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Kyoto University
A note on the heteroclinic $\Omega$-explosions

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Abstract

Palis and Takens have obtained many results about the chaotic dynamics with homoclinic tangency. In 1987, they classified one-parameter families of diffeomorphisms $\{\varphi_\mu : \mu \in \mathbb{R}\}$ on surfaces which display a homoclinic $\Omega$-explosion at $\mu = 0$ and derived several theorems about it\(^1\). We also consider one-parameter families of diffeomorphisms which have heteroclinic tangencies for $\mu = 0$, are persistently hyperbolic for $\mu < 0$ and have transversal heteroclinic orbits for positive $\mu$. Using their method, we classified the bifurcations with heteroclinic $\Omega$-explosion. The theory developed by Palis and Takens regarding the measure of $\{\mu\}$ for which $\varphi_\mu$ is non-hyperbolic can also be applied to our classification.

1 Introduction

There are some excellent results about the bifurcation due to homoclinic and heteroclinic tangencies such as Smale’s homoclinic theorem\(^2\), Robinson’s creation of infinitely many sinks in a homoclinic tangency \(^3\),\(^4\), Newhouse theory \(^3\),\(^5\) and the results by Mora, Viana, Benedics and Carleson\(^6\),\(^7\). All of them studied one-parameter families $\{\varphi_\mu : \mu \in \mathbb{R}\}$ of diffeomorphisms with homoclinic tangency at $\mu = 0$ and in the interval $(-\delta, \delta)$ of a parameter range. It is well-known that those families create infinitely many bifurcations no matter how small this interval may be. Particularly, Palis and Takens successfully in studied the nonhyperbolic dynamics (i.e. bifurcating diffeomorphism) by investigating those families, and they suggested the possibility of extending their results to heteroclinic tangencies\(^8\) (Fig.1).

To recall a heteroclinic point, let $M$ be a 2-manifold, $\varphi : M \to M$ a $C^2$ diffeomorphism and $p_1, p_2$ ($p_1 \neq p_2$) fixed points for $\varphi$. We will be particularly interested in the case where each $p_i$ is a (hyperbolic) saddle fixed point i.e. $\varphi(p_i) = p_i$ and $d\varphi(p_i)$ has two real eigenvalues $\lambda_i$ and $\sigma_i$ with $0 < |\lambda_i| < 1 < |\sigma_i|$. For simplicity we assume that these eigenvalues are positive, so $0 < \lambda_i < 1 < \sigma_i$. We call $\lambda_i$ (resp. $\sigma_i$) the contracting (resp. expanding) eigenvalue. In this situation from the invariant manifold’s theorem\(^3\) we know that:
- $W^s(p_i)$ and $W^u(p_i)$; the stable and unstable separatrices of $p_i$, are $C^2$.
- there are $C^1$ linearizing coordinates in a neighborhood of $p_i$, i.e. $C^1$ coordinates $x_1, x_2$ such that $p_i = (0, 0)$ and such that $\varphi(x_1, x_2) = (\lambda x_1, \sigma x_2)$.

We consider the situation (shown in Fig.1) where the stable manifold $W^s(p_1)$ of $p_1$ intersects the unstable manifold $W^u(p_2)$ of $p_2$ transversally at some point $z$. We call $z$ a transversal heteroclinic point. If the separatrices are tangential at $z$, then we call it a heteroclinic point of tangency.

Let us recall a few concepts. The point $z$ is nonwandering for $\varphi$ if for any given neighborhood $U$ of $z$ and any integer $n_0 > 0$, there is an integer $n$ such that $|n| > n_0$ and $\varphi^n(U) \cap U \neq \emptyset$. The union of the nonwandering points is called the nonwandering set, which is denoted by $\Omega(\varphi)$. A closed subset $\Lambda \subset \Omega(\varphi)$ is called a basic set if it is hyperbolic (see [9] for its definition) and transitive (i.e. it has a dense orbit), and its subset of periodic orbits is dense. Palis and Takens defined $\varphi$ to be hyperbolic if $\Omega(\varphi)$ is a hyperbolic set, and persistently hyperbolic if every $\tilde{\varphi}$ which is $C^2$ near $\varphi$ is also hyperbolic. They showed in [8] that if $\varphi$ is persistently hyperbolic, then it cannot have cycles and hence is $\Omega$-stable$^{10}$.

2 heteroclinic $\Omega$-explosion and its criterions

We say that a one parameter family $\varphi_\mu$: $M \to M$ of $C^2$ diffeomorphisms on a closed surface $M$ has a heteroclinic $\Omega$-explosion at $\mu = 0$ if:

1. For $\mu < 0$, $\varphi_\mu$ is persistently hyperbolic and has two saddle fixed points $p_{1\mu}$, $p_{2\mu}$.
2. For $\mu = 0$, the nonwandering set $\Omega(\varphi_0)$ consists of $\tilde{\Omega}(\varphi_0) = \lim_{\mu \nearrow 0} \Omega(\varphi_\mu)$ together with heteroclinic orbits of tangency $O$ associated with a fixed saddle point $p_{i_\mu}$, so that $\Omega(\varphi_0) = \tilde{\Omega}(\varphi_0) \cup O$ and the product of the eigenvalues of $d\varphi(\mu)$ is different from 1;
3. The separatrices have quadratic tangency along $O$ unfolding generically, i.e., by changing the coordinates, $W^s(p_{i\mu})$ is given by $x_2 = 0$ and $W^s(p_{i\mu})$ by $x_2 = ax_1^2 + b\mu (a, b \neq 0)$ in the neighborhood of $p_{i\mu}^{11}$. 

We shall classify these heteroclinic $\Omega$-explosions according to the following criterions concerning $\varphi_0$ (hereafter denoted by $\varphi$).

a) The signs of the eigenvalues of $d\varphi(p_i)$

The signs of eigenvalues of $d\varphi(p_i)$, $\lambda_i$ and $\sigma_i$, are simply denoted by $(+, +)$, $(+, -)$, $(-, +)$, $(-, -)$. We consider, in this paper, eight possible combinations, $(+, +)(+, +), (+, +)(-, -), (-, +)(-, +), (-, -)(-, -)$. We find that the following eight classes of stable and unstable separatrices are possible:

- $(+)$, $(-)$
- $(-)$, $(+)$
- $(+)$, $(+)$
- $(-)$, $(-)$
- $(+)$, $(-)$
- $(-)$, $(+)$
- $(+)$, $(+)$
- $(-)$, $(-)$

Each class gives rise to a different behavior of the $\Omega$-explosion.
+1(-, +) 2 and (-, +) 2 , which can be summarized as $\lambda_1 \sigma_1 \lambda_2 \sigma_2 > 0$. We say that $\varphi$ is orientation-preserving, resp. orientation-reversing if $\det(d\varphi) > 0$, resp. $\det(d\varphi) < 0$.

b) The sides of tangencies

This concerns the positions of $W^s(p_i)$ in the neighborhood of $p_j$. For example if the configuration of $W^u(p_2)$ and $W^s(p_2)$ in the neighborhood of $p_1$ is as shown in Fig.2, then we have four cases of the configuration of $W^s(p_1)$ and $W^s(p_1)$ in the neighborhood of $p_2$ as in Fig.3.

We define configurations $(B)-(D)$ by replacing $p_2$ in $(b)-(d)$ by $p_1$.

c) The mode of connection

It concerns the distinction that still can be made if the sides of tangencies are prescribed; for example if they are as in (A)-(a) of the above cases, then the global unstable separatrix of $p_1$ can be connected with the relevant part of $W_{loc}(p_2)$ in the two different ways shown in Fig.4:

In all cases which may exist according to the above criterions, we want to determine whether a heteroclinic $\Omega$-explosion can occur, and if so, to determine

- if the ambient manifold $M$ can be orientable or not;
- if the saddle point $p_i$ can be part of a non-trivial basic set or not.

We say that the expanding, resp. contracting, eigenvalue is dominating if the product of the eigenvalues is bigger than one, resp. smaller than one.

We consider, as an example, a one parameter family $\varphi_\mu$ with heteroclinic $\Omega$-explosion, such that the signs of eigenvalues of $d\varphi(p_i)$ ($i = 1, 2$) are + + and such that the sides of tangencies and the mode of connection are as in Fig.4.

In order to prove this we take as before $q \in W^s(p_2)$ and $r \in W^u(p_1)$ both in $O$ of $\varphi$. We can also take a diffeomorphism $\varphi^n$ from the neighborhood of $r$ to the neighborhood of $q$ for some integer $n > 0$. We see that $\varphi^n$ is orientation-preserving in the first case in Fig.4 (i.e. the orientation of $B'$ which is an oriented box near $q'$ is the same as that of $B$ which is an oriented box near $r$). From the assumption, $\varphi$ is an orientation-preserving map. From these it follows that the first case in Fig.4 can occur on any closed surface. However the second case in Fig.4 can not occur if the ambient surface is orientable because $\varphi^n$ is orientation-reversing in this picture.

We know that a transversal homoclinic orbit and heteroclinic orbit are contained in a horseshoe (that is often called a maximal invariant set or ‘non-trivial basic set’)[8,10]. If a saddle fixed point $p_i$ is in a non-trivial basic set $\Lambda$, then we express the (local) position of $\Lambda$ with respect to $p_i$ by the quadrant which is separated by $W^u_{loc}(p_i)$ and $W^s_{loc}(p_i)$ and contains points of $\Lambda \cap U$ for any small neighborhood $U$ of $p_i$. Furthermore, for each point $x \in \Lambda \cap U$ we can construct a pair of leaves, which are called stable and unstable leaves. For example, if the side of tangencies is as
in Fig.2-(A) then the only quadrant where there might be a basic set is the upper right quadrant.

If at least one of the eigenvalues of $d\varphi(p_i)$ is negative, then $p_i$ can not be in a non-trivial basic set, because such a basic set would occupy at least two quadrants.

Neither heteroclinic $\Omega$-explosion nor homoclinic $\Omega$-explosion can occur when 'premature tangencies', 'premature creation of horseshoes' or 'infinitely many circles' appear.

3 General classification and possible cases

In this section, we systematically go through all the cases as classified by: signs of eigenvalues, side of tangencies, mode of connection, and where necessary, eigenvalue condition. Furthermore we show examples of $\varphi$ with heteroclinic $\Omega$-explosion for all the possible cases. Since each case realized on an orientable surface can also be realized on a non-orientable surface, we give only those examples on a non-orientable surface that cannot be realized on an orientable surface.

(1) The case $(+, +)_1-$(+, +)$_2$

The sides of tangencies are as in Fig.2. Before classifying them according to the different connections, we observe that (B) (and (b)) cannot occur due to 'premature tangencies' and that (A) and (D) ((a) and (d)) can be interchanged by replacing $\varphi$ by $\varphi^{-1}$. So we only have to consider the cases (A)-(a), (A)-(c), (A)-(d), and (C)-(c). We examine the connection of the four cases.

We first consider the two modes of connections shown in Fig.5 for the case (A)-(a).

Case (A)-(a)-1 can only occur if the expanding eigenvalue is dominating so that 'premature creation of horseshoes' don't occur. It can occur when $p_i$ is part of a non-trivial basic set or when $p_i$ is an isolated point in $\tilde{\Omega}(\varphi)$. $M$ can be orientable or non-orientable in this case.

Case (A)-(a)-2 can occur if $M$ is a non-orientable surface. Here $p_i$ is part of a non-trivial basic set to avoid 'infinitely many circles'. This non-trivial basic set must be in the upper right quadrant. But this quadrant is not allowed. So this case cannot occur.

We next consider the cases shown in Fig.6 for the case (A)-(c).

Case (A)-(c)-1 can occur if $M$ is non-orientable and the expanding eigenvalue is dominating and $p_1$ and $p_2$ are not part of non-trivial basic set.

Case (A)-(c)-2 can not be observed for the same reason as in the case (A)-(a)-2. We consider the two cases shown in Fig.7 for the case (A)-(d) too.
Case (A)-(d)-1 can only occur if the expanding eigenvalue is dominating and \( p_1 \) is isolated in \( \tilde{\Omega}(\varphi) \). Therefore it occurs in a non-orientable surface.

Case (A)-(d)-2 can not also occur because of the same reason as in the case (A)-(a)-2.

Now we consider the two cases shown in Fig.8 for the case (C)-(c).

Case (C)-(c)-1 can only occur if expanding eigenvalue is dominating and \( p_1 \) is isolated in \( \tilde{\Omega}(\varphi) \). Therefore it occurs in a non-orientable surface.

Case (C)-(c)-2 Again because of the "infinitely many circles" it can only occur when \( p_1 \) is contained in a non-trivial basic set, which must be in the upper left quadrant. The surface \( M \) can be orientable or non-orientable in this case.

Here we show examples of all the cases that were not excluded above. Some examples in [8] and [12] were helpful in constructing the following.

Case (A)-(a)-1: \( p_1 \) is not part of non-trivial basic set.

For example, we start with a diffeomorphism \( \Phi \) as indicated in Fig.9 with \( \Omega(\Phi) \) consisting of two sources and three sinks besides the saddles \( p_1 \), \( p_2 \).

Let \( l \) (resp. \( \tilde{l} \)) be a curve segment from \( W^u(p_1) \) to \( W^s(p_2) \) (resp. from \( W^s(p_1) \) to \( W^u(p_2) \)), and \( U \) (resp. \( \tilde{U} \)) be a small neighborhood of \( l \) (resp. \( \tilde{l} \)), and let \( \sigma_\mu \) (resp. \( \overline{\sigma}_\mu \)) be a 1-parameter family of diffeomorphisms with support in \( U \) (resp. \( \tilde{U} \)) (so that for \( \mu \leq -1 \), \( \sigma_\mu \) (resp. \( \overline{\sigma}_\mu \)) is the identity and for \( \mu > -1 \), it pushes \( W^u(p_1) \) (resp. \( W^s(p_1) \)) down so that for \( \mu = 0 \) we have a generically unfolding tangency of \( \sigma_\mu(W^u(p_1)) \) and \( W^s(p_2) \) (resp. \( \overline{\sigma}_\mu(W^s(p_1)) \) and \( W^u(p_2) \)). Here we can obtain \( \varphi_\mu \) defined by \( \varphi_\mu = \overline{\sigma}_\mu \circ \sigma_\mu \circ \Phi \) as in Fig. 10.

Case (A)-(a)-1: \( p_1 \) is part of non-trivial basic set. Non-trivial basic sets are obtained from the horseshoes by modification of their stable manifolds and unstables manifolds. We can obtain \( \varphi_\mu \) as in Fig. 11.

Case (C)-(c)-1: This example is given in the projective plane \( P^2 = D^2 \cup M \) where the boundary of \( D^2 \) is identified with the boundary of the M"obius band \( M \) as in Fig. 12.

Case (C)-(c)-2: We can realize this case by the use of a "type-3 horseshoe" [Palis & Takens, 1987] in the projective plane as in Fig. 13.

\[ (2) \quad (+, +)(- -), (- -)(+ +), (- -) \]

When the eigenvalues of \( d\varphi(p_1) \) are negative, we can take two possibilities of the sides of tangencies in the neighborhood \( p_1 \) (we may assume that \( i = 2 \)) as in Fig. 14.

The case (b') cannot occur because of "premature tangencies". For \( (+, +)(+, -), (-) \), we only have the cases (A)-(a') and (C)-(a').
We consider the two modes of connection shown in Fig.15 for the case (A)-(a'):

Case (A)-(a')-1 can only be observed if the expanding eigenvalue is dominating, because otherwise 'premature creation of horseshoes' would occur. Therefore it can occur when \( p_2 \) is an isolated point in \( \tilde{\Omega}(\varphi) \) whether \( p_1 \) is part of a non-trivial basic set or not. \( M \) can be orientable or non-orientable in this case.

Case (A)-(a')-2 Here \( p_2 \) is part of a non-trivial basic set to avoid 'infinitely many circles'. This must be in the upper right quadrant. However this quadrant is not allowed. So this case cannot occur.

The above reasoning applied to the cases for (C)-(a') implies that the only possible case is the one shown in Fig. 16.

On any surface \( M \) case (C)-(a') can occur if the expanding eigenvalue is dominating and \( p_1 \) and \( p_2 \) are not part of a non-trivial basic set.

When the eigenvalues of both \( d\varphi(p_1) \) and \( d\varphi(p_2) \) are negative, heteroclinic \( \Omega \)-explosion can not occur because, as in the case of homoclinic \( \Omega \)-explosion\(^1\), premature creation of horseshoes occurs whether the expanding eigenvalue is dominating or not.

We show an example of the Case (A)-(a')-1 in Fig. 17.

We can also obtain an example of the Case (C)-(a') by slightly changing Fig. 17 so that \( p_1 \) is not part of a non-trivial basic set.

(3) The case \((+, -)_1(+, -)_2\)

In this case there are two possible combinations of the sides of tangencies as in Fig. 19. The case \((b')\) cannot occur because of 'premature tangencies'. So we may only consider the connection of \((A')-(a')\) as in Fig. 20.

Case \((A')-(a')_1\) can only occur if expanding eigenvalue is dominating and \( p_1 \) is isolated in \( \tilde{\Omega}(\varphi) \). It can occur in any surface.

Case \((A')-(a')_2\) can occur that \( M \) is non-orientable and if the expanding eigenvalue is dominating. Neither \( p_1 \) or \( p_2 \) can be part of non-trivial basic set.

The other cases can not generate heteroclinic \( \Omega \)-explosion.

We show some examples for the above two cases in Fig.21-22.

The other three cases \((+, -)_1(-, +)_2, (-, +)_1(+, -)_2 \) and \((-, +)_1(-, +)_2\) can be treated essentially in the same way as (3). This completes our classification of the heteroclinic \( \Omega \)-explosion.

One of the theorems in [1] says that there is a set of \( \mu \geq 0 \) for which the nonwandering set of \( \varphi_{\mu} \) is not hyperbolic but that it is very 'small' in following sense.

Let \( \varphi_{\mu} \) be a family with homoclinic \( \Omega \)-explosion at \( \mu = 0 \). If \( d^s(\Lambda)+d^u(\Lambda)<1 \), where \( \Lambda \) is the basic set of \( \varphi_0 \) associated with the homoclinic tangency, then
\[
\lim_{\delta \to 0} \frac{m(\mu \in [0, \delta]: \varphi_\mu \text{ is not persistently hyperbolic})}{\delta} = 0
\]
where \(m\) denotes Lebesgue measure and \(d^s\) (resp. \(d^u\)) is the stable (resp. unstable) limit capacity. They suggest that the same theory will hold for heteroclinic \(\Omega\)-explosions. We are now trying to it and apply the theorem to our classification of heteroclinic \(\Omega\)-explosions.

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