{v_0\}$ の外では不分岐で，δ では分岐するものが存在する。B' の signature は (δ', δ).
$k'(\tau) = \kappa(\tau) + 2$, $\tau \in \delta'$, $\kappa'(\tau) = k(\tau) - 2$, $\tau \in \delta$ とおくと，[7], Theorem 16.1 により，χ は $S_{k', \kappa'}(B')$ に occur する。(B' が v_0 で分岐したとき，i.e., $|\delta|$ が odd の case, π_{v_0} の形が効く。) $g' \in W(\chi, B', \overline{\mathbb{Q}})$, $g' \neq 0$ をとると，[15], Theorem 5.4 により，

$$\langle g, g \rangle \langle g', g' \rangle \sim \langle f, f \rangle \sim \langle h, h \rangle \langle g', g' \rangle$$

が成立ち，結論を得る。

<u>Remark</u>. $S_{k,\kappa}(B)$ の定義は [14], II におけるものより幾分一般にしている。従って上の証明で [15], Theorem 5.4 をそのままには適用できないが，new form の理論と Theorem 6.1 により，この点は処理できる。

χ と J_F の non-empty subset δ が与えられたとき，ある signature (δ, δ') の F 上の quaternion algebra B があって，$S_{k,\kappa}(B)$ に χ が occur すれば，

$$Q(\chi, \delta) = Q(\chi, B)$$

として invariant $Q(\chi, \delta) \in \mathbb{C}^\times / \overline{\mathbb{Q}}^\times$ は B のとり方に依存せずに定義できる。これは Shimura [16], p. 286 にある

予想を、$k(\tau) \geq 2$, $^\forall \tau \in \delta$ の仮定の下に確かめたことになっている。

<u>Theorem 6.4.</u> χ は Hecke operator の eigenvalue の system,
$f \in S_{m,0}(M_2(F)) \cap W(\chi, M_2(F), \overline{\mathbb{Q}})$, $f \neq 0$,
$g \in S_{k,K}(B) \cap W(\chi, B, \overline{\mathbb{Q}})$, $g \neq 0$
とする。B の signature を (δ, δ') とする。F_1 を F の l 次の totally real cyclic extension とし、$B_1 = B \otimes_F F_1$,
$\tau \in J_{F_1}$ に対し、

$$\tilde{m}(\tau) = m(\tau|F), \quad \tilde{k}(\tau) = k(\tau|F), \quad \tilde{K}(\tau) = K(\tau|F)$$

とおく。$\tilde{f} \in S_{\tilde{m},0}(M_2(F_1), \overline{\mathbb{Q}})$ を f の base change lift、
$\tilde{f} \in W(\tilde{\chi}, M_2(F_1), \overline{\mathbb{Q}})$ とする。このとき、$\tilde{g} \in S_{\tilde{k},\tilde{K}}(B_1)$,
$\tilde{g} \in W(\tilde{\chi}, B_1, \overline{\mathbb{Q}})$ が存在して

$$\langle \tilde{g}, \tilde{g} \rangle \sim \langle g, g \rangle^{l}$$

が、$^\forall \tau \in \delta$ に対して $k(\tau) \geqq 3$ ならば成立つ。

証明は、F の CM-extension K と $K_A^\times$ の Hecke character をとり、[15], Theorem 5.7 を適用することで得られる。証明の最後の段階で、[13], Theorem 1.1 にある CM-periods の関係を用いる。$k(\tau) \geqq 3$, $^\forall \tau \in \delta$ の条件がつくのは、[15], Prop. 5.2 にある non-vanishing result を使うためで、

この条件は多分不要である。また Theorem 6.4 はもう少し精密にできる。

χ が $S_{k,\pi}(B,\overline{\mathbb{Q}})$ に occur しなくても、適当に ℓ 次拡大をとると（$\ell=2$ でよい）$\tilde{\chi}$ は $S_{\tilde{k},\tilde{\pi}}(B_1,\overline{\mathbb{Q}})$ に occur するから

$$Q(\chi,\delta) = Q(\tilde{\chi},\tilde{\delta})^{1/\ell} \qquad (\text{in } \mathbb{C}^\times/\overline{\mathbb{Q}}^\times)$$

とおく。Theorems 6.3, 6.4 により、$k(\tau) \geq 3$ $\forall \tau \in \delta$ の条件下で、$Q(\chi,\delta) \in \mathbb{C}^\times/\overline{\mathbb{Q}}^\times$ は常に consistent に定義できることがわかる。（ここに $\tilde{\delta}$ は制限写像 $J_{F_1} \to J_F$ による δ の full inverse image である。）

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