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連続区分線形写像の一般形

小室元政 (Motomasa KOMURO)(西東京科学大学理工学部)

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1 連続区分線形写像の定義

定義 1 Define an $n-1$ dimensional hyperplane U in n-dimensional euclidian space $\mathbf{R}^n$ by

$$U = U(\alpha, \beta) = \{x \in \mathbf{R}^n :< \alpha, x >= \beta\}$$

where $\alpha \in \mathbf{R}^n - \{0\}, \beta \in \mathbf{R}$ and $< \cdot, \cdot >$ denotes the usual inner product. We suppose that elements of $\mathbf{R}^n$ are column vectors. For $\alpha_1, \cdots, \alpha_k \in \mathbf{R}^n - \{0\}$ and $\beta_1, \cdots, \beta_k \in \mathbf{R}$, define

$$\tilde{\alpha} = (\alpha_1, \cdots, \alpha_k) \in M(n \times k), \tilde{\beta} = (\beta_1, \cdots, \beta_k) \in M(1 \times k)$$

where $M(m \times n)$ denotes the set of all $m \times n$ matrices with real components.For $(\tilde{\alpha}, \tilde{\beta})$ a union of hyperplanes

$$B = B(\tilde{\alpha}, \tilde{\beta}) = \bigcup_{i=1}^{k} U(\alpha_i, \beta_i)$$

is called *a linear boundary* (or simply, *boundary*) defined by $(\tilde{\alpha}, \tilde{\beta})$. For $(\tilde{\alpha}, \tilde{\beta})$ define a function $\omega : \mathbf{R}^n \to \{0,1\}^k$ by

$$\omega(x) = (\mathrm{sgn}(< \alpha_1, x > -\beta_1), \cdots, \mathrm{sgn}(< \alpha_k, x > -\beta_k))$$

where

$$\mathrm{sgn}(t) = \begin{cases} 0 & (t \leq 0) \\ 1 & (t > 0) \end{cases}$$

The set of signs of regions is a subset of $\{0,1\}^k$ defined by

$$\Omega = \Omega(\tilde{\alpha}, \tilde{\beta}) = \{\omega \in \{0,1\}^k : \omega = \omega(x) \text{ for some } x \in \mathbf{R}^n\}.$$

The polyhedral region (or simply, *region*) with a sign $\omega \in \Omega$ is

$$R_\omega = \{x \in \mathbf{R}^n : \omega(x) = \omega\} \text{ for } \omega \in \Omega.$$

The union $\bigcup\{R_\omega : \omega \in \Omega\}$ is a partition of $\mathbf{R}^n$;

$$\mathbf{R}^n = \bigcup_{\omega \in \Omega} R_\omega; \text{ and}$$
$$R_\omega \cap R_{\omega'} = \emptyset \text{ if } \omega \neq \omega'$$

定義 2 A mapping $f : \mathbf{R}^n \to \mathbf{R}^m$ is *piecewise-affine* if there is a linear boundary $B = B(\tilde{\alpha}, \tilde{\beta})$ such that

(i) f is differentiable at all points which do not belong to B;

(ii) for each $\omega \in \Omega(\tilde{\alpha}, \tilde{\beta})$, the derivative $Df(x)$ is constant in the interior of R_ω, i.e. $x, x' \in \mathrm{int}(R_\omega) \Rightarrow Df(x) = Df(x')$.

If $f : \mathbf{R}^n \to \mathbf{R}^m$ is piecewise-affine, then for each $\omega \in \Omega(\tilde{\alpha}, \tilde{\beta})$, there are $A_\omega \in M(m \times n)$ and $q_\omega \in \mathbf{R}^m$ such that

$$\begin{aligned} f(x) &= A_\omega x + q_\omega \text{ for } x \in \mathrm{int}(R_\omega) \\ A_\omega &= Df(x) \text{ for } x \in \mathrm{int}(R_\omega) \end{aligned}$$

When f is piecewise-affine, we will say that f is *piecewise-linear* (abbrev. *PL*), according to custom. In general, a PL map $f : \mathbf{R}^n \to \mathbf{R}^m$ may be discontinuous at points on B. If f is continuous on B, and so, on $\mathbf{R}^n$, f is called *a continuous piecewise-linear map* (abbrev. *CPL map*).

2 一般形

定義 3 A continuous piecewise linear map from $\mathbf{R}^n$ to $\mathbf{R}$ is called a continuous piecewise linear function of $\mathbf{R}^n$. A continuous piecewise linear function is abbreviated as CPL function. The set of all CPL functions of $\mathbf{R}^n$ is denoted by $\mathrm{CPL}(\mathbf{R}^n)$.

If we denote a continuous piecewise linear map $f : \mathbf{R}^n \to \mathbf{R}^m$ by

$$f(x) = (f_1(x), \cdots, f_m(x)), \quad x \in \mathbf{R}^n,$$

each f_i is a continuous function of $\mathbf{R}^n$.

Now we will consider to express a CPL function using by a absolute value function $|\cdot| : \mathbf{R} \to \mathbf{R}$;

$$|x| = \begin{cases} x & (x \geq 0) \\ -x & (x < 0) \end{cases}$$

定義 4 Define a set of formal expression of variable $x \in \mathbf{R}^n$, $L_k(\mathbf{R}^n)$, $(k \geq 0)$, inductively as follows;

$$\begin{aligned} L_0(\mathbf{R}^n) &= \{f(x) = < a, x > + b : a \in \mathbf{R}^n, b \in \mathbf{R}\} \\ L_k(\mathbf{R}^n) &= \{f_0(x) + \sum_{i=1}^{N} \varepsilon_i |f_i(x)| : f_i(x) \in L_{k-1}(\mathbf{R}^n) \quad (0 \leq i \leq N), \\ &\qquad \varepsilon_i \in \{-1, 1\} \quad (1 \leq i \leq N), \quad N \geq 0\} \end{aligned}$$

where $N = 0$ means that the summation is not taken. Then the following holds;

$$L_0(\mathbf{R}^n) \subset L_1(\mathbf{R}^n) \subset \cdots \subset L_k(\mathbf{R}^n) \subset \cdots$$

Hence $L_k(\mathbf{R}^n)$ is the set of all linear expression with at most k-ply absolute value function. Define

$$L_\infty(\mathbf{R}^n) = \bigcup_{k=0}^{\infty} L_k(\mathbf{R}^n).$$

An element of $L_\infty(\mathbf{R}^n)$ is called an *expression* of CPL function of $\mathbf{R}^n$.

定義 5 Define a mapping S from $L_\infty(\mathbf{R}^n)$ to $CPL(\mathbf{R}^n)$ by

$$S(f)(x) = F(x) \quad \text{for} \quad f(x) \in L_\infty(\mathbf{R}^n)$$

where $F(x) \in \mathbf{R}$ is a value that a formal expression $f(x)$ takes when $x \in \mathbf{R}^n$ is substituted to $f(x)$.

Remark. For $x \in \mathbf{R}$, $f_1(x) = 1 - |x| + |1 - |x||$ and $f_2(x) = |x+1| + |2x| + |x-1|$ are considered as two different elements of $L_2(\mathbf{R})$. However, if we substitute any $x \in \mathbf{R}$ to them, we have $f_1(x) = f_2(x)$, so they are same function as element of CPL($\mathbf{R}$). That is, $S(f_1)(x) = S(f_2)(x)$. In general, when $f_1(x) = f_2(x)$ for all $x \in \mathbf{R}^n$ while they are different elements of $L_\infty(\mathbf{R}^n)$, we say that they are *different expression of same CPL function.*

定義 6 For $f(x) = < a, x > +b \in L_0(\mathbf{R}^n)$, the $b \in \mathbf{R}$ is called *a constant term* of $f(x)$. Inductively, for $f(x) \in L_k(\mathbf{R}^n)$, if

$$f(x) = f_0(x) + \sum_{i=1}^{N} \varepsilon_i |f_i(x)|, \quad f_i(x) \in L_{k-1}(\mathbf{R}^n) \quad (0 \le i \le N),$$

each constant term of $f_i(x)$ is called *a constant term* of $f(x)$.

定義 7 For $f(x) \in L_k(\mathbf{R}^n)$, define an expression $\bar{f}(x,y)$ by multiplying $-y \in \mathbf{R}$ by all constant terms of $f(x)$. Clearly $\bar{f}(x,y)$ has at most k-ply absolute value function, hence

$$\bar{f}(x,y) \in L_k(\mathbf{R}^{n+1}) \quad (x,y) \in \mathbf{R}^n \times \mathbf{R} = \mathbf{R}^{n+1}.$$

Define a function $F_{k,n}$ from $L_k(\mathbf{R}^n)$ to $L_k(\mathbf{R}^{n+1})$ by

$$F_{k,n}(f) = \bar{f}.$$

Remark. Assume $f_1(x), f_2(x) \in L_k(\mathbf{R}^n)$ are two different expression of same function, i.e.

$$f_1(x) = f_2(x) \quad \text{for all} \quad x \in \mathbf{R}^n.$$

Then $\bar{f}_1(x,y)$ and $\bar{f}_2(x,y)$, which are given by multiplying $-y \in \mathbf{R}$ by all constant terms of $f_1(x)$ and $f_2(x)$, may be different function.

For example, $f_1(x) = 1 - |x| + |1 - |x||$ and $f_2(x) = |x+1| + |2x| + |x-1|$ satisfies

$$f_1(x) = f_2(x) \quad \text{for all} \quad x \in \mathbf{R}.$$

Then, since

$$\bar{f}_1(x,y) = -y - |x| + |-y-|x||, \quad \text{and}$$
$$\bar{f}_2(x,y) = |x-y| + |2x| + |x+y|,$$

we have

$$\bar{f}_1(0,1) = 0, \quad \text{and} \quad \bar{f}_2(0,1) = 2,$$

i.e. $\bar{f}_1(x,y)$ and $\bar{f}_2(x,y)$ are different function.

However, it is proved that if $y \leq 0$, then

$$\bar{f}_1(x,y) = \bar{f}_2(x,y) \quad \text{for all } x \in \mathbf{R}^n, \quad y \leq 0.$$

定義 8 For $f(x) \in L_k(\mathbf{R}^n)$, define an expression $\tilde{f}(x,y)$ by multiplying

$$\frac{1}{2}\{y + |y|\} \quad (y \in \mathbf{R})$$

by all constant terms of $f(x)$. Clearly $\tilde{f}(x,y)$ has at most (k+1)-ply absolute value function, hence

$$\tilde{f}(x,y) \in L_{k+1}(\mathbf{R}^{n+1}) \quad (x,y) \in \mathbf{R}^n \times \mathbf{R} = \mathbf{R}^{n+1}.$$

Define a function $G_{k,n}$ from $L_k(\mathbf{R}^n)$ to $L_{k+1}(\mathbf{R}^{n+1})$ by

$$G_{k,n}(f) = \tilde{f}.$$

定義 9 Using two functions $F_{k,n}$ and $G_{k,n}$, we define a function $T_{k,n}$ as follows;

$$T_{k,n} : L_k(\mathbf{R}^n) \times L_k(\mathbf{R}^n) \to L_{k+1}(\mathbf{R}^{n+1});$$
$$T_{k,n}(f,g) = F_{k,n}(f) + G_{k,n}(g).$$

定義 10 Define subsets $L_n^a(\mathbf{R}^n)$,$L_n^b(\mathbf{R}^n)$ and $L_n^c(\mathbf{R}^n)$ of $L_n(\mathbf{R}^n)$ as follows inductively;

$$L_1^a(\mathbf{R}) := \{ax + \frac{b}{2}\{x + |x|\} : a, b, x \in \mathbf{R}\}$$
$$L_1^c(\mathbf{R}) := \{c + \sum_{i=1}^{N} f_i(x - x_i) : f_i(x) \in L_1^s(\mathbf{R}), c \in \mathbf{R}, x_i \in \mathbf{R}, N \geq 1\}$$
$$L_1^b(\mathbf{R}) := \{f(x) \in L_1^c(\mathbf{R}) : S(\tilde{f})(x,y) = 0 \quad \text{for} \quad \text{all} x \in \mathbf{R} \quad \text{and} \quad y = 0\}$$

where $\tilde{f}(x,y) = G_{1,1}(f)$.

$$L_2^a(\mathbf{R}^2) := T_{1,1}(L_1^c(\mathbf{R}), L_1^b(\mathbf{R}))$$
$$L_2^c(\mathbf{R}^2) := \{c + \sum_{i=1}^{N} f_i(x - x_i) : f_i(x) \in L_2^s(\mathbf{R}^2), c \in \mathbf{R}, x_i \in \mathbf{R}^2, N \geq 1\}$$
$$L_2^b(\mathbf{R}^2) := \{f(x) \in L_2^c(\mathbf{R}^2) : S(\tilde{f})(x,y) = 0 \quad \text{for} \quad \text{all} x \in \mathbf{R}^2 \quad \text{and} \quad y = 0\}$$

where $\tilde{f}(x,y) = G_{2,2}(f)$.

$$L_n^a(\mathbf{R}^n) := T_{n-1,n-1}(L_{n-1}^c(\mathbf{R}^{n-1}), L_{n-1}^b(\mathbf{R}^{n-1}))$$
$$L_n^c(\mathbf{R}^n) := \{c + \sum_{i=1}^{N} f_i(x - x_i) : f_i(x) \in L_n^s(\mathbf{R}^n), c \in \mathbf{R}, x_i \in \mathbf{R}^n, N \geq 1\}$$
$$L_n^b(\mathbf{R}^n) := \{f(x) \in L_n^c(\mathbf{R}^n) : S(\tilde{f})(x,y) = 0 \quad \text{for} \quad \text{all} x \in \mathbf{R}^n \quad \text{and} \quad y = 0\}$$

where $\tilde{f}(x,y) = G_{n,n}(f)$.

定理 1 *Any CPL function of* $\mathbf{R}^n$, $f(x) \in \mathrm{CPL}(\mathbf{R}^n)$, *has an expression in* $L_n^c(\mathbf{R}^n)$.

Example 1. Define a new notation $[x]^\varepsilon$ for $x \in \mathbf{R}$ and $\varepsilon \in \{0,1\}$ by

$$[x]^\varepsilon = \begin{cases} \frac{1}{2}\{x + |x|\} & (\varepsilon = 1) \\ x & (\varepsilon = 0) \end{cases}$$

Assume that all a's belong to $\mathbf{R}^n$, all b's belong to $\mathbf{R}$ and all ε's belong to $\{0,1\}$.

(1) $L_1^a(\mathbf{R})$ consists of all expression with following form;

$$a_0 x + a_1 [x]^\varepsilon \quad \text{for} \quad x \in \mathbf{R}$$

$L_1^c(\mathbf{R})$ consists of all expression with following form;

$$\sum_{i=1}^{N} a_i [x + b_i]^{\varepsilon_i} \quad \text{for} \quad x \in \mathbf{R}$$

Clearly

$$L_1(\mathbf{R}) = L_1^c(\mathbf{R})$$

holds.

(2) $L_2^a(\mathbf{R}^2)$ consists of all expressions with following form;

$$\sum_{i=1}^{N} a_i [x + b_i [y]^{\varepsilon_{i2}}]^{\varepsilon_{i1}} \quad \text{for} \quad (x,y) \in \mathbf{R}^2$$

$L_2^c(\mathbf{R}^2)$ consists of all expression with following form;

$$\sum_{i=1}^{N} a_i [x + c_i + b_i [y + d_i]^{\varepsilon_{i2}}]^{\varepsilon_{i1}} \quad \text{for} \quad (x,y) \in \mathbf{R}^2$$

(3) $L_3^a(\mathbf{R}^3)$ consists of all expression with following form;

$$\sum_{i=1}^{N} a_i [x + c_i + b_i [y + d_i]^{\varepsilon_{i2}}]^{\varepsilon_{i1}} \quad \text{for} \quad (x,y,z) \in \mathbf{R}^3$$

$L_3^c(\mathbf{R}^n)$ consists of all expression with following form;

$$\sum_{i=1}^{N} a_i[x + c_i[z]^{\varepsilon_{i3}} + b_i[y + d_i[z]^{\varepsilon_{i3}}]^{\varepsilon_{i2}}]^{\varepsilon_{i1}} \quad \text{for} \quad (x, y, z) \in \mathbf{R}^3$$

Example 2. (1) $f_1(x) \in L_1^c(\mathbf{R}), \quad f_2(x) \in L_1^b(\mathbf{R})$;

$$f_1(x) = a_1x + (a_2 - a_1)[x] + (a_3 - a_2)[x-1] + c_1$$
$$f_2(x) = -a_4 + a_4[x+1] - a_4[x] + c_2$$

(2) $F(x,y), G(x,y) \in L_2^c(\mathbf{R}^2)$;

$$\begin{aligned} F(x,y) &= \tilde{f}_1(x,y) + \tilde{f}_2(x,y) \\ &= a_1x + (a_2 - a_1)[x] + (a_3 - a_2)[x+y] - c_1y \\ &\quad -a_4[y] + a_4[x + [y]] - a_4[x] + c_2[y] \\ &= a_1x - c_1y + (a_2 - a_1 - a_4)[x] + (-a_4 + c_2)[y] \\ &\quad +(a_3 - a_2)[x+y] + a_4[x + [y]] \end{aligned}$$

$$\begin{aligned} G(x,y) &= -c_1'y + a_3'[x] + c_1'[y] + (a_3' - a_2')[x+y] \\ &\quad +(a_2' - a_3')[x + [y]] \end{aligned}$$

(3) $H_1(x,y) \in L_2^c(\mathbf{R}^2)$, $H_2(x,y) \in L_2^b(\mathbf{R}^2)$;

$$\begin{aligned} H_1(x,y) &= F(x+1, y-1) + G(x-1, y+1) + c_3 \\ &= a_1[x+1] - c_1(y-1) + (a_2 - a_1 - a_4)[x+1] \\ &\quad +(-a_4 + c_2)[y-1] + (a_3 - a_2)[x+y] + a_4[x+1+[y-1]] \\ &\quad -c_1'(y+1) + a_3'[x-1] + c_1'[y+1] + (a_3' - a_2')[x+y] \\ &\quad +(a_2' - a_3')[x-1+[y+1]] + c_3 \end{aligned}$$

$$\begin{aligned} H_2(x,y) &= F'(x+1, y-1) + G'(x-1, y+1) + d_3 \\ &= -d_1(y-1) + b_3[x+1] + d_1[y-1] \\ &\quad +(b_3 - b_2)[x+y] + (b_2 - b_3)[x+1+[y-1]] \\ &\quad +d_1(y+1) - b_3[x-1] - d_1[y+1] + (b_2 - b_3)[x+y] \\ &\quad +(b_3 - b_2)[x-1+[y+1]] + d_3 \end{aligned}$$

(4)

$$\begin{aligned} \bar{H}_1(x,y,z) &= F(x-z, y+z) + G(x+z, y-z) - c_3z \\ &= a_1[x-z] - c_1(y+z) + (a_2 - a_1 - a_4)[x-z] \end{aligned}$$

$$+(-a_4+c_2)[y+z]+(a_3-a_2)[x+y]+a_4[x-z+[y+z]]$$
$$-c_1'(y-z)+a_3'[x+z]+c_1'[y-z]+(a_3'-a_2')[x+y]$$
$$+(a_2'-a_3')[x+z+[y-z]]-c_3z$$

$$\tilde{H}_2(x,y,z)=F(x+[z],y-[z])+G(x-[z],y+[z])+d_3[z]$$
$$=-d_1(y-[z])+b_3[x+[z]]+d_1[y-[z]]$$
$$+(b_3-b_2)[x+y]+(b_2-b_3)[x+[z]+[y-[z]]]$$
$$+d_1(y+[z])-b_3[x-[z]]-d_1[y+[z]]+(b_2-b_3)[x+y]$$
$$+(b_3-b_2)[x-[z]+[y+[z]]]+d_3[z]$$

(5) $K(x,y,z)\in L_3^a(\mathbf{R}^3)$;

$$K(x,y,z)=\bar{H}_1(x,y,z)+\tilde{H}_2(x,y,z)$$
$$=a_1[x-z]-c_1(y+z)+(a_2-a_1-a_4)[x-z]$$
$$+(-a_4+c_2)[y+z]+(a_3-a_2)[x+y]+a_4[x-z+[y+z]]$$
$$-c_1'(y-z)+a_3'[x+z]+c_1'[y-z]+(a_3'-a_2')[x+y]$$
$$+(a_2'-a_3')[x+z+[y-z]]-c_3z$$
$$-d_1(y-[z])+b_3[x+[z]]$$
$$+d_1[y-[z]]$$
$$+(b_3-b_2)[x+y]+(b_2-b_3)[x+[z]+[y-[z]]]$$
$$+d_1(y+[z])-b_3[x-[z]]-d_1[y+[z]]+(b_2-b_3)[x+y]$$
$$+(b_3-b_2)[x-[z]+[y+[z]]]+d_3[z]$$