数理解析研究所講究録
第 810 巻 1992 年 79-84

A HAMILTONIAN SYSTEM ON THE FUCHSIAN MODULI SPACE

KATSUNORI IWASAKI†

1. Introduction.

The purpose of the present talk is to define a completely integrable Hamiltonian system on the moduli space of Fuchsian differential equations on a Riemann surface of arbitrary genus. This Hamiltonian system is a generalization of the celebrated Painlevé equation and is a quitely interesting system of nonlinear partial differential equations which contains as special solutions a broad class of classical special functions such as hypergeometric functions of several variables. It also seems very interesting in connection with correlation functions in the quantum field theory.

The present talk is divided into two parts. The first part is the topological part in which we shall consider the moduli space of monodromy representations. By the Poincaré-Lefschetz duality, the moduli space of monodromy representations carries a natural Poisson structure. We shall further give it a local system structure topologically, which, together with the Poisson structure, defines a Hamiltonian dynamical system on it.

The second part is the analytic part in which we shall consider the moduli space of Fuchsian differential equations. By the Riemann-Hilbert correspondence, the local system structure on the moduli space of monodromy representations pulls back to one on the moduli space of Fuchsian differential equations and defines the monodromy preserving foliation on it. This foliation is an analytic and hence concrete realization of the Hamiltonian dyanmical system defined topologically before.

Many authors have already considered the monodromy preserving deformation. However, we would like to consider it from a different point of view, regarding it as something like a non-linear de Rham-Hodge theory.

2. Toplogical part — moduli of monodromy representations.

First we are going to the topological part of the theory, in which we shall consider the moduli space of monodromy representations. Let C be a closed oriented surface of genus $g \geq 0$, G a complex semi-simple Lie group with nontrivial discrete center $Z(G)$, $\mathbb{P}G = G/Z(G)$ its projectivization. Let $B(m)$ be the space of mutually distinct ordered m points in C. Given $\mathbf{p} = (p_1, ..., p_m) \in B(m)$, put $C_{\mathbf{p}} = C \setminus \{p_1, ..., p_m\}$.

Consider the set $R(\mathbf{p})$ of all representations of the fundamental group $\pi_1(C_{\mathbf{p}})$ into the Lie group $\mathbb{P}G$ up to conjugacy $\sim$. The set $\mathrm{Hom}(\pi_1(C_{\mathbf{p}}); \mathbb{P}G)$ is given the compact-open topology and $R(\mathbf{p}) = \mathrm{Hom}(\pi_1(C_{\mathbf{p}}); \mathbb{P}G)/\sim$ is given the quotient topology. This topological space is a quitely interesting object to be studied. Furthermore, consider the disjoint union $R(m) = \bigcup_{\mathbf{p}\in B(m)} R(\mathbf{p})$. We have the natural projection $R(m) \to B(m)$. The space $R(m)$

† Department of Mathematical Sciences, the University of Tokyo, 3-8-1 Komaba, Meguro-ku, Tokyo, 153 Japan

is naturally a local system over $B(m)$, whose characteristic homomorphism is given by a natural action of the braid group $\pi_1(B(m))$ on $R(\mathbf{p})$.

As we shall see below, the moduli space of monodromy representations with fixed local momodromy data admits a natural Poisson structure. Here a *Poisson manifold* P is a manifold whose structure sheaf $\mathcal{O}_P$ admits a Lie algebra structure $\{\cdot,\cdot\}$ such that, for any germ f at $p \in P$, $\{f,\cdot\}$ acts on the stalk $\mathcal{O}_{P,p}$ as a derivation. A Poisson manifold is a manifold on which a Hamiltonian dynamics can be considered.

In order to describe the Poisson structure, we need a more rigorous formulation. For this purpose, the use of fundamental groupoids is more convenient than that of fundamental groups. Let Π be the fundamental groupoid functor. This means the following : Given a topological space X, let ΠX be the fundamental groupoid of X, which is, by definition, the set of all homotopy equivalence classes of arcs in X. This forms a groupoid in a natural manner. Given a continuous map $f : X \to Y$, let $\Pi f : \Pi X \to \Pi Y$ be the associated groupoid homomorphism defined by $\Pi f(\gamma) = f \cdot \gamma$ for an arc $\gamma \in \Pi X$. Π is a covariant functor of the category of topological spaces into that of groupoids.

Let G and $\mathbb{P}G$ be as before. Given a topological space X, let $Q_G(X) = \mathrm{Hom}(\Pi X; \mathbb{P}G)$ be the set of all groupoid homomorphisms of ΠX into $\mathbb{P}G$, $G(X) = \mathrm{Map}(X; \mathbb{P}G)$ the group of all maps of X into $\mathbb{P}G$. The group $G(X)$ acts on $Q_G(X)$ from the left by $G(X) \times Q_G(X), (\phi, \rho) \mapsto \phi \cdot \rho$. Here $\phi \cdot \rho$ is defined by $\phi \cdot \rho(\gamma) = \phi(p) \cdot \rho(\gamma) \cdot \phi(q)^{-1}$, where $\gamma \in \Pi X$ is an arc with initial point q and terminal point p. Put $R_G(X) = X(G) \setminus Q_G(X)$. Given a continuous map $f : X \to Y$, let $R_G(f) = G(X)\setminus\mathrm{Hom}(\Pi f; \mathbb{P}G) : R_G(Y) \to R_G(X)$ be the corresponding groupoid homomorphism. One observes that $R_G(\cdot)$ is a contravariant functor of the category of topological spaces into that of sets.

Apply this formalism to our situation. As before, let C be a closed oriented surface of genus $g \geq 0$. Given $\mathbf{p} = (p_1, ..., p_m) \in B(m)$, let $C^*_{\mathbf{p}}$ be the real blow-up of C at the points $p_1, ..., p_m$. The boundary of $C^*_{\mathbf{p}}$ is homeomorhpic to the disjoint union of m copies of the unit circle S^1 : $\partial C^*_{\mathbf{p}} \simeq S^1 \bigcup ... \bigcup S^1$ (m-times). The inclusion $\iota : \partial C^*_{\mathbf{p}} \to C^*_{\mathbf{p}}$ induces the morphism $r = R_G(\iota) : R_G(C_{\mathbf{p}}) \to R_G(\partial C_{\mathbf{p}})$. Note that r is the ristriction map of representations on X into those on the boundary $\partial C^*_{\mathbf{p}}$. We have $R_G(\partial C^*_{\mathbf{p}}) \simeq R_G(S^1 \bigcup ... \bigcup S^1) = R_G(S^1) \times ... \times R_G(S^1) = C(G) \times ... \times C(G)$, where $C(G)$ denotes the set of all conjugate classes of elements in G.

Given $\theta = (\theta_1, ..., \theta_m) \in C(G) \times ... \times C(G)$, let $R(\mathbf{p}; \theta)$ be the fiber of the restriction map $r : R_G(C_{\mathbf{p}}) \to R_G(\partial C_{\mathbf{p}})$ over θ. We shall see that

PROPOSITION. *$R(\mathbf{p}; \theta)$ carries a natural symplectic structure.*

To see this, we consider the tangent space of $R(\mathbf{p}; \theta)$ at a point ρ. For symplicity of notation, put $X = C_{\mathbf{p}}$. Let $\mathfrak{g}$ be the Lie algebra of G, $\mathrm{Ad} : G \to GL(\mathfrak{g})$ the adjoint representation of G. Given $\rho \in R(\mathbf{p}; \theta)$, let L_ρ be the flat $\mathfrak{g}$-bundle over X associated to the representation $Ad \cdot \rho$. The cohomology exact sequence of the pair $(X, \partial X)$ with coefficients in L_ρ is

$$
\begin{aligned}
0 = H^0(X; L_\rho) &\xrightarrow{j^*} H^0(\partial X; L_\rho) \\
\xrightarrow{\delta^*} H^1(X, \partial X; L_\rho) &\xrightarrow{i^*} H^1(X; L_\rho) \xrightarrow{j^*} H^1(\partial X; L_\rho) \\
\xrightarrow{\delta^*} H^2(X, \partial X; L_\rho) &= 0.
\end{aligned}
$$

The tangent map $(dr)_\rho : T_\rho R_G(X) \to T_{r(\rho)} R_G(\partial X)$ admits the natural identification

$$\begin{array}{ccc} T_\rho R_G(X) & \xrightarrow{(dr)_\rho} & T_\rho R_G(\partial X) \\ \| & & \| \\ H^1(X; L_\rho) & \xrightarrow[j^*]{} & H^1(\partial X; L_\rho). \end{array}$$

Since $R(\mathbf{p}; \theta)$ is the r-fiber over $\theta \in R_G(\partial X)$, we have

$$\begin{aligned} T_\rho R(\mathbf{p}; \theta) &= \mathrm{Ker}[H^1(X; L_\rho) \xrightarrow{j^*} H^1(\partial X; L_\rho)] \\ &\underset{i^*}{\cong} \frac{H^1(X, \partial X; L_\rho)}{\delta^* H^0(\partial X; L_\rho)}, \end{aligned}$$

Here the second equality $\cong$ follows from the cohomology exact sequence.

Recall now the Poincaré-Lefschetz duality. Since G is semisimple, the Killing form $\mathfrak{g} \otimes \mathfrak{g} \to \mathbb{C}$ is nondegenerate. It extends to a bilinear form $L_\rho \otimes L_\rho \to \mathbb{C}_X$ compatible with the flat structure of L_ρ, where $\mathbb{C}_X$ is the constant system with fiber $\mathbb{C}$. Consider the pairing

$$\begin{aligned} H^1(X; L_\rho) \otimes H^1(X, \partial X; L_\rho) &\xrightarrow{\text{cup product}} H^2(X, \partial X; L_\rho \otimes L_\rho) \\ &\xrightarrow{\text{Killing form}} H^2(X, \partial X; \mathbb{C}_X) = \mathbb{C}. \end{aligned}$$

The Poincaré-Lefschetz duality asserts that this pairing is perfect. We see that the orthogonal complement of $\mathrm{Ker}[H^1(X; L_\rho \xrightarrow{j^*} H^1(\partial X)]$ with respect to this pairing is $\delta^* H^0(\partial X; L_\rho)$. Hence the Poincaré-Lefschetz duality, together with the two way descriptions of the tangent space $T_\rho R(\mathbf{p}; \theta)$, yields a non-degenerate skew-symmetric bilinear form on $T_\rho R(\mathbf{p}; \theta)$ Thus we have obtained an almost symplectic structure on $R(\mathbf{p}; \theta)$. We can show that this is integrable and hence a symplectic structure.

Consider the disjoint union $R(m; \theta) = \bigcup_{\mathbf{p} \in B(m)} R(\mathbf{p}; \theta)$. Let $R(m; \theta) \to B(m)$ be the natural projection. As mentioned before, $R(m; \theta)$ admits a natural local system structure over $B(m)$. Moreover, we can show the following :

PROPOSITION. *There exists a Poisson structure on $R(m; \theta)$ such that each fiber of the projection is a symplectic leaf.*

Therefore we have arrived at the following situation : The moduli space $R(m; \theta)$ of monodromy representations with fixed local monodromy data θ admits a natural Poisson structure arising from the Poincaré-Lefschetz duality, as well as a natural local system structure over $B(m)$. This local system structure defines a foliation on $R(m; \theta)$. One can ask the following question :

QUESTION. *Is this foliation a Hamiltonian dynamical system ? If so, what are the Hamiltonians ? Describe this dynamical system as concretely as possible.*

To consider how we can answer this question is the second step of the theory.

3. Analytic part — moduli of Fuchsian differential equations.

A local system structure of $R(m;\theta)$ over $B(m)$ is given in Section 2 by a purely topological argument. We would like to consider it more deeply and concretely. A basic idea to do this is to represent this local system structure in another auxiliary space. Such an auxiliary space $E(m;\theta)$ should be not a topological object but an object with finer structure, i.e. an *analytic object.* So we will try to find a diagram

$$\begin{array}{ccc} E(m;\theta) & \longrightarrow & R(m;\theta) \\ \downarrow & & \downarrow \\ B(m) & = & B(m) \end{array}$$

and to translate everything on the right-hand side into something on the left-hand side by pulling back through the horizontal arrow $\longrightarrow$. The analytic structure on the left-hand side enables us to understand things more clearly.

As is well-known, a closed oriented surface C admits a complex structure, i.e. a structure of Riemann surface. Fixing a complex structure on C, hereafter we shall regard C as a Riemann surface. In order to define auxiliary space $E(m;\theta)$ on the left-hand side, we make use of this complex structure. Explicitly, $E(m;\theta)$ will be a moduli space of Fuchsian differential equations on the Riemann surface C, $E(m;\theta) \to B(m)$ will be the natural projection assigning to each Fuchsian differential equation its regular singular points and $E(m;\theta) \to R(m;\theta)$ will be the Riemann-Hilbert correspondence.

Now we shall define the moduli space $E(m;\theta)$. For simplicity of exposition, hereafter, we restrict our attention to the simplest case where the Lie group G is $SL(2;\mathbb{C})$ and $\mathbb{P}G = PSL(2;\mathbb{C})$. In order to define $E(m;\theta)$, we shall establish notation.

Consider second order differential operators L on C of Schrödinger type. This means that, around any point of C, L can be represented by $L = -\frac{d^2}{dx^2} + Q$ in terms of a local coordinate x, where Q is a meromorphic function defined locally. We always assume that L is of Fuchsian type. Intrinsically we consider L as a differential operator $L : \mathcal{M}(\xi) \longrightarrow \mathcal{M}(\xi \otimes \kappa^{\otimes 2})$, where ξ is a suitable holomorphic line bundle over C, κ the canonical line bundle over C, and $\mathcal{M}(\cdot)$ denotes the sheaf of meromorphic sections. In order that there exist differential operators L of Schrödinger type on ξ, the line bundle ξ must satisfy the topological constraint on its Chern class : $c_1(\xi) = 1 - g$. Fix a line bundle ξ satisfying this constraint. Hereafter we mean by a *differential operator* a differential operator of the form mentioned above.

We are ready to define $E(m;\theta)$. Put $n = m + 3g - g$ and assume $n > 0$. Given $\theta = (\theta_1, ..., \theta_m) \in (\mathbb{C} \setminus \mathbb{Z})^m$, let $E(m;\theta)$ be the set of all differential operators L with ordered $m + n$ regular singular points such that, for $i = 1, .., m$, L has characrteristic exponents $\frac{2}{1}(1 \pm \theta_i)$ at the i^{th} singular point and the last n singular points of L are apparent and of ground state. Let $\pi : E(m;\theta) \to B(m+n)$ be the projection assigning to each differential equation its ordered singular points. Moreover, let $\varphi : B(m+n) \to B(m), \mathbf{r} = (p_1, ..., p_m, q_1, ..., q_n) \mapsto \mathbf{p} = (p_1, ..., p_m)$ be the projection into the first m components. Put $\varpi = \varphi \cdot \pi : E(m;\theta) \to B(m)$. Since apparent singular points have no effect on the projective monodromy representation, the projective monodromy map $PM : E(m;\theta) \to R(m;\theta)$ is well-defined. We have obtained the commutative diagram

$$\begin{tikzcd}
 & E(m;\theta) \arrow[r, "\text{projective monodromy map}"] \arrow[dl, "\pi"'] \arrow[dd, "\varpi"] & R(m;\theta) \arrow[ddl] \\
B(m+n) \arrow[dr, "\varphi"'] & & \\
 & B(m) &
\end{tikzcd}$$

THEOREM. *$E(m;\theta)$ admits a natural structure of albebraic variety of pure dimension $m+2n$ such that π and ϖ are rational surjection and the projective monodromy map is a holomorphic map.*

$E(m;\theta)$ may have singularities. Where are singularities of $E(m;\theta)$?What kinds of properties does the smooth part of $E(m;\theta)$ has ? In order to answer this question, consider the holomorphic line bundles $\xi_{\mathbf{r}}, (\mathbf{r} \in B(m+n))$ over C defined by $\xi_{\mathbf{r}} = \kappa^{\otimes 2} \otimes [p_1 + \ldots + p_m - (q_1 + \ldots + q_n)]$, where $\mathbf{r} = (p_1, \ldots, p_m, q_1, \ldots, q_n)$. Put $h^i(\mathbf{r}) = \dim H^i(C; \mathcal{O}(\xi_{\mathbf{r}})), (i = 0, 1)$. Since $n = m + 3g - 3$, the Chern class of $\xi_{\mathbf{r}}$ is $g - 1$. Hence the Riemann-Roch formula implies the Fredholm alternative $h^0(\mathbf{r}) = h^1(\mathbf{r})$. Let $A(m)$ be the algebraic subset of $B(m+n)$ consisting of all points $\mathbf{r}$ such that $h^0(\mathbf{r}) > 0$. If $g = 0$, then $A(m)$ is empty. If $g = 1$, then $A(m)$ can be written down explicitly by using Abel's theorem. Put $X(m) = B(m+n) \setminus A(m)$. This is a nonempty Zariski open subset of $B(m+n)$. Let $\mathcal{E}(m;\theta)$ be the inverse image of $X(m)$ by the projection $\pi : E(m;\theta) \to B(m+n)$. The above commutative diagram now yields the new one :

$$\begin{tikzcd}
 & \mathcal{E}(m;\theta) \arrow[r, "\text{projective monodromy map}"] \arrow[dl, "\pi"'] \arrow[dd, "\varpi"] & R(m;\theta) \arrow[ddl] \\
X(m) \arrow[dr, "\varphi"'] & & \\
 & B(m) &
\end{tikzcd}$$

An answer to the above question is given in the following :

THEOREM. *$\mathcal{E}(m;\theta)$ is smooth and hence a complex manifold. It admits a natural Poisson structure. The projection $\varpi : \mathcal{E}(m;\theta) \to B(m)$ is still surjective. Furthermore, the projective monodromy map $PM : \mathcal{E}(m;\theta) \to R(m;\theta)$ is locally biholomorphic. The Poisson structure on $\mathcal{E}(m;\theta)$ coincides with the pull-back of that on $R(m;\theta)$ by the projective monodromy map.*

A key to the theorem is a certain kind of Cousin's problem associated to the family of line bundles $\xi_{\mathbf{r}}, (\mathbf{r} \in X(m))$ and the Fredholm alternative $h^0(\mathbf{r}) = h^1(\mathbf{r})$.

Since PM is locally biholomorphic on $\mathcal{E}(m;\theta)$, the local system structure on $R(m;\theta)$ pulls back to one on $\mathcal{E}(m;\theta)$ through PM and defines a foliation on it. This is the *monodromy preserving foliation* on $\mathcal{E}(m;\theta)$. Let Ω be the fundamental two form associated to the Poisson structure on $\mathcal{E}(m;\theta)$. Then the monodromy preserving foliation is characterized by the following theorem.

THEOREM. *The monodromy preserving foliation is the Ω-Lagrangian foliation on $\mathcal{E}(m;\theta)$ which is transverse to each fiber of $\varpi : \mathcal{E}(m;\theta) \to B(m)$.*

This shows that the monodromy preserving foliation is a Hamiltonian dynamical system with $B(m)$ as the space of time variables. In terms of local coordinates, this theorem gives us a system of completely integrable Hamiltonian system. Explicit formulas for the Hamiltonians, as well as further developments of the theory, may be found in [**1**][**2**].

REFERENCES

1. K. Iwasaki, *Moduli and deformation for Fuchsian projective connections on a Riemann surface*, J. Fac. Sci., Univ. Tokyo, Sect. IA Math. **38** (1991), 431–531.
2. K. Iwasaki, *Fuchsian moduli on a Riemann surface — its Poisson structure and Poincaré-Lefschetz duality*, to appear in Pacific J. Math..