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Periodic solutions of a singular Hamiltonian system of 2-body type

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0. Introduction and results

In this short note, we study the existence of periodic solutions of a Hamiltonian system

$$\ddot{q} + \nabla V(q,t) = 0, \qquad (HS)$$

where $q = (q_1, \cdots, q_N) \in \mathbf{R}^N$ $(N \geq 3)$ and $V(q,t): \mathbf{R}^N \times \mathbf{R} \to \mathbf{R}$ is a given potential. We deal with the case where a potential has a singularity and is related to 2-body problem.

More precisely, we assume $V(q,t)$ satisfies

(V1) $V(q,t) \in C^2((\mathbf{R}^N \setminus \{0\}) \times \mathbf{R}, \mathbf{R})$ is T-periodic in t;

(V2) $V(q,t) < 0$ and $V(q,t)$, $\nabla V(q,t) \to 0$ as $|q| \to \infty$ uniformly in t;

(V3) $V(q,t)$ is of a form:

$$V(q,t) = -\frac{1}{|q|^\alpha} + W(q,t),$$

where $\alpha > 0$ and $W(q,t) \in C^2((\mathbf{R}^N \setminus \{0\}) \times \mathbf{R}, \mathbf{R})$ satisfies

$$|q|^\alpha W(q,t), \ |q|^{\alpha+1} \nabla W(q,t), \ |q|^{\alpha+2} \nabla^2 W(q,t),$$
$$|q|^\alpha W_t(q,t) \to 0 \qquad \text{as } q \to 0 \text{ uniformly in } t.$$

We consider the following two problems:

(i) *Prescribed Period Problem* (PP): For a given $T > 0$, we study the existence of T-periodic solutions of (HS), i.e., solutions of (HS) such that

$$q(t+T) = q(t) \qquad \text{for all } t. \qquad (HS.P)$$

(ii) *Prescribed Energy Problem* (PE): Assume V is independent of t. For a given $H \in \mathbf{R}$, we study the existence of periodic solutions of (HS) such that

$$\frac{1}{2} |\dot{q}(t)|^2 + V(q(t)) = H \qquad \text{for all } t. \qquad (HS.E)$$

(Here we do not fix the period of $q(t)$.)

We study via variational methods these problems. Recently it is observed that the order α of the singularity of $V(q,t)$ at $q = 0$ plays an important role for the existence of periodic solutions for both of problems. We consider the following cases separately; (i) the *strong force* case $\alpha \geq 2$ for (PP) and $\alpha > 2$ for (PE) (ii) the *weak force* case $\alpha \in (0,2)$.

For the Prescribed Period problem (PP), we use the following variational formulation. Let $E = \{q \in H^1_{loc}(\mathbf{R}, \mathbf{R}^N);\ q(t)$ is T-periodic in $t\}$ is a space of T-periodic functions with norm $\|q\|_E^2 = \int_0^T [\mid \dot{q}(t) \mid^2 + \mid q(t) \mid^2]\, dt$ and set

$$\Lambda = \{q \in E;\ q(t) \neq 0 \text{ for all } t\}.$$

We define the functional $I(q) : \Lambda \to \mathbf{R}$ by

$$I(q) = \int_0^T [\frac{1}{2} \mid \dot{q} \mid^2 - V(q(t),t)]\, dt.$$

Then there is one-to-one correspondence between critical points $q \in \Lambda$ of $I(q)$ and T-periodic solutions of (HS), (HS.P). Therefore we try to find critical points of $I(q)$.

If (V1)–(V3) holds and $\alpha \geq 2$, more generally, under the conditions of (V2)–(V3) and

(V1') $V(q,t) \in C^1((\mathbf{R}^N \setminus \{0\}) \times \mathbf{R}, \mathbf{R})$ is T-periodic in t

and the following strong force condition (SF) of Gordon [**Go**]:

(SF) there is a neighborhood Ω of 0 and $U(q) \in C^1(\Omega \setminus \{0\}, \mathbf{R})$ such that

$$U(q) \to \infty, \qquad q \to 0,$$
$$-V(q,t) \geq \mid \nabla U(q) \mid^2 \qquad \text{for all } q \in \Omega \setminus \{0\} \text{ and } t,$$

we can see the functional $I(q)$ satisfies the Palais-Smale condition and we can apply minimax methods to obtain critical points of $I(q)$. We refer to [**BR**, **Gr**, **AC1**]. Our main purpose is to study the weak force case $\alpha \in (0,2)$. We remark that the Palais-Smale condition does not hold in this case. Our result is as follows:

Theorem 0.1 ([**T2**]). *Suppose (V1)–(V3) and $\alpha \in (1,2)$. Then the prescribed period problem (HS), (HS.P) possesses at least one periodic solution.*

For the Prescribed energy problem (PE), we can expect the existence of periodic solutions only under the situations

(i) $H > 0$ if $\alpha > 2$, or

(i) $H < 0$ if $\alpha \in (0,2)$.

Actually, if $V(q) = -\dfrac{1}{\mid q \mid^\alpha}$, we can easily see that periodic solutions of (HS), (HS.E) exist if and only if (i) or (ii) holds. In the strong force case $\alpha > 2$, the Palais-Smale condition holds under additional assumptions and we refer to Ambrosetti and Coti Zelati [**AC2**] for the existence result. We study the case $\alpha \in (0,2)$. Here we assume

(V4) there is $\overline{\alpha} \in (0, \alpha]$ such that

$$\nabla V(q)q \geq -\overline{\alpha} V(q) \quad \text{for all } q \in \mathbf{R}^N \setminus \{0\}$$

in addition to (V1)–(V3).

Theorem 0.2 ([**T3**]). *Suppose V is independent of t, $H < 0$ and (V1)–(V4). Moreover assume $\alpha \in (1,2)$ if $N \geq 4$ and $\alpha \in (4/3, 2)$ if $N = 3$. Then the prescribed energy problem (HS), (HS.E) possesses at least one periodic solution.*

We remark that in case of weak force the existence of *generalized solutions*, which may enter the singularity 0, is obtained by [**BR**] for the prescribed period problem (PP) and by [**AC2**] for the prescribed energy problem (PE). We also remark the result very closely related to Theorem 0.1 is obtained by Coti Zelati and Serra [**CS**] independently.

In what follows, we sketch outline of the proof of Theorem 0.1. The proof of Theorem 0.2 is done essentially in a same way (but more complicated) to Theorem 0.1 and we refer to [**T3**].

1. Perturbed functionals

We take the following approach, which is used by [**BR**] first time.

1° First we introduce a perturbed potential $V_\epsilon(q,t) = V(q,t) - \frac{\epsilon}{|q|^2}$. The corresponding functional

$$I_\epsilon(q) = \int_0^T [\frac{1}{2} \mid q \mid^2 - V_\epsilon(q,t)]\, dt$$
$$= \int_0^T [\frac{1}{2} \mid q \mid^2 - V(q,t) + \frac{\epsilon}{\mid q \mid^2}]\, dt$$

satisfies a variant of the Palais-Smale condition and we can apply a minimax method of [**BR**] to get approximate solution $q_\epsilon(t)$ for each $\epsilon \in (0,1]$.

2° Second we try to pass to the limit as $\epsilon \to 0$ and we try to obtain a solution as a limit of $q_\epsilon(t)$

More precisely, we use the following minimax method; we set

$$\Gamma = \{\gamma \in C(S^{N-2}, \Lambda);\ \deg \widetilde{\gamma} \neq 0\} \tag{1.1}$$

where $\widetilde{\gamma} : S^1 \times S^{N-2} \simeq ([0,T]/\{0,\, T\}) \times S^{N-2} \to S^{N-1}$ is defined by

$$\widetilde{\gamma}(t,x) = \frac{\gamma(x)(t)}{\mid \gamma(x)(t) \mid}$$

and $\deg \widetilde{\gamma}$ denote the Brower degree of $\widetilde{\gamma}$. We define

$$b_\epsilon = \inf_{\gamma \in \Gamma} \max_{x \in S^{N-2}} I_\epsilon(\gamma(x)). \tag{1.2}$$

Then we have

Proposition 1.1 ([**BR**]). *For any $\epsilon \in (0,1]$, b_ϵ is a critical value of $I_\epsilon(q)$. That is, there is a critical point $q_\epsilon(t)$ of $I_\epsilon(q)$ such that $I_\epsilon(q_\epsilon) = b_\epsilon$. Moreover, there are constants M, $m > 0$ independent of $\epsilon \in (0,1]$ such that*

$$m \le b_\epsilon \le M \qquad \text{for all } \epsilon \in (0,1]. \tag{1.3}$$

∎

Using the uniform estimate (1.3), we can get

Proposition 1.2 ([**BR**]). *There is a constant $C > 0$ independent of $\epsilon \in (0,1]$ such that*

$$\|q_\epsilon\|_E \le C \qquad \text{for all } \epsilon \in (0,1].$$

∎

Therefore we can choose a subsequence — still we denote by $\epsilon \to 0$ — such that $q_\epsilon \to q_0 \in E$ weakly in E and strongly in L^∞. If $q_0(t) \ne 0$ for all t, in other words, if $q_0 \in \Lambda$, we can easily see $q_0(t)$ is a periodic solution of (HS), (HS.P). The difficulty is to prove $q_0 \in \Lambda$.

Even if $q_0 \notin \Lambda$, we can see

(i) Set $D = \{t;\, q_0(t) = 0\}$. Then $\operatorname{meas} D = 0$;

(ii) $q_0(t) \in C^2(\mathbf{R}\setminus D, \mathbf{R}^N) \cap C(\mathbf{R}, \mathbf{R}^N)$;

(iii) $q_0(t)$ satisfies (HS) in $\mathbf{R}\setminus D$.

Bahri and Rabinowitz [**BR**] called such a limit function $q_0(t)$ *generalized solution* of (HS), (HS.P). They constructed generalized solutions under the conditions (V1'), (V2) and

(V3') $V(q,t) \to -\infty$ as $q \to 0$ uniformly in t.

To prove $q_0(t)$ does not enter the singularity 0, we use a combination of a re-scaling argument and an estimate of Morse indices.

2. Re-scaling argument

Suppose $q_0(t)$ enters the singularity 0 at $t_0 \in (0,T]$, i.e., $q_0(t_0) = 0$. Then there is a sequence $t_\epsilon \in (0,T]$ such that

1° $t_\epsilon \to t_0$;

2° $|\, q_\epsilon(t) \,|$ takes its local minimum at t_ϵ.

<u>Case 1:</u> First we study the behavior of $q_\epsilon(t)$ near the singularity 0 more precisely via a re-scaling argument. We set

$$\delta_\epsilon = |\, q_\epsilon(t_\epsilon) \,|,$$
$$x_\epsilon(s) = \delta_\epsilon^{-1} q_\epsilon(\delta_\epsilon^{(\alpha+2)/2} s + t_\epsilon).$$

Then $x_\epsilon(s)$ satisfies $|x_\epsilon(0)|=1$ and

$$\ddot{x}_\epsilon + \frac{\alpha x_\epsilon}{|x_\epsilon|^{\alpha+2}} + \delta_\epsilon^{\alpha+1}\nabla W(\delta_\epsilon x_\epsilon(s), \delta_\epsilon^{(\alpha+2)/2}s + t_\epsilon) + \frac{2\epsilon}{\delta_\epsilon^{2-\alpha}}\frac{x_\epsilon}{|x_\epsilon|^4} = 0.$$

We study the behavior of $x_\epsilon(s)$ instead of $q_\epsilon(t)$.

After taking a suitable subsequence — still we denote by ϵ —, we may assume that

$$d = \lim_{\epsilon\to 0}\frac{\epsilon}{\delta_\epsilon^{2-\alpha}} \in [0,\infty] \tag{2.1}$$

exists. We consider the following two cases separately.

Case 1: $d<\infty$;

Case 2: $d=\infty$.

Case 1: First we deal with Case 1.

Proposition 2.1. *Suppose $d<\infty$. After taking a subsequence — still denoted by ϵ —, $x_\epsilon(s)$ converges to a function $y_{\alpha,d}(s)$ in $C^2_{loc}(\mathbf{R},\mathbf{R}^N)$, where $y_{\alpha,d}(s)$ is the solution of*

$$\begin{aligned} &\ddot{y} + \frac{\alpha y}{|y|^{\alpha+2}} + \frac{2dy}{|y|^4} = 0, \quad \text{in } \mathbf{R}, \\ &y(0) = e_1, \\ &\dot{y}(0) = \sqrt{2(1+d)}e_2. \end{aligned} \tag{2.2}$$

Here, $e_1, e_2, \cdots, e_N \in \mathbf{R}^N$ are vectors satisfying $e_i\cdot e_j = \delta_{ij}$. ∎

Case 2: In this case, we introduce another re-scaled function

$$z_\epsilon(s) = \delta_\epsilon^{-1} q_\epsilon(\epsilon^{-1/2}\delta_\epsilon^2 s + t_\epsilon).$$

Then $z_\epsilon(s)$ satisfies

$$\ddot{z}_\epsilon + \frac{\alpha\delta_\epsilon^{2-\alpha}}{\epsilon}\frac{z_\epsilon}{|z_\epsilon|^{\alpha+2}} + \frac{\alpha\delta_\epsilon^{2-\alpha}}{\epsilon}\delta_\epsilon^{\alpha+1}\nabla W(\delta_\epsilon z_\epsilon, \epsilon^{-1/2}\delta_\epsilon^2 s + t_\epsilon) + \frac{2z_\epsilon}{|z_\epsilon|^4} = 0.$$

We have

Proposition 2.2. *Suppose $d=\infty$. Then, after taking a subsequence — still denoted by ϵ —, we have*

$$z_\epsilon(s) \to z_0(s) = e_1\cos\sqrt{2}s + e_1\sin\sqrt{2}s \qquad \text{in } C^2_{loc}(\mathbf{R},\mathbf{R}^N).$$

Here, $e_1, e_2, \cdots, e_N \in \mathbf{R}^N$ are vectors satisfying $e_i \cdot e_j = \delta_{ij}$. ∎

We remark $z_0(s)$ is a solution of

$$\ddot{z} + \frac{2z}{|z|^4} = 0, \quad \text{in } \mathbf{R},$$
$$z(0) = e_1,$$
$$\dot{z}(0) = \sqrt{2}e_2.$$

3. Estimates of Morse index

Using the propositions 2.1 and 2.2, we have the following estimate of Morse indices.

Proposition 3.1. *Suppose $q_0(t)$ enters the singularity 0 and set*

$$\nu = \#\{t \in (0,t];\ q_0(t) = 0\}.$$

Then

$$\liminf_{\epsilon \to 0} \operatorname{index} I''_\epsilon(q_\epsilon) \geq (N-2)i(\alpha)\nu \tag{3.1}$$

where

$$i(\alpha) = \max\{k \in \mathbf{N};\ k < \frac{2}{2-\alpha}\}.$$ ∎

Before we sketch the proof of Proposition 3.1, we give a proof of Theorem 0.1.

Proof of Theorem 0.1. First we remark that the following estimate of Morse index follows from the minimax characterization (1.1)–(1.2) of b_ϵ.

Proposition 3.2 (c.f.[**BL, LS, T1**]). *$q_\epsilon(t) \in \Lambda$ satisfies*

$$\operatorname{index} I''_\epsilon(q_\epsilon) \leq N-2 \qquad \textit{for all } \epsilon \in (0,1]. \tag{3.2}$$

∎

Comparing (3.1) and (3.2), we have

$$i(\alpha)\nu \leq 1. \tag{3.3}$$

Since $i(\alpha) \geq 2$ for $\alpha \in (1,2)$ and $i(\alpha) = 1$ for $\alpha \in (0,1]$, we find

$$\nu = 0, \qquad \text{if } \alpha \in (1,2),$$
$$\nu \leq 1, \qquad \text{if } \alpha \in (0,1].$$

Therefore in case $\alpha \in (1,2)$, we obtain $q_0(t) \neq 0$ for all t and it is a classical solution. ∎

Sketch of the proof of Proposition 3.1. Suppose $q_0(t_0) = 0$ and choose $t_\epsilon \in (0, T]$ as above. We deal with only the Case 1: $d < \infty$. The Case 2: $d = 0$ can be treated similarly. For $L > 0$, $\varphi(s) \in H_0^1(-L, L; \mathbf{R})$ and $j = 1, 2, \cdots, N$, we set

$$h_{\epsilon,j}(t) = \delta_\epsilon \varphi(\delta_\epsilon^{-(\alpha+2)/2}(t - t_\epsilon))e_j.$$

After the change of variable, we take a limit as $\epsilon \to 0$ and obtain

$$\begin{aligned}&\delta_\epsilon^{-(2-\alpha)/2} I''_\epsilon(q_\epsilon)(h_{\epsilon,j}, h_{\epsilon,j})\\ &\to \int_{-L}^{L} [\,|\dot{\varphi}|^2 - \frac{\alpha |\varphi|^2}{|y_{\alpha,d}|^{\alpha+2}} + \frac{\alpha(\alpha+2)(y_{\alpha,d}, e_j)^2 |\varphi|^2}{|y_{\alpha,d}|^{\alpha+4}}\\ &\qquad - \frac{2d|\varphi|^2}{|y_{\alpha,d}|^4} + \frac{8d(y_{\alpha,d}, e_j)^2|\varphi|^2}{|y_{\alpha,d}|^6}]\, ds.\end{aligned}$$

Recalling $y_{\alpha,d}(s) \in \operatorname{span}\{e_1, e_2\}$ for all s, we can see

$$\liminf_{\epsilon \to 0} \operatorname{index} I''_\epsilon(q_\epsilon) \geq (N-2)i(\alpha, d) \tag{3.4}$$

where

$$i(\alpha,d) = \sup_{L>0} \begin{pmatrix} \text{the number of negative eigenvalues} \\ \text{of the following eigenvalue problem:} \\ -\ddot{\varphi} - (\dfrac{\alpha}{|y_{\alpha,d}|^{\alpha+2}} + \dfrac{2d}{|y_{\alpha,d}|^4})\varphi = 0, \\ \varphi(L) = \varphi(-L) = 0. \end{pmatrix}.$$

We repeat the above argument at all other $t'_0 \in (0, T]$ such that $q_0(t'_0) = 0$ and we find

$$\liminf_{\epsilon \to 0} \operatorname{index} I''_\epsilon(q_\epsilon) \geq (N-2)i(\alpha)\nu$$

where

$$i(\alpha) = \min_{d \geq 0} i(\alpha, d).$$

Now Proposition 3.1 follows from the following proposition. ∎

Proposition 3.3.

$$i(\alpha, d) = \max\{k \in \mathbf{N};\ k < \frac{2\sqrt{1+d}}{2-\alpha}\}. \tag{3.5}$$

Thus $i(\alpha) = \max\{k \in \mathbf{N};\ k < \frac{2}{2-\alpha}\}$.

Proof. The case $d = 0$ is proved in [T2]. The case $d > 0$ is proved similarly. The key of the proof is the Sturm comparison theorem and the following property of $y_{\alpha,d}(s)$. We use the polar coordinate and write

$$y_{\alpha,d}(s) = r(s)(e_1 \cos\theta(s) + e_2 \sin\theta(s))$$

where $r(s) > 0$ and $\theta(s) \in \mathbf{R}$ with $\theta(0) = 0$. Then we have

(i) $s\dot{r}(s) > 0$ for all $s \neq 0$ and $r(s) \to \infty$ and $s \to \pm\infty$;

(ii) $\dot{\theta}(s) > 0$ for all s;

(iii) $\theta(s) \to \pm\dfrac{2\pi\sqrt{1+d}}{2-\alpha}$ as $s \to \pm\infty$.

∎

4. Remarks

In case $\alpha \in (0, 1]$, it seems that the existence of classical periodic solutions is not known. However by (3.3) we can see there is a generalized solution of (HS), (HS.P) that enters at most one time in its period. By (3.4) and (3.5), we also have

$$d \leq (2-\alpha)^2 - 1 \tag{4.1}$$

where d is defined in (2.1).

We get the following additional information under slightly stronger conditions: (V1), (V2) and

(V3") $V(q,t)$ is of a form:

$$V(q,t) = -\frac{1}{|\, q \,|^\alpha} + W(q,t),$$

where $\alpha > 0$ and $W(q,t) \in C^2((\mathbf{R}^N \setminus \{0\}) \times \mathbf{R}, \mathbf{R})$ satisfies

$$|\, q \,|^{\alpha-\rho}\, W(q,t),\ |\, q \,|^{\alpha-\rho+1}\, \nabla W(q,t),\ |\, q \,|^{\alpha-\rho+2}\, \nabla^2 W(q,t),$$
$$|\, q \,|^{\alpha-\rho}\, W_t(q,t) \to 0 \qquad \text{as } q \to 0 \text{ uniformly in } t$$

for some $\rho \in (0, \alpha)$.

We assume $q_0(t)$ is a generalized solution such that $q_0(t_0) = 0$. Beaulieu [**B**] proved that the limits

$$a_\pm = \lim_{t \to t_0 \pm 0} \frac{q_0(t)}{|\, q_0(t) \,|} \in S^{N-1}$$

exist. We have

Theorem 4.1 ([**T4**]). *Assume (V1), (V2), (V3") and let $q_\epsilon(t)$ be a critical point of $I_\epsilon(q)$ which is obtained through a minimax method (1.1)–(1.2). Suppose $q_0(t) = \lim_{\epsilon \to 0} q_\epsilon(t)$ is a generalized solution such that $q_0(t_0) = 0$ and let $a_\pm = \lim_{t \to t_0 \pm 0} \frac{q_0(t)}{|q_0(t)|} \in S^{N-1}$. Then we have*

$$\text{the angle between } a_+ \text{ and } a_- = \frac{2\pi\sqrt{1+d}}{2-\alpha} \quad \text{modulo } 2\pi$$

where $d \in [0, (2-\alpha)^2 - 1]$ is defined in (2.1). ∎

In particular, in case $\alpha = 1$ we have $d = 0$ and $a_+ = a_-$.

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