On the b-Function of Nonisolated Hypersurface Singularities (Algebraic Analysis and Number Theory)

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On the b-Function of Nonisolated Hypersurface Singularities

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Let $f$ be a germ of holomorphic function of $n$ variables, and $b_f(s)$ the $b$-function (i.e. Bernstein polynomial) of $f$. It is the monic generator of the ideal consisting of polynomials $b(s)$ which satisfy the relation

\[(0.1)\quad b(s)f^s = Pf^{s+1} \text{ in } \mathcal{O}_X[f^{-1}][s]f^s\]

for $P \in \mathcal{D}_X[s]$, where $\mathcal{D}_X$ denotes the germs of holomorphic differential operators on $X := (\mathbb{C}^n, 0)$, and $\mathcal{D}_X[s] = \mathcal{D}_X \otimes_{\mathbb{C}} \mathbb{C}[s]$. Substituting $s = -1$, we can check easily that $b_f(s)$ is divisible by $s + 1$. Let $\tilde{b}_f(s) = b_f(s)/(s+1)$, $R_f$ the roots of $\tilde{b}_f(-s)$, $\alpha_f = \min R_f$, and $m_\alpha(f)$ the multiplicity of a root $\alpha$ of $\tilde{b}_f(-s)$. By Kashiwara [7], we have

\[(0.2)\text{ Theorem. } \alpha_f > 0, \text{ and } R_f \subset \mathbb{Q}.\]

Assume $f$ has isolated singularity and $n > 1$. Let $H_f'' = \Omega^2_X / df \wedge d\Omega^{n-2}_X$, following Brieskorn [2]. Then $H_f''$ is a free $\mathbb{C}(\{t\})$-module of rank $\mu$ (the Milnor number of $f$), and has a regular singular meromorphic connection. Let $\tilde{H}_f'' = \sum_{i \geq 0} (t\partial_t)^i H_f'' \subset H_f''[t^{-1}]$ (the saturation of $H_f''$). By Malgrange [13], we have

\[(0.3)\text{ Theorem. } \tilde{b}_f(s) \text{ is the minimal polynomial of the action of } -\partial_t \text{ on } \tilde{H}_f'' / t\tilde{H}_f''.\]

Combined with a result of Varchenko [29] (and [26]), this implies (see also [17]):

\[(0.4)\text{ Theorem. } R_f \subset [\alpha_f, n - \alpha_f].\]

\[(0.5)\text{ Theorem. } m_\alpha(f) \leq n - \alpha_f - \alpha + 1 (\leq n - 2\alpha_f + 1).\]

In the isolated singularity case, we proved also (see [16]):

\[(0.6)\text{ Proposition. } Y = f^{-1}(0) \text{ has rational singularity if and only if } \alpha_f > 1.\]

Using the theory of mixed Hodge Modules [18] [19] [20], we extend these to the nonisolated singularity case (see [23] [24]), i.e.

\[(0.7)\text{ Theorem. } (0.4-6) \text{ are valid also in the nonisolated singularity case, where we assume } Y \text{ reduced in } (0.6).\]

Note that (0.5) is an improvement of $m_\alpha(f) \leq n - \delta_{\alpha,1}$ (where $\delta_{\alpha,1}$ is Kronecker's delta) which is shown in [14] as a corollary of the relation with Deligne's vanishing cycle sheaf
§ 1. Microlocal b-Function

(1.1) Let $\delta(t-f)$ denote the delta function on $X' := X \times (\mathbb{C}, 0)$ with support $\{f = t\}$, where $t$ is the coordinate of $\mathbb{C}$. Then, setting $s = -\partial_t t$, $f^s$ and $\delta(t-f)$ satisfy the same relation (see for example [13]). So $f^s$ in (0.1) can be replaced by $\delta(t-f)$, and $f^{s+1}$ by $t\delta(t-f)$. We define the microlocal b-function $b(t)(s)$ by the monic generator of the ideal consisting of polynomials $b(s)$ which satisfy the relation

$$b(s)\delta(t-f) = P^{\partial_t^{-1}}\delta(t-f)$$

for $P \in D_{\mathbb{C}}[\partial_t^{-1}, s]$. Here we can also allow for $P$ a microdifferential operator [7] [9] [10] [25] satisfying a condition on the degree of $t$ and $\partial_t$ (see [24, (1.4)]).

We can show (see [24, (1.5)]) :

(1.2) Proposition. $b(s) = (s+1)b(t)(s)$.

(1.3) Let $R_X = D_{\mathbb{C}}[t, \partial_t], \tilde{R}_X = D_{\mathbb{C}}[t, \partial_t, \partial_t^{-1}]$, and

$$B_t = \mathcal{O}_X[\partial_t]6(t-f), \quad \tilde{B}_t = \mathcal{O}_X[\partial_t, \partial_t^{-1}]6(t-f),$$

where $\mathcal{O}_X[\partial_t]6(t-f)$ is a free module of rank one over $\mathcal{O}_X[\partial_t]$ with a basis $\delta(t-f)$ (similarly for $\tilde{B}_t$). Then $B_t, \tilde{B}_t$ have naturally a structure of $R_X$-module and $\tilde{R}_X$-module respectively.

Let $V^p$ be the filtration on $R_X, \tilde{R}_X$ by the differences of the degrees of $t$ and $\partial_t$ i.e.,

$$V^pR_X = \sum_{i-j=p} D_{\mathbb{C}}[t, \partial_t] \partial_t^i (\text{same for } \tilde{R}_X).$$

We define a decreasing filtration $G$ on $B_t$, $\tilde{B}_t$ by

$$G^pB_t = V^pR_X\delta(t-f), \quad G^p\tilde{B}_t = V^p\tilde{R}_X\delta(t-f),$$

and an increasing filtration $\mathcal{F}$ by
Then we have

(1.3.5) \[ \partial_t^i : F_{p+i}B \rightarrow F_{p+i}B \]

(1.3.6) \[ D_X[s](F_pB) \subset G^{-p}B \]

(1.4) Remark. \( b_f(s) \) and \( \delta_f(s) \) are the minimal polynomial of the action of \( s := -\partial_t \) on \( Gr^0G_{\tilde{B}_f} \) and \( Gr^0\tilde{B}_f \) respectively, because \( s \) belongs to the center of \( Gr^0R_X = Gr^0\tilde{R}_X = D_X[s] \).

§2. Filtration V

(2.1) Let \( V \) denote the filtration of Kashiwara [8] and Malgrange [14] on \( B \) indexed by \( \mathbb{Q} \). Here we index \( V \) decreasingly so that the action of \( \partial_t - \alpha \) on \( Gr^\alpha B \) is nilpotent, where \( Gr^\alpha = V^\alpha/V^{>\alpha} \) with \( V^{>\alpha} = \bigcup \beta \). By [7] (see also (0.2) above), we have

(2.2.1) \[ F_0B \subset V^0B \]

We can show (see [24, (2.2) and (2.4)])

(2.2) Lemma. We have a decreasing filtration \( V \) on \( B \) such that

(2.2.1) \[ V^\alpha B = V^\alpha B + \mathcal{O}_X[\partial_t^{-1}]\partial_t^{-1}(t-f) \quad \text{for} \quad \alpha \leq 1, \]

(2.2.2) \[ \partial_t^j : V^\alpha B \rightarrow V^{\alpha-j}B \quad \text{for any} \quad j, \alpha. \]

(2.3) Proposition. We have

(2.3.1) \[ Gr^\alpha B = D_X(F_pGr^\alpha B) \quad \text{if} \quad F_{-p-1}Gr^\alpha B = 0. \]

(2.4) Proof of (0.4) in the general case. We have \( G^1Gr^\alpha B \supset D_X(F_{-1}Gr^\alpha B) \) by (1.3.6). So it is enough to show \( Gr^\alpha B = D_X(F_{-1}Gr^\alpha B) \) for \( \alpha > n - \alpha_f \) by (1.4), because it implies \( Gr^\alpha Gr^\alpha B = Gr^\alpha Gr^\alpha B = 0 \). By definition of \( \alpha_f \), we have

(2.4.1) \[ F_0Gr^\alpha B = G^0Gr^\alpha B = 0 \quad \text{for} \quad \alpha < \alpha_f \]

using (1.3.6). So the assertion follows from (2.3) applied to \( p = -1 \).

By a similar argument, we prove (0.5) using also the monodromy filtration \( W \). Here \( W \) is uniquely characterized by the properties (see [41]):

(2.4.1) \[ NW_i \subset W_{i-2}, \quad N^j : Gr^W_j \rightarrow Gr^W_j \quad (j > 0), \]

where \( N = s + \alpha \) on \( Gr^\alpha B \). See [24, (2.8)] for the details. We can show also the following:
(2.5) Remark. Let $\varphi_f^\alpha \mathbb{C}_x$ be Deligne’s vanishing cycle sheaf [3], and $T_u, T_s$ denote respectively the unipotent and semisimple part of the monodromy $T$ on $\varphi_f^\alpha \mathbb{C}_x$. Let $\varphi_f^\alpha \mathbb{C}_x = \text{Ker}(T_s - \exp(-2\pi i \alpha))$ (as a shifted perverse sheaf [1]), and $N = \log T_u / 2\pi i$. Then we have $N^{r+1} = 0$ on $\varphi_f^\alpha \mathbb{C}_x$ for $\alpha \in [\alpha_f, \alpha_f + 1)$ and $r = [n - \alpha_f - \alpha]$. In particular, $N^{r+1} = 0$ on $\varphi_f^\alpha \mathbb{C}_x$ for $r = [n - 2\alpha_f]$. See [24, (0.6)].

(2.6) Remark. If $	ext{Sing} f^{-1}(0)$ is isolated and $f$ is a quasi-homogeneous polynomial of weight $(w_1, \cdots, w_n)$ (i.e. $f$ is a linear combination of monomials $x_1^{m_1} \cdots x_n^{m_n}$ such that $m_1 w_1 + \cdots + m_n w_n = 1$), then it is well-known that $m_\alpha(f) = 1$ for $\alpha \in R_f$, and $\alpha$ belongs to $R_f$ if and only if the coefficient of $t^\alpha$ in

\[(2.6.1) \quad \prod_i (t^{w_i} - t)(1 - t^{w_i})\]

is nonzero. This follows for example from Steenbrink [28] (using [13] [29]) and also from Brieskorn or Kashiwara (unpublished). In particular, we have $\max R_f = n - \alpha_f$ in this case.

(2.7) Remark. If $f$ has nondegenerate Newton boundary, we can show $\alpha_f = \frac{1}{t}$ for $(t, \cdots, t) \in \partial \Gamma_+(f)$ (see [24, (3.3)]), where $\Gamma_+(f)$ is the Newton polygon of $f$. In the isolated singularity case, it is known that the equality holds. (See also [22].)

(2.8) Remark. Let $g$ be a holomorphic function on a germ of complex manifold $Y$. Let $Z = X \times Y$, and $h = f + g \in \mathcal{O}_Z$. We define $R_g, R_h$ as in the introduction. Then $R_h R_g \subset R_h + \mathbb{Z}_{\geq 0}, R_h \subset R_g R_h + \mathbb{Z}_{\geq 0}$. Furthermore, if there is a holomorphic vector field $\xi$ such that $\xi g = g$, then $R_h R_g = R_h$, and $m_\gamma(h) = \max_{\alpha + \beta = \gamma} \{m_\alpha(f) + m_\beta(g) - 1\}$. See [24, (4.3–4)]. The last assertion is proved in [30] if $f$ and $g$ have isolated singularities.

§3. Rational Singularity

(3.1) Let $Y$ be a reduced complex analytic space. We say that $Y$ has rational singularity, if the natural morphism

\[(3.1.1) \quad \mathcal{O}_Y \rightarrow \mathbf{R} \pi_* \mathcal{O}_Y,\]

is an isomorphism for a resolution of singularity $\pi : Y' \rightarrow Y$. If $Y$ is Cohen-Macaulay and pure dimensional, it is equivalent to the bijectivity of the trace morphism

\[(3.1.2) \quad \pi_* \omega_{Y'} \rightarrow \omega_Y\]

by duality [15], because $\mathbf{R}^i \pi_* \omega_{Y'} = 0$ for $i > 0$ by [6] (this follows also from [11] [21]) where $\pi$ is assumed projective. Here $\omega_Y$ denotes the dualizing sheaf (i.e., the dualizing complex [15] shifted by the dimension to the right). The trace morphism (3.1.2) is injective, and its image is independent of the choice of resolution, because (3.1.2) is an isomorphism if
$Y$ is smooth. We will denote by $\tilde{\omega}_Y$ the image of (3.1.2).

(3.2) Assume $Y$ is a reduced divisor $D$ on the germ of complex manifold $X$ in the introduction. Let $f$ be a reduced defining equation of $D$.

Using the coordinate system $(x_1, \ldots, x_n)$ of $X$, we have the involution of $D_X$ such that $(PQ)^* = Q^*P^*$, $(x_i)^* = x_i$, $(\partial/\partial x_i)^* = -\partial/\partial x_i$. So the right $D$-module $\omega_X$ is identified with the left $D$-module $O_X$ using the basis $dx = dx_1 \wedge \cdots \wedge dx_n$ of $\omega_X$, and we get isomorphisms

$$\tilde{B}_f = \omega_X[\partial_t]6(t-f), \quad \underline{B}_f = \omega_X[\partial_t, \partial_t^{-1}]6(t-f).$$

We can show (see [23]):

(3.3) Theorem. We have a commutative diagram

$$\begin{array}{ccc}
0 & \rightarrow & \tilde{\omega}_D \\
\downarrow & & \downarrow \\
F_0W_0Gr_\psi^1B_f & \rightarrow & F_0B_f/Gr^\psi_\psi^2B_f \\
\downarrow & & \downarrow \\
F_0(\underline{B}_f/V^\psi B_f) & \rightarrow & F_0(\underline{B}_f/V^\psi B_f) \\
0 & \rightarrow & 0
\end{array}$$

(3.3.1)

such that the vertical morphisms are isomorphisms.

(3.4) Remark. The horizontal short exact sequences correspond to the short exact sequence of mixed Hodge modules [19]:

$$0 \rightarrow \mathbb{Q}_{D}^{H}[n-1] \rightarrow \psi_f\mathbb{Q}_{X}^{H}[n] \rightarrow \varphi_f\mathbb{Q}_{X}^{H}[n] \rightarrow 0.$$

In fact, taking $Gr_\psi$ of (3.3.1), we get $F_{1-n}$ of the underlying filtered $D$-module of (3.4.1) (using (2.2.1)), because the underlying filtered $D$-modules $\psi_\psi\omega_X$, $\varphi_\varphi\omega_X$ of $\psi_\psi\mathbb{Q}_{X}^{H}[n]$, $\varphi_\varphi\mathbb{Q}_{X}^{H}[n]$ are defined by

$$\psi_\psi\omega_X = \oplus_{0<\alpha \leq 1}Gr_\psi^\alpha\underline{B}_f, \quad \varphi_\varphi\omega_X = \oplus_{0<\alpha \leq 1}Gr_\psi^\alpha\underline{B}_f.$$  

Here we have a shift of the filtration $F$ coming from the transformation of left and right filtered $D$-modules (see [23]). Furthermore, $\tilde{\omega}_D$ is $F_{1-n}$ of the underlying filtered $D$-module of the intersection complex $IC_{D}^{\bullet}\mathbb{Q}^{H}$ which is a quotient of $\mathbb{Q}_{D}^{H}[n-1]$.

As a corollary of (3.3), we get (0.6) and the following

(3.5) Corollary. We have a canonical isomorphism

$$F_{1-n}(\varphi_\varphi\omega_X) = \oplus_{0<\alpha \leq 1} Gr_\psi^\alpha(\omega_D/\tilde{\omega}_D),$$

such that $Gr_\psi^\alpha(\omega_D/\tilde{\omega}_D)$ corresponds to the $exp(-2\pi i \alpha)$-eigenspace of $\varphi_\varphi\omega_X$ with respect to the action of monodromy.

References