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Invariant measures for certain multi-dimensional maps

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Abstract
We investigate singular points of the invariant density for a class of multi-dimensional maps with finite range structure. In particular, we concentrate on maps with countably many discontinuity points which do not satisfy Renyi's condition and do not necessarily satisfy the Markov property. Such maps occur from number theory quite naturally. Under some conditions, we show that indifferent periodic points must be singular points of the invariant density.

1 Introduction
We consider multi-dimensional piecewise smooth maps which are almost expanding. These maps generally do not have the Markov property, but they have a similar structure, which we call a "finite range structure" (FRS) and leads to a nice countable state symbolic dynamics [22]. Many examples of such maps come from number theory (see section 4). The maps we study are typically only $C^1$-smooth, and they need not satisfy Renyi's condition (uniformly bounded distortion for all iterates).

In [4], sufficient conditions for the existence of absolutely continuous invariant measures were given for systems with FRS. Before [4], analyses of absolutely continuous invariant measures appealed to Renyi's condition and to the Markov property (e.g. [1],[2],[3],[6],[8],[9],[11],[15],[19],[23],[25]), both of which may fail (with interesting consequences, as we will see) for systems with FRS. If Renyi's condition holds, then the invariant density obtained is bounded. However without this condition, the invariant density of the finite measure may be unbounded (see section 4). We study singularities of the invariant density (see the definition in section 3) and relate them to the existence of non-hyperbolic periodic orbits. We also provide a sufficient condition for the validity of Rohlin's entropy formula.

We now establish some notation and recapitulate some definitions. We say a map $T$ on a bounded domain $X \subset \mathbb{R}^d$ has a "finite range structure" (FRS) if there exists a countable partition $Q = \{X_a\}_{a \in I}$ of $X$ and a collection of finitely many subsets of $X, \{U_0, U_1, \ldots, U_N\}$ such that
1. each $X_a$ is a measurable, connected subset with piecewise smooth boundary and $\text{int} X_a \neq \emptyset$,

2. each $U_k$ has positive Lebesgue measure,

3. for each $X_a, T|_{X_a}$ is injective, of class $C^1$ with $\det DT|_{X_a} \neq 0$,

4. if $\text{int} X_{a_1} \cap \text{int}(T^{-1}X_{a_2}) \cap \ldots \cap \text{int}(T^{-(n-1)}X_{a_n}) \neq \emptyset$, let $X_{a_1\ldots a_n} = X_{a_1} \cap T^{-1}X_{a_2} \cap \ldots \cap T^{-(n-1)}X_{a_n}$. Then $T^n X_{a_1\ldots a_n} = U_k$ for some $k \in \{0, 1, \ldots, N\}$.

Remark A Here a partition means a collection of disjoint sets. In 2, $U_k$ can intersect $U_j$ for $j \neq k$, and in particular one of the $U_k$ can be equal to $X$. In 3, $\det DT|_{\partial X_a} = 0$ is possible. When we say a function is $C^1$ on $X_a$, we mean it agrees on $X_a$ with a $C^1$ function defined on a neighbourhood of $X_a$ in $\mathbb{R}^d$.

If there exists a constant $C(>1)$ such that

$$\frac{\sup_{x \in X_{a_1\ldots a_n}} |\det DT^n(x)|}{\inf_{x \in X_{a_1\ldots a_n}} |\det DT^n(x)|} < C$$

for all $n > 0$ and all $X_{a_1\ldots a_n}$, then we say $T$ satisfies Renyi's condition. If $\text{int}(X_a \cap TX_b) \neq \emptyset$ implies $X_a \subset TX_b$, then we say $T$ has the Markov property.

In section 3, we explain that indifferent periodic points must be singular points of the invariant density, and we discuss the characterization of non-singular points. In section 4, we apply our theorems to examples on which precise discussions are shown in [21]. In section 5, we consider Rohlin's entropy formula. Proofs of our results of section 3 and of section 5 are given in [21].

## 2 Notation and preliminary results

We call $X_{a_1\ldots a_n}$ a cylinder of rank $n$ with respect to $T$. $\mathcal{L}^n$ denotes the family of all cylinders $X_{a_1\ldots a_n}$ of rank $n$ and $\mathcal{L} = \bigcup_{n=1}^{\infty} \mathcal{L}^n$. For constant $C > 1$, $X_{a_1\ldots a_n}$ is called a $R(C,T)$-cylinder of rank $n$ if

$$\frac{\sup_{x \in X_{a_1\ldots a_n}} |\det DT^n(x)|}{\inf_{x \in X_{a_1\ldots a_n}} |\det DT^n(x)|} < C.$$ 

$R(C,T)$ denotes the set of all $R(C,T)$-cylinders. We say that a cylinder $X_{a_1\ldots a_n}$ satisfies the local Renyi condition for $C$ if for all cylinders $X_{b_1\ldots b_m}$ such that $X_{b_1\ldots b_m a_1\ldots a_n} \in \mathcal{L}^{m+n}, X_{b_1\ldots b_m a_1\ldots a_n} \in R(C,T)$. We say that $T$ satisfies the local Renyi condition if there exists a constant $C(>1)$ such that $R(C,T)$ is not empty and for all $X_{a_1\ldots a_n} \in R(C,T)$ satisfies the local Renyi condition for $C$. We call the constant $C$ a local Renyi constant. For $z \in X$ and $n \in \mathbb{N}$, we define

$$C(n, z) \equiv \frac{\sup_{y \in X_{a_1\ldots a_n}} |\det DT^n(y)|}{\inf_{y \in X_{a_1\ldots a_n}} |\det DT^n(y)|}.$$
where $X_{a_1...a_n}(x)$ is the unique cylinder of rank $n$ containing $x$. As $C(n,x)$ is constant on $X_{a_1...a_n}(x)$, we sometimes denote by $C(a_1...a_n)$ the constant. For point $x \in X$, if there exists a constant $C > 1$ such that for $\forall n > 0$, $\exists i_n(>n)$ so that $C(i_n,x) < C$, we call the point $x$ a limit point of $R(C,T)$-cylinders. For $C > 1$, we define

$$D_i^{(C)} \equiv \{X_{d_1...d_i} \in \mathcal{L}^i; X_{d_1...d_i} \not\in R(C,T) \text{ for all } j = 1, 2, ..., i\}$$

$$B_i^{(C)} \equiv \bigcup_{X_{b_1...b_i} \in D_i^{(C)}} X_{b_1...b_i}$$

and

$$\sigma_{n=0}^\infty \lambda(D_n^{(C)}) < \infty$$

where $\lambda$ is the normalized Lebesgue measure.

In particular, for $i = 0$ we define $D_0 = X$. We sometime write $\psi_a$ for $(T|_{X_a})^{-1}$ and $\psi_{a_1...a_n}$ for $(T^n|_{X_{a_1...a_n}})^{-1}$.

**Theorem 2.1** Let $T : X \to X$ have a FRS and satisfy the local Renyi condition, and let $Q = \{X_a\}_{a \in I}$ satisfy the generator condition, i.e., $\sum_{m=0}^\infty T^{-m}Q = \epsilon$ (where $\epsilon$ is the partition into points). Assume that the local Renyi constant $C(>1)$ satisfies the following:

1. (transitivity condition) for all $j \in \{0, 1, ..., N\}$, there exists a cylinder $X_{a_1...a_j} \in R(C,T)$ such that $X_{a_1...a_j} \subset U_j$ and $T^jX_{a_1...a_j} = X$,

2. $\sum_{n=0}^\infty \lambda(D_n^{(C)}) < \infty$, where $\lambda$ is the normalized Lebesgue measure.

Then there exists a finite, ergodic invariant measure $\mu$ which is equivalent to $\lambda$, and with respect to $\mu T$ is exact.

(cf [4],[20]).

**Remark B** If we replace the condition 2 by the weaker condition

$$\lim_{n \to \infty} \lambda(D_n^{(C)}) = 0$$

then we still have an ergodic invariant measure which is equivalent to $\lambda$. This measure need not be finite (although it will be $\sigma$-finite)(see[4]).

**Remark C** The finite measure $\mu$ of Theorem 2.1 does not depend on $C$ (in particular the invariant density of $\mu$ does not depend on $C$) (cf [24]).
3 Singularities of the invariant density

In this section, we assume that $T$ satisfies all assumptions of Theorem 2.1. We say a point $x_0 \in X$ is an indifferent periodic point if there exists a $p > 0$ such that $T^p x_0 = x_0$ and $|\det DT^p(x_0)| = 1$. A point $x \in X$ is called a singular point of a measurable function $f$ if $\forall \epsilon > 0$ the essential supremum of $|f|$ on $B_\epsilon(x)$ is infinite, where $B_\epsilon(x)$ is a $\epsilon$-neighbourhood of $x$.

Theorem 3.1 Suppose $x_0$ is an indifferent periodic point, and

(M) $x_0 \notin (\bigcup_{j=0}^{N} \partial U_j \setminus \bigcup_{a \in I} \partial X_a)$.

Then $x_0$ is a singular point of the invariant density of $\mu$.

Remark D If $T$ satisfies the Markov property, then $\partial U_j \subset \bigcup_{a \in I} \partial X_a$ for every $j$ and so the condition (M) is automatically satisfied.

The following results are needed to prove Theorem 3.1.

Proposition 3.1 If $x_0$ is an indifferent periodic point, then for the local Renyi constant $C$, $x_0 \in \bigcap_{n=0}^{\infty} D_n^{(C)}$.

Lemma 3.1 If $x_0$ is an indifferent periodic point, then $\lim_{n \to \infty} C(n, x_0) = \infty$.

Lemma 3.2 If $x_0$ is any point such that $\lim_{n \to \infty} C(n, x_0) = \infty$, then for the local Renyi constant $C$, there exists a number $N_0(C)$ such that $T^n x_0 \in \bigcap_{n=0}^{\infty} D_n^{(C)}$ for all $n \geq N_0(C)$. In particular, if $x_0$ is a periodic point, then $x_0 \in \bigcap_{n=0}^{\infty} D_n^{(C)}$.

Let $S$ be the set of all singular points of the invariant density of $\mu$, and $P$ be the set of all periodic points for $T$. We remark that a point $x_0$ in $P$ with period $p$ satisfies $|\det DT^p(x_0)| \geq 1$. In fact, the generator condition does not allow the case that $|\det DT^p(x_0)| < 1$. Now we ask, is the converse of Theorem 3.1 true? Some examples in the next section show that the answer is no! A singular point of the invariant density is not necessarily a periodic point (Examples 7,8). A singular point of the invariant density which is periodic is not necessarily an indifferent periodic point(Examples 5,6,7,8). In general, it is unclear how to characterize singular points of the invariant density. For limit points of $R(C,T)$ -cylinders, we can obtain the following answer:

Theorem 3.2 A limit point $x$ of $R(C,T)$-cylinders is a singular point of the invariant density of $\mu$ if and only if

$$\sum_{n=0}^{\infty} \sum_{X_{d_1} \ldots d_a \in D_a} |\det D\psi_{d_1 \ldots d_a}(x)| = \infty$$
For some class of one-dimensional piecewise $C^2$-smooth Bernoulli maps, it is possible to characterize completely the singular points of the invariant density by indifferent fixed points, that is, Renyi's condition holds iff there is no indifferent fixed point (Thaler [18]). However, under $C^2$-smoothness, the existence of an indifferent fixed point leads to an infinite ergodic absolutely continuous invariant measure([17]). On the other hand, in our setting the invariant measure which we obtain is a finite measure. In fact, our one-dimensional example in section 4 which satisfies all assumptions of Theorem 2.1 does not have $C^2$-smoothness on any neighborhood of an indifferent fixed point(Examples 3,7). In all multi-dimensional examples of the next section, at a singular point $x_0$ of the invariant density the derivative has at least one eigenvalue of modulus one, and $\sup_{n>0} C(n, x_0)$ is infinite. So we can ask, for example

**Question 1** Can a repelling periodic point (i.e., all eigenvalues of the derivative have modulus strictly greater than one) of a piecewise $C^1$ map satisfying conditions of Theorem 2.1, be a singular point of the invariant density?

The following result is a possible tool for approaching Question 1 when the domain of $T$ is one-dimensional.

**Corollary 3.1 (One dimensional case)** Let $T^p x_0 = x_0, |(T^p)'(x_0)| > 1$ and assume that there exists $\epsilon > 0$ such that $T$ restricted to a $\epsilon$-neighborhood of $x_0$ is of class $C^2$. Then $\sup_{n>0} C(n, x_0) \equiv C_0 < \infty$. If $C_0 \leq C$, then conditions of Theorem 3.2 are satisfied, and so

$$\sum_{n=0}^{\infty} \sum_{x_{d_1...d_n} \in \mathcal{D}_n} |\det D\psi_{d_1...d_n}(x_0)| < \infty$$

iff $x_0$ is a non-singular point of the invariant density of $\mu$.

We use Corollary 3.1 for analyze Example 3.

**Remark E** A possible class of maps satisfying the conditions of Corollary 3.1 are piecewise expanding maps which have smoothness of the class $1 + \alpha$ at the endpoints. Such behavior occurs for Lorenz-type maps.

**Question 2** For a repelling periodic point, is $\sup_{n>0} C(n, x_0)$ finite? (In the case of direct product of one-dimensional maps, there is a partial answer (Corollary 3.1)).

**Question 3** When $\mathcal{P}$ consists of repelling periodic points, can the invariant density have singular points?
4 Examples and applications

In this section, we will first show some examples which satisfy the assumptions of Theorem 3.1, and thus have ergodic finite invariant measures with unbounded densities whose singular points are indifferent periodic points. Examples 1, 2, and 4 are number theoretical two-dimensional maps. Example 3 is a one-dimensional map which does not relate to number theory. This is also one of examples for which we can verify the condition of Theorem 3.2 to use this result. As we mentioned in section 3, this example suggests that the appearance of indifferent periodic points does not necessarily lead to infiniteness of our invariant measure without (piece-wise) $C^2$-smoothness. In the case of two-dimensional maps, even if $C^2$-smoothness is valid, from Example 4 we can say the same fact as in the case of one-dimensional maps. Next we will show some two-dimensional examples which suggest that the singular points of the invariant densities are not necessarily indifferent periodic points. All of examples 5, 6, and 7 have singularities of invariant densities at periodic points with period $p$ which are not indifferent, but at these points the derivative of $p$-th powers have at least one eigenvalue of modulus one. The last example, 8 shows that singular points of the invariant density are not necessarily periodic.

Example 1 (A skew product two-dimensional map which is related to Diophantine approximation in inhomogeneous linear class)

Let $X = \{(x_1, x_2) \in R^2 : 0 \leq x_2 \leq 1, -x_2 \leq x_1 \leq -x_2 + 1\}$. Define $T$ on $X$ by

\[ T(x_1, x_2) = (1/x_1 - [(1-x_2)/x_1] - [-(x_2/x_1)], -[-(x_2/x_1)] - (x_2/x_1)). \]

The invariant density of the finite invariant measure of $T$ is:

\[ h(x_1, x_2) = \frac{1}{2\log 2(1-x_1^2)} \]

([5]), so the singular points of the density $h(x_1, x_2)$ are $(1, 0)$ and $(-1, 1)$. These points are periodic points with period 2 and are indifferent.

Remark F In this example, the local Renyi constant is unique. In fact, under the assumptions of Theorem 3.1, the appearance of the indifferent periodic points gives us to the following condition for the local Renyi constant $C$: $C \leq \inf\{C(n, z) : n > 0, z$ is an indifferent periodic point $\}$.

Example 2 (A real two-dimensional map which is related to a complex continued fraction expansion)

Let $X = \{z = x_1 \alpha + x_2 \overline{\alpha} : -(1/2) \leq x_1, x_2 \leq 1/2\} (\alpha = 1 + i)$ and define $T$ on $X$ by $Tz = 1/z - [1/z]_1$, where $[z]_1$ denotes $[x_1 + 1/2] \alpha + [x_2 + 1/2] \overline{\alpha}$ for a complex
number \( z = x_1 \alpha + x_2 \overline{\alpha} \). Let the index set \( I \) be; \( I = \{ n\alpha + m\overline{\alpha} : m, n \in \mathbb{Z} \} \setminus \{0\} \). \( T \)
induce a continued fraction expansion of \( z \in X \),
\[
z = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ldots}}}.
\]
where each \( a_i \) is contained in \( I \). Figure (1). \( T \) has indifferent periodic points \( \pm 2, \pm 2i \), and the invariant density which was obtained by S. Tanaka([16]) has singularities at these points and no others. (Figure(2)). He showed by his own method that finiteness and ergodicity of the invariant measure. On the other hand, \( T \) satisfies all assumptions of Theorem 2.1 and 3.1. Further dynamical properties (for example weak Bernoulli property) are discussed in [20]. So about this example, we omit further details.

Example 3 (A one-parameter family of maps on the interval \([0, 1]\))

Let \( X = [0, 1] \) and for \( \alpha \) with \( 0 < \alpha < 1 \) define
\[
f_{\alpha}(x) = \begin{cases} 
\frac{x}{(1-x\alpha)^{1/\alpha}} & \text{on } X_0 = [0, (1/2)^{1/\alpha}) \\
\frac{1}{(1/2)^{1/\alpha} - 1} + \frac{1}{1 -(1/2)^{1/\alpha}} & \text{on } X_1 = [(1/2)^{1/\alpha}, 1] 
\end{cases}
\]

In the case of a direct product of one-dimensional maps, if one of the Invariant densities of the maps gives an infinite invariant measure, the direct product has an infinite invariant measure, too. The next example is defined by using a one-dimensional map with an infinite invariant measure, but the map itself has a finite invariant measure with unbounded density.

Example 4 (Two-dimensional map which is related to Brun’s algorithm, “Brun map”)

First, we define a one-dimensional map \( T_1 : [0, 1] \to [0, 1] \) by
\[
T_1(x) = \begin{cases} 
\frac{x}{1-x} & \text{on } X_0 = [0, 1/2) \\
\frac{1}{x} - 1 & \text{on } X_1 = [1/2, 1] 
\end{cases}
\]
(Figure (3)).

Now, we define Brun’s map \( T \). Let \( X = \{(x_1, x_2) \in \mathbb{R}^2; \leq x_2 \leq x_1 \leq 1\} \), and let for \( i \in \{0, 1, 2\}, X_i = \{(x_1, x_2) \in X; x_i + x_1 \geq 1 \geq x_{i+1} + x_1\} \), where we put \( x_0 = 1 \) and \( x_3 = 0 \). \( T \) is defined by
\[
T(x_1, x_2) = \begin{cases} 
(T_1(x_1), \frac{x_2}{1-x_1}) & \text{on } X_0 \\
(T_1(x_1), \frac{x_2}{x_1}) & \text{on } X_1 \\
(\frac{x_1}{x_1}, T_1(x_1)) & \text{on } X_2 
\end{cases}
\]
Note that $T(0,0) = (0,0)$ and $|\det DT(0,0)| = 1$, so $(0,0)$ is the indifferent fixed point for $T$, and $T$ is a piecewise $C^2$-map. This map is one of examples of Markovian MCF algorithm with the weak convergence property (Lagarias [7]), and Schweiger determined the unique absolutely continuous invariant measure which is ergodic. This invariant measure is finite and the invariant density $h(x_1, x_2)$ is the following([12], [13],[14]);

$$h(x_1, x_2) = \frac{1}{2x_1(1 + x_2)}.$$  

So $(0,0)$ is the only singular point of $h(x_1, x_2)$. In fact, $T$ satisfies all assumptions of Theorem 2.1 and 3.1.

**Example 5 (A modification of Brun’s map with a finite partition)**

Now we define a modification of Brun’s map such that two pieces of the partition touch at the fixed point 0. The domain $X$ is the same as in Example 4. We devide $X_0$ into two pieces $X_\alpha = \{(x_1, x_2) \in X_0; x_1 \geq 2x_2\}$ and $X_\beta = \{(x_1, x_2) \in X_0; x_1 < 2x_2\}$, and define $T^*$ on the se pieces by

$$T^*(x_1, x_2) = \begin{cases} (T_1(x_1), \frac{2x_1}{1-x_1}) & \text{on } X_\alpha \\ (T_1(x_1), \frac{2x_1-x_1}{1-x_1}) & \text{on } X_\beta \end{cases}$$

On $X_1$ and $X_2$, $T^*$ is defined as in Example 4. This changing in the definition of $T^*$ allows us to have the non-indifferent fixed point 0. In fact $|\det DT^*(0,0)| = 2$.

We remark that the dynamical properties of $T^*$ in which we are interested are not changed essentially[21], and still the fixed point 0 is a singular point of the invariant density.

**Example 6 (A modification of Brun’s map with a countable partition)**

Let the domain $X$ be as in Example 4, and let devide $X_0$ into countably many pieces,

$$X_{\alpha_k} = \{(x_1, x_2) \in X_0; \frac{2x_1}{k+2} \leq x_2 < \frac{2x_1}{k+1}\}(k > 0).$$

On each $X_{\alpha_k}$, define $T^{**}(x_1, x_2) = (\frac{x_1}{1-x_1}, \frac{(k+1)x_2-kx_2}{1-x_1})$. On $X_1$ and $X_2$, the definition of $T^{**}$ are the same as in Example 4.

**Example 7 (A product of one-dimensional maps with a countable partition)**

Let $X = [0,1]^2$, and define $T$ on $X$ by

$$T(x_1, x_2) = (T_1(x_1), T_2(x_2)),$$

where

$$T_1(x_1) = \frac{x_1(2 - \sqrt{x_1})^2}{4(1 - \sqrt{x_1})^2} - \left[\frac{x_1(2 - \sqrt{x_1})^2}{4(1 - \sqrt{x_1})^2}\right].$$
and $T_2(z_2) = 2z_2 - [2z_2]$. (Here $\lfloor \cdot \rfloor$ denotes the Gauss part of $z$.) $T(0, 0) = (0, 0), |\det DT(0, 0)| = 2$, so $(0, 0)$ is a non-indifferent fixed point. First we remark the properties of $T_1$. $T_1(0) = 0, (T_1)'(0) = 1$ and $T_1''(0) = \infty$. $0$ is an indifferent fixed point of $T^1$, and on any neighbourhood of 0, $T_1$ is only of class $C^1$, not of class $C^2$. In fact $T_1$ has a finite invariant measure with an unbounded density $h(z_1) = 1/\sqrt{21}(\text{Thaler} [17])$. The indifferent fixed point 0 is exactly the singular point of the invariant density $h(z_1)$. Since the invariant measure of $T_2$ is the Lebesgue measure, the invariant density $h(z_1, z_2)$ of $T$ is given by $h(z_1, z_2) = 1/\sqrt{21}$ and this gives us a finite invariant measure. Hence this example shows that a periodic point which is a singular point of the invariant density is not necessarily indifferent and singular points of the invariant density are not necessarily periodic.

Example 8 (A product of one-dimensional maps with countably many cylinders in $\mathcal{D}_1$)

Let $X = [0, 1]^2$, and we define a product of the one-dimensional map $f_\alpha(0 < \alpha < 1)$ in Example 3 and the Gauss transformation which is related to the simple continued fraction expansion, that is $T$ is defined by

$$ T(x_1, x_2) = \begin{cases} \left( \frac{x_1}{(1-x_1^{1/\alpha})^{1/\alpha}}, \frac{1}{x_2} \right) & \text{on } X_{(0, l)}(l \in \mathbb{N}) \\ \left( \frac{x_1}{1-(1/2)^{1/\alpha}}, \frac{1}{x_2} \right) & \text{on } X_{(1, l)} \end{cases} $$

where $X_{(0, l)} = \{(x_1, x_2) \in X; 0 \leq x_1 < \frac{1}{2^{l+1}}, \frac{1}{l+1} \leq x_2 \leq \frac{1}{l}\}$ and $X_{(1, l)} = \{(x_1, x_2) \in X; \frac{1}{l} \leq x_1 < 1, \frac{1}{l+1} \leq x_2 < \frac{1}{l}\}$.

$T$ has an invariant density $h_1(x_1)h_2(x_2)$, where $h_1(x_1)$ is the invariant density of $f_\alpha$ and $h_2(x_2)$ is the invariant density of the Gauss transformation,(it is well-known that $h_2(x_2) = \frac{1}{\log 2(1+x_2)}$.) We have already known from Theorem 3.1 that $h_1$ gives a finite invariant measure and at the indifferent fixed point of $f_\alpha; 0$, $h_1$ is unbounded. On the other hand, the Gauss transformation satisfies Renyi's condition,so $h_2(x_2)$ is bounded from above and below. As a result, we can obtain a finite ergodic invariant measure whose density is unbounded on $\{0\} \times [0, 1]$.

5 Rohlin's entropy formula

When Renyi's condition is satisfied Rohlin's entropy formula is true ([13]). In our setting, the invariant density has singular points, however the entropy formula is still true under some conditions

Theorem 5.1 (Rohlin's entropy formula) Let $T$ satisfy all assumptions of Theorem 2.1. Assume further

1. $\log |\det DT(\ )| \in \mathcal{L}^1(X, \lambda)$
2. \( \|D_1 \| < \infty \)

3. there is a constant \( K > 0 \) such that
\[
\sup_{X_a \in B_1} \left( \sum_{n=0}^{\infty} \sum_{X_{d_1 \ldots d_n} \in D_n} \inf_{x \in T^* X_{d_1 \ldots d_n} \cap X_a} |\det D\psi_{d_1 \ldots d_n}(x)| \right) < K
\]

4. there is a number \( l > 0 \) such that \( \sup_n C(n,x) = O(n^l) \).
Then \( h(T) = \int_X \log |\det DT(x)| d\mu(x) \).

Remark G

If \( \frac{d\mu}{d\lambda}|_{B_1} \) is bounded from above, then the condition 3 is valid. We can verify the condition 3 explicitly for our new class of examples (Examples 1,2,3), so we can apply the theorem for these examples (Cf. [20]).

Lemma 5.1 Under the assumptions 1,2, and 3, we have
\[
H(Q) \equiv -\sum_{a \in I} \mu(X_a) \log \mu(X_a) < \infty.
\]

Lemma 5.2 Under the assumptions 1,2, and 3, we have
\[
\log |\det DT(x)| \in L^1(X,\mu).
\]

Lemma 5.3 4 allows us to have
\[
\lim_{n \to \infty} \frac{1}{n} \log \frac{1}{\lambda(X_{a_1 \ldots a_n}(x))} = \lim_{n \to \infty} \frac{1}{n} \log |\det DT^n(x)|.
\]

6 Appendix

Let define for \( C > 1 \),
\[
B_1^{(C)} = \{ X_b \in L^1; X_b \in R(CT) \text{ and } X_b \text{ satisfies the local Renyi condition} \}
\]
\[
D_1 = L^1 \setminus B_1^{(C)}.
\]
\[
B_2^{(C)} = \{ X_{b_1 b_2} \in L^2; X_{b_1} \in D_1^{(C)} \},
\]
\[
D_2^{(C)} = \{ X_{d_1 d_2} \in L^2; X_{d_1}, X_{d_2} \in D_1^{(C)} \},
\]
and inductively define
\[
B_n^{(C)} = \{ X_{b_1 \ldots b_n} \in L^n; X_{b_1 \ldots b_{n-1}} \in D_n^{(C)} \},
\]
\[
B_1^{(C)} = \{ X_{b_1} \in L^\infty; X_{b_1 \ldots b_{n-1}} \in D_n^{(C)} \},
\]
\[ \mathcal{D}_n^{(c)} = \{X_{d_1 \ldots d_n} \in \mathcal{L}^n; X_{d_1 \ldots d_i} \in \mathcal{D}_i^{(c)} \text{ for } i = 1, 2, \ldots n\}. \]

Notice that
\[ B_n^{(c)} \subset \{X_{b_1 \ldots b_n} \in \mathcal{L}^n; X_{b_1 \ldots b_{n-1}} \in \mathcal{D}_{n-1}^{(c)}, X_{b_1 \ldots b_n} \in R(C.T)\}. \]

Under the above new definition of \( \mathcal{D}_n \) and \( B_n \), still we have Theorem 2.1 and Theorem 3.1. Here we show sketch of the proof (see [4]).

Let \( T_R : \bigcup_{i=1}^\infty B_i \to \bigcup_{i=1}^\infty B_i \) be the jump transformation, that is, \( T_Rx = T^jx \) for \( x \in B_i \). The index set of the partition with respect to \( T_R \) is \( J = \bigcup_{n=1}^\infty \{(a_1 \ldots a_n) \in I^n; X_{a_1 \ldots a_n} \in B_n\} \). So each cylinders with respect to \( T_R \) have sequences of symbols in \( J, (a_1 \ldots a_n), a_i \in J \).

**Theorem 6.1 (cf. [4])** Let \( T \) satisfy all assumptions of Theorem 2.1 under the new definitions of \( \mathcal{D}_n \) and \( B_n \). Then \( T_R \) still satisfies all conditions for the existence of a finite ergodic invariant measure with a bounded density, that is the following conditions are valid:

1. (generator condition) the partition for \( T_R \) with the index set \( J \) is a generating partition,
2. (transitivity condition) each \( U_j \) contains a cylinder \( X_{a_1 \ldots a_{\ell_j}} \) such that \( T_R^{\ell_j}X_{a_1 \ldots a_{\ell_j}} \),
3. Renyi’s condition.

For 1, we have to show that the \( \sigma \)-algebra generated by cylinders with respect to \( T_R \) coincides with the one with respect to \( T \). Under the new definition, we have still \( D_n = B_{n+1} \cup D_{n+1} \), and \( D_n = \bigcup_{k=0}^\infty B_{n+k} \) (mod 0), so it is almost the same as in [4] to show that every cylinder with respect to \( T \) is a disjoint union of cylinders with respect to \( T_R \). To show this, we used only the “local Renyi condition”. In fact, if \( X_{a_1 \ldots a_n} \in \mathcal{D}_n \), we can show immediately. If \( X_{a_1 \ldots a_n} \notin \mathcal{D}_n \), then still there is a maximal number \( k_0 \in [1, n] \) such that \( X_{a_1 \ldots a_{k_0}} \) is a cylinder with respect to \( T_R \). If \( X_{a_{k_0+1} \ldots a_n} \notin \mathcal{D}_{n-k_0} \), then there is a \( l \) with \( 0 < l < n \) such that \( X_{a_{k_0+1} \ldots a_l} \in B_{l-(k_0+1)+1} \) and the local Renyi condition allows us to have \( X_{a_1 \ldots a_{l}} \in \mathcal{L}_{T_R} \), i.e., \( X_{a_1 \ldots a_l} \) is a cylinder with respect to \( T_R \). This contradicts to the maximality of \( k_0 \). So \( X_{a_{k_0+1} \ldots a_n} \in \mathcal{D}_{n-k_0} \). Thus \( X_{a_{k_0+1} \ldots a_n} = \bigcup_{k=1}^\infty B_{n-k_0+k} \cap X_{a_{k_0+1} \ldots a_n} \) and hence \( X_{a_1 \ldots a_n} = X_{a_{1} \ldots a_{k_0}} \cap T^{-k_0} \bigcup_{k=1}^\infty B_{n-k_0+k} \cap X_{a_{k_0+1} \ldots a_n} \). For 2 and 3, it is immediate to show the ergodicity of \( T \) with respect to \( \lambda, T \)-invariance of \( \mu \) and finiteness of \( \mu \) as in [4].

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**References**


(Figure 1) The Partition $Q$.
$D_1 = \{X_0, X_\theta, X_\delta, X_{-\theta}, X_{-\delta}, X_{2i}, X_{-2i}, X_2, X_{-2}\}$
(Figure 2)
(Figure 3)
(Figure 4)