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Semi-algebraic aspect of the theory of Teichmüller space

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Abstract: We expose a semi-algebraic construction of Teichmüller space due to J. Morgan and P.B. Shalen, and Brumfiel's compactification of Teichmüller space by using the real spectrum in the sense of Coste.

§ 1. Semi-algebraic description of Teichmüller space.

In this section, we review the semi-algebraic construction of Teichmüller space due to Morgan-Shalen [MS].

Let Γ be the closed surface group of genus g $(\geqq 2)$:

$$\Gamma = \langle \alpha_i, \beta_i \ (1 \leq i \leq g) \mid \prod_{i=1}^{g} [\alpha_i, \beta_i] = id \rangle$$

We can embed $Hom(\Gamma, SL_2(\mathbb{R}))$ as an algebraic subset of $\mathbb{R}^{8g}$ by using these generators α_i, β_i $(1 \leq i \leq g)$:

$$\begin{array}{ccccccc} Hom(\Gamma, SL_2(\mathbb{R})) & \xrightarrow{\sim} & R(\Gamma) & \subset & SL_2(\mathbb{R})^{2g} & \subset & \mathbb{R}^{8g} \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & & & & \\ \rho & \mapsto & (\rho(\alpha_1), \rho(\beta_1), \cdots, \rho(\alpha_g), \rho(\beta_g)) & & & & \end{array}$$

Let $A(R(\Gamma))$ be the affine coordinate ring of $R(\Gamma)$. For any $g \in \Gamma$, we define a function $\tau_g \in A(R(\Gamma))$ by

$$\tau_g(\rho) := tr\, \rho(g) \qquad (\forall \rho \in R(\Gamma))$$

<u>Claim.</u> (Helling [He], Horowitz [Ho], Culler-Shalen [CS])

$\mathbb{Z}$-subalgebra of $A(R(\Gamma))$ generated by τ_g $(\forall g \in \Gamma)$ is finitely generated

i.e. $\exists\, g_1, \cdots, g_\ell \in \Gamma$ s.t.

$$\mathbb{Z}[\tau_g \mid g \in \Gamma] = \mathbb{Z}[\tau_{g_1}, \cdots, \tau_{g_\ell}]$$

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Let $X(\Gamma)$ be the algebraic subset of $\mathbb{R}^\ell$ whose affine coordinate ring is $\mathbb{R} \otimes_{\mathbb{Z}} \mathbb{Z}[\tau_g \mid g \in \Gamma] = \mathbb{R}[\tau_{g_1}, \cdots, \tau_{g_\ell}]$. Then we can define the polynomial map $t : R(\Gamma) \to X(\Gamma)$ as follows :

$$\begin{array}{cccc} t : & R(\Gamma) & \longrightarrow & X(\Gamma) \\ & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ & \rho & \mapsto & (\tau_{g_1}(\rho), \cdots, \tau_{g_\ell}(\rho)) = (\mathrm{tr}\,\rho(g_1), \cdots, \mathrm{tr}\,\rho(g_\ell)) \end{array}$$

We call $R(\Gamma)$ a space of representations and $X(\Gamma)$ a space of characters. Next claim is due to Culler - Shalen [C-S].

<u>Claim.</u>

1) There exists a closed algebraic subset Δ of $X(\Gamma)$ such that

$$t^{-1}(\Delta) = \left\{ \rho \in R(\Gamma) \;\middle|\; \begin{array}{l} \rho(\Gamma) \subset SL_2(\mathbb{R}) \text{ is an abelian subgroup or} \\ \text{has an invariant line in } \mathbb{R}^2 \text{ by } \rho(\Gamma) \curvearrowright \mathbb{R}^2 \end{array} \right\}$$

2) For any $\rho \in R(\Gamma) \setminus t^{-1}(\Delta)$ (i.e. ρ is non-abelian irreducible rep.),

$$t^{-1}(t(\rho)) = PGL_2(\mathbb{R})\text{-conj. class of } \rho. \qquad //$$

We define $DR(\Gamma)$, $DX(\Gamma)$ as follows.

$$\begin{aligned} DR(\Gamma) &:= \{ \rho \in R(\Gamma) \mid \rho \text{ is faithful and } \rho(\Gamma) \subset SL_2(\mathbb{R}) : \text{discrete} \} \\ &= \{ \rho \in R(\Gamma) \mid \rho \text{ is totally hyperbolic i.e. for } {}^\forall M (\neq id) \in \rho(\Gamma), \\ &\qquad\qquad |\mathrm{tr}\, M| > 2 \} \end{aligned}$$

$$DX(\Gamma) := t(DR(\Gamma))$$

Then 2) of the following claim is due to Weil [W] and Jørgensen [J].

<u>Claim.</u>

1) $DR(\Gamma) \subset R(\Gamma) \setminus t^{-1}(\Delta)$

$t^{-1}(DX(\Gamma)) = DR(\Gamma)$

$DR(\Gamma)$ is a trivial $PGL_2(\mathbb{R})$-bundle over $DX(\Gamma)$.

(i.e. $DX(\Gamma) = DR(\Gamma)/PGL_2(\mathbb{R}) = Aut(SL_2(\mathbb{R}))$.)

2) $DR(\Gamma)$ (resp. $DX(\Gamma)$) consists of finite many connected components of $R(\Gamma)$ (resp. $X(\Gamma)$). Therefore $DR(\Gamma)$, $DX(\Gamma)$ are semi-algebraic sets i.e. defined by finite many polynomial equations and inequations over $\mathbb{R}$. //

Next claim which is due to Patterson, tells the relation between $DX(\Gamma)$ and Teichmüller space.

<u>Claim</u> (Patterson[P]).

Let $\eta : \Gamma \to PSL_2(\mathbb{R})$ be a discrete faithful representation. Let $A_i, B_i \in SL_2(\mathbb{R})$ be any representatives of $\eta(\alpha_i)$, $\eta(\beta_i)$ of $PSL_2(\mathbb{R})$ $(1 \le i \le g)$. Then

$$\prod_{i=1}^{g} [A_i, B_i] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in SL_2(\mathbb{R})$$

In other words, η can be always lifted to $\rho \in DR(\Gamma)$.

$$\begin{tikzcd} & SL_2(\mathbb{R}) \arrow[d] \\ \Gamma \arrow[r, "\eta"'] \arrow[ur, dashed, "\rho"] & PSL_2(\mathbb{R}) \end{tikzcd}$$

<u>Corollary</u>

1) $Hom(\Gamma, \mathbb{Z}/2\mathbb{Z})$ acts on $R(\Gamma)$ as the group which changes the sign of $\rho(\alpha_i), \rho(\beta_i) \in SL_2(\mathbb{R})$ for $\rho \in R(\Gamma)$ $(1 \le i \le g)$.

Then the action $\mathrm{Hom}(\Gamma, \mathbb{Z}/2\mathbb{Z}) \curvearrowright DR(\Gamma)$ induces the action $\mathrm{Hom}(\Gamma, \mathbb{Z}/2\mathbb{Z}) \curvearrowright DX(\Gamma)$ through the map t, and we can consider $DR(\Gamma)/\mathrm{Hom}(\Gamma, \mathbb{Z}/2\mathbb{Z})$, $DX(\Gamma)/\mathrm{Hom}(\Gamma, \mathbb{Z}/2\mathbb{Z})$ as following sets.

$DR(\Gamma)/\mathrm{Hom}(\Gamma, \mathbb{Z}/2\mathbb{Z})$ = the set of discrete faithful $PSL_2(\mathbb{R})$-rep. of Γ.

$DX(\Gamma)/\mathrm{Hom}(\Gamma, \mathbb{Z}/2\mathbb{Z})$ = the set of $PGL_2(\mathbb{R})$-conj. classes of discrete faithful $PSL_2(\mathbb{R})$-rep. of Γ.

We call $T(\Gamma) := DX(\Gamma)/\mathrm{Hom}(\Gamma, \mathbb{Z}/2\mathbb{Z})$ Teichmüller space of Γ.

2) $\mathrm{Hom}(\Gamma, \mathbb{Z}/2\mathbb{Z})$ permutes the set of connected components of $DX(\Gamma)$ freely. Therefore,

$$\# |\mathrm{Hom}(\Gamma, \mathbb{Z}/2\mathbb{Z})| (= 2^{2g} = 4^g) \text{ divides } \# |DX(\Gamma)|$$

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From the above argument, $T(\Gamma)$ can be considered as some components of $DX(\Gamma)$ and therefore has a semi-algebraic structure.

$$\begin{array}{l} DR(\Gamma) \\ t \downarrow \quad PGL_2(\mathbb{R})\text{-trivial bundle.} \\ DX(\Gamma) \\ \quad \downarrow \quad \text{unramified } \mathrm{Hom}(\Gamma, \mathbb{Z}/2\mathbb{Z})\text{-covering.} \\ T(\Gamma) \end{array}$$

§2. Real spectrum.

In this section we review the theory of real spectrum due to Coste [BCR].

2.1. Real spectrum.

Let $X \subseteq \mathbb{R}^N$ be a real algebraic set and $A(X) := \mathbb{R}[X_1, \cdots, X_N]/I(X)$ be the affine coordinate ring of X.

A subset $\alpha \subset A(X)$ is called prime cone if it satisfies the following conditions:

(i) For any $a, b \in \alpha$, $a+b \in \alpha$ (i.e. $\alpha + \alpha \subset \alpha$)

(ii) For any $a, b \in \alpha$, $a \cdot b \in \alpha$ ($\alpha \cdot \alpha \subset \alpha$)

(iii) For any $f \in A(X)$, $f^2 \in \alpha$ ($A(X)^2 \subset \alpha$)

(iv) $-1 \notin \alpha$

(V) If $a \cdot b \in \alpha$ for $a, b \in A(X)$, then $a \in \alpha$ or $-b \in \alpha$.

Prime cone has the following properties

Claim.

1) $\alpha \cup -\alpha = A(X)$ (where $-\alpha := \{-a \mid a \in \alpha\}$)

2) $\mathrm{Supp}(\alpha) := \alpha \cap -\alpha$ is a prime ideal of $A(X)$.

3) Let $k(\alpha)$ be the quotient field of $A(X)/\mathrm{Supp}(\alpha)$. Then $P := \{\frac{\bar{a}}{\bar{b}} \in k(\alpha) \mid a \cdot b \in \alpha\}$ is a positive cone.

(i.e. $P+P \subset P \wedge P \cdot P \subset P \wedge k(\alpha)^2 \subset P \wedge -1 \notin P \wedge P \cup -P = k(\alpha)$)

We define the real spectrum by the set of prime cone of $A(X)$.

$$\mathrm{Spec}_r\, A(X) := \{ \alpha \subset A(X) \mid \alpha \text{ is a prime cone of } A(X) \}$$

Moreover we define the topology on $\mathrm{Spec}_r A(X)$ as follows:
If we put $\mathcal{U}(f) := \{ \alpha \in \mathrm{Spec}_r A(X) \mid f \in \alpha \setminus \mathrm{Supp}(\alpha) \}$ $(f \in A(X))$,
then $\bigcap_{i=1}^{n} \mathcal{U}(f_i)$ $(f_i \in A(X))$ is an open basis of $\mathrm{Spec}_r A(X)$.

Claim.

1) With this topology, $\mathrm{Spec}_r A(X)$ is quasi-compact.
2) $\mathrm{Spec}_r^m A(X) := \{ \alpha \in \mathrm{Spec}_r A(X) \mid \alpha \text{ is a closed point} \}$ is a compact Hausdorff space.
3) X (with induced Euclidean topology) can be embedded topologically in $\mathrm{Spec}_r^m A(X)$:

$$\begin{array}{ccc} X & \hookrightarrow & \mathrm{Spec}_r^m A(X) \\ \cup\!\!\!\!\,\shortmid & & \cup\!\!\!\!\,\shortmid \\ \vec{x} & \mapsto & \alpha := \{ f \in A(X) \mid f(\vec{x}) \geq 0 \} \end{array}$$

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Therefore, we can consider X as a subset of the compact Hausdorff space $\mathrm{Spec}_r^m A(X)$.

2.2. The real spectrum compactification of closed semi-alg. sets.

A subset $S \subset X$ is called a semi-algebraic subset of X if there exist finite many $f_i, g_{ij} \in A(X)$ $(1 \leq i \leq l,\ 1 \leq j \leq m(i))$

such that

$$S = \bigcup_{i=1}^{\ell} \{ \vec{x} \in X \mid f_i(\vec{x}) = 0 \wedge g_{i1}(\vec{x}) > 0 \wedge \cdots \wedge g_{i\,m(i)}(\vec{x}) > 0 \}.$$

A subset $C \subset \mathrm{Spec}_r A(X)$ is called a constructible subset of $\mathrm{Spec}_r A(X)$ if there exist f_i, g_{ij} $(1 \le i \le \ell,\ 1 \le j \le m(i))$ such that

$$C = \bigcup_{i=1}^{\ell} \{ \alpha \in \mathrm{Spec}_r A(X) \mid f_i(\alpha) = 0 \wedge g_{i1}(\alpha) > 0 \wedge \cdots \wedge g_{i\,m(i)}(\alpha) > 0 \}$$

(where $f_i(\alpha), g_{ij}(\alpha)$ are the image of f_i, g_{ij} of the map $A(X) \twoheadrightarrow A(X)/\mathrm{Supp}(\alpha) \hookrightarrow k(\alpha)$. Because $k(\alpha)$ has a positive cone $P = \{ \frac{\bar{a}}{\bar{b}} \in k(\alpha) \mid a \cdot b \in \alpha \}$, we can define an order $\leq$ on $k(\alpha)$ by $x \leq y \iff y - x \in P$ (for $x, y \in k(\alpha)$))

Let $\mathcal{S}$ be the collection of semi-algebraic subsets of X and $\mathcal{C}$ be the collection of constructible subsets of $\mathrm{Spec}_r A(X)$. Next claim tells the relation between $\mathcal{S}$ and $\mathcal{C}$.

<u>Claim</u>

the map $\begin{array}{ccc} \mathcal{C} & \to & \mathcal{S} \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ C & \mapsto & C \cap X \end{array}$ is bijective and open (resp. closed)

constructible set goes to open (resp. closed) semi-algebraic set. //

Let $C(S) \in \mathcal{C}$ be the constructible set corresponding to $S \in \mathcal{S}$. If $W \in \mathcal{S}$ is a closed semi-alg. subset of X, then we can define the real spectrum compactification of W as the closure

$\tilde{W}$ of W in $\mathrm{Spec}_r^m A(X)$ (where we assume X as a subset of $\mathrm{Spec}_r^m A(X)$).

<u>Claim</u> (Structure of $\tilde{W}$)

1) $\tilde{W} = C(W) \cap \mathrm{Spec}_r^m A(X)$

2) $\tilde{B}(W) := \tilde{W} \setminus W = \{\alpha \in \tilde{W} \mid (\sum_{i=1}^{N} x_i^2 - r)(\alpha) > 0 \ (\text{for } {}^\forall r \in \mathbb{R})\}$

3) $W \subset \tilde{W}$: open and dense.

4) If $W_1, \cdots, W_s$ are the connected components of W, then $\tilde{W}_1, \cdots, \tilde{W}_s$ are the connected components of $\tilde{W}$. //

Next we consider the mapping between semi-alg. sets. For $S_1, S_2 \in \mathcal{S}$, a mapping $f: S_1 \to S_2$ is called semi-algebraic if the graph of f in $S_1 \times S_2$ is a semi-alg. subset. In this case, if V_1, V_2 be semi-alg. sets of S_1, S_2, then $f(V_1) \subset S_2$, $f^{-1}(V_2) \subset S_1$ are also semi-alg. subsets.

<u>Claim.</u>

If $f: S_1 \to S_2$ ($S_1, S_2 \in \mathcal{S}$) is a semi-alg. continuous map, then there exists uniquely the map $C(f): C(S_1) \to C(S_2)$ which is continuous in the real spectrum topology and satisfies the following functorial condition:

For any semi-alg. subset V of S_2

$$C(f^{-1}(V)) = C(f)^{-1}(C(V))$$

$$\begin{tikzcd} C(S_1) \arrow[r, "C(f)"] & C(S_2) \\ S_1 \arrow[u, hook] \arrow[r, "f"'] & S_2 \arrow[u, hook] \end{tikzcd}$$

In particular if f is a semi-alg. homeomorphism, then $C(f)$ is also homeomorphism and moreover if $S_1, S_2 \in \mathcal{S}$ are closed semi-alg. sets, then $C(f)$ induces the homeomorphism $\tilde{f}: \tilde{S}_1 \cong \tilde{S}_2$. Therefore if f is an semi-algebraic automorphism of a closed semi-alg. set W, then f is always extended to the automorphism of its real spectrum compactification $\tilde{W}$. //

This result will be used later in the context where W is $DX(\Gamma)$ and f is an element of $\mathrm{Hom}(\Gamma, \mathbb{Z}/2\mathbb{Z})$ and where W is $T(\Gamma)$ and f is an element of the mapping class group $\mathrm{Out}^+(\Gamma)$.

§3. The real spectrum compactification of Teichmüller space (after Brumfiel [B]).

In section 1, we have seen that Teichmüller space $T(\Gamma)$ can be considered as a semi-alg. subset of $X(\Gamma)$, more exactly some components of $DX(\Gamma)$. In this section we apply the theory of the real spectrum compactification of closed semi-alg. set to $DX(\Gamma)$ or $T(\Gamma)$. Thus, $DX(\Gamma) \subset X(\Gamma)$ can be compactified as $\widetilde{DX(\Gamma)} \subset \mathrm{Spec}_r^m A(X(\Gamma))$ and because $\mathrm{Hom}(\Gamma, \mathbb{Z}/2\mathbb{Z})$ acts on $DX(\Gamma)$ semi-algebraically, this action extends on $\widetilde{DX(\Gamma)}$, therefore we can define $\widetilde{T(\Gamma)}$ by

$$\widetilde{T(\Gamma)} := \widetilde{DX(\Gamma)} \big/ \mathrm{Hom}(\Gamma, \mathbb{Z}/2\mathbb{Z})$$

3.1. Representation theoretic characterization of boundary points of $\widetilde{T(\Gamma)}$.

By using the argument in §2, the diagram

$$\begin{array}{ccc} R(\Gamma) & \xrightarrow{t} & X(\Gamma) \\ \cup & & \cup \\ DR(\Gamma) & \twoheadrightarrow & DX(\Gamma) \end{array} \qquad (A(X(\Gamma)) \xrightarrow{t_*} A(R(\Gamma)))$$

induces the following maps.

$$\begin{array}{ccc} \mathrm{Spec}_r A(R(\Gamma)) \; (= C(R(\Gamma))) & \xrightarrow{c(\;)} & \mathrm{Spec}_r A(X(\Gamma)) \; (= C(X(\Gamma))) \\ & \beta \longmapsto t_*^{-1}(\beta) & \\ \cup & & \cup \\ C(DR(\Gamma)) & \twoheadrightarrow & C(DX(\Gamma)) \\ \cup & & \cup \\ \widetilde{DR(\Gamma)} & & \widetilde{DX(\Gamma)} \end{array}$$

with $C(t)^{-1}(C(DX(\Gamma))) = C(t^{-1}(DX(\Gamma))) = C(DR(\Gamma))$.

If $\alpha \in \widehat{DX(\Gamma)} \setminus DX(\Gamma)$, then for any $\beta \in C(t)^{-1}(\alpha)$, there exists homomorphism from $k(\alpha)$ to $k(\beta)$ as follows.

$$\begin{tikzcd} A(R(\Gamma)) \arrow[r, two heads] & A(R(\Gamma))/_{Supp(\beta)} \arrow[r, hook] & k(\beta) \\ A(X(\Gamma)) \arrow[r, two heads] \arrow[u] & A(X(\Gamma))/_{Supp(\alpha)} \arrow[r, hook] \arrow[u, hook] & k(\alpha) \arrow[u, hook] \end{tikzcd}$$

By using Tarski principle (this is not defined here), we can prove that there exists $\beta \in C(t)^{-1}(\alpha)$ such that $k(\beta)/_{k(\alpha)}$ is algebraic. In this case, we can also prove the next claim.

<u>Claim.</u>

If $k(\beta)/k(\alpha)$ is algebraic, then $\beta \in \widehat{DR(\Gamma)} \setminus DR(\Gamma)$. //

Moreover, $\beta \in Spec_r A(R(\Gamma))$ induces the following map:

$$A(R(\Gamma)) \twoheadrightarrow A(R(\Gamma))/_{Supp(\beta)} \hookrightarrow k(\beta)$$

and this means that β can be considered as $k(\beta)$-valued point of $R(\Gamma)$

i.e. $\beta \in Hom(A(R(\Gamma)), k(\beta)) = Hom(\Gamma, SL_2(k(\beta)))$

Thus, β is a representation $\beta : \Gamma \to SL_2(k(\beta))$. C. Frohman proved that if $\beta \in \widehat{DR(\Gamma)}$, then $\beta : \Gamma \to SL_2(k(\beta))$ is discrete faithful (moreover, totally hyperbolic) [B].

Summarizing,

Claim.

For any $[\alpha] \in \widetilde{T(\Gamma)} \setminus T(\Gamma)$ $(\alpha \in \widetilde{DX(\Gamma)} \setminus DX(\Gamma))$, there exists a representation $\beta : \Gamma \to SL_2(k(\beta))$ over $[\alpha]$ which is discrete, faithful and belongs to $\widetilde{DR(\Gamma)} \setminus DR(\Gamma)$.

3.2. Comparison with the Thurston compactification.

Let $(F, \leq)$ be an ordered field. We call $b \in F^+ := \{x \in F \mid x > 0\}$ is a big element if for any $a \in F$, there exists $m \in N$ such that $a < b^m$. (For example, any $r\ (>1) \in \mathbb{R}$ is a big element of $\mathbb{R}$.)

If an ordered field $(F, \leq)$ has a big element, we can define the logarithm $\log_b : F^+ \to \mathbb{R}$ by using the Dedekind cut of $\mathbb{Q}$:

$$\frac{m'}{n} \leq \log_b(a) \leq \frac{m}{n} \quad \text{if } b^{m'} \leq a^n \leq b^m \quad \left(\begin{array}{l} a, b \in F^+,\ m, m', n \in \mathbb{Z} \\ n > 0 \end{array}\right)$$

This function has properties which are satisfied by the ordinary logarithm on $\mathbb{R}^+$. For example,

(a) $\log_b(b^m) = m \quad (\forall m \in \mathbb{Z})$

(b) $\log_b(a \cdot a') = \log_b(a) + \log_b(a')$

(c) If $0 < a < a'$ $(a, a' \in F^+)$, then $\log_b(a) \leq \log_b(a')$

(d) If b, b' are big elements of F and $a \in F^+$, then

$$\log_{b'}(a) = \log_{b'}(b) \log_b(a) \quad \text{and} \quad \log_b(b') > 0$$

Let $\mathfrak{S} \subset A(X)$ be a subset which satisfies the following properties:

(i) $\mathfrak{S}$ contains generator system of $A(X)$ as $\mathbb{R}$-algebra.

(ii) For any $\vec{x} \in W$ and any $f \in \mathfrak{S}$, $|f(\vec{x})| \geq 1$

(iii) For any $\vec{x} \in W$, there exists $f \in \mathfrak{S}$ such that $|f(\vec{x})| > 1$.

If there exists such $\mathfrak{S} \subset A(X)$, we can define the continuous map θ from W to the infinite dimensional projective space $\mathbb{P}^{\mathfrak{S}}$ by using logarithm:

$$\begin{array}{rccc} \theta : & W & \longrightarrow & \mathbb{P}^{\mathfrak{S}} \\ & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ & \vec{x} & \longmapsto & \theta(\vec{x}) := (\log|f(\vec{x})|)_{f \in \mathfrak{S}} \end{array}$$

(where θ does not depend on the base of logarithm.)

θ can be extended uniquely to the map from the real spectrum compactification of W.

<u>Claim.</u>

θ can be extended continuously to $\tilde{\theta}$ by the same formula.

$$\begin{array}{rccc} \tilde{\theta} : & \widetilde{W} & \longrightarrow & \mathbb{P}^{\mathfrak{S}} \\ & \rotatebox{90}{$\in$} & & \\ & \alpha & \longmapsto & \tilde{\theta}(\alpha) := (\log|f(\alpha)|)_{f \in \mathfrak{S}} \end{array} \quad (\text{where } f(\alpha) \in k(\alpha))$$

Next we apply the above consideration to $\widetilde{T(\Gamma)}$.

Let $\mathcal{S}$ be the set of conjugacy classes of the primary elements of Γ

where primary element means that it is not a power of any other element of Γ, and put $\mathfrak{T} := \{\tau_g \mid [g] \in \mathcal{S}\}$.

Then $\mathfrak{T}$ satisfies the conditions (i), (ii), (iii), therefore we can consider the following map θ ;

$$\begin{array}{ccl} \theta : T(\Gamma) & \longrightarrow & \mathbb{P}^{\mathfrak{T}} \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ [\rho] & \mapsto & (\log|\tau_g(\rho)|)_{[g]\in\mathcal{S}} = (\log|\mathrm{tr}\,\rho(g)|)_{[g]\in\mathcal{S}} . \end{array}$$

It is known that θ is homeomorphic and the closure of $\theta(T(\Gamma))$ in $\mathbb{P}^{\mathfrak{T}}$ is essentially the Thurston compactification $\widehat{T(\Gamma)}$ of $T(\Gamma)$.

Moreover $\mathrm{Out}^+(\Gamma)$ (subgroup of $\mathrm{Out}(\Gamma)$ of index 2) acts on $\mathfrak{T}$, therefore on $\mathbb{P}^{\mathfrak{T}}$ by the change of coordinates. On the other hand, the action $\mathrm{Out}^+(\Gamma)$ on $\mathrm{Spec}_r A(X(\Gamma))$ is also induced by the action of $\mathrm{Out}^+(\Gamma)$ on $A(X(\Gamma)) = \mathbb{R}[\tau_g \mid \tau_g \in \mathfrak{T}] = \mathbb{R}[\mathfrak{T}]$. This leads to the last claim.

<u>Claim.</u>

1) There exists surjective continuous map $\tilde{\theta}$ from $\widetilde{T(\Gamma)}$ to $\widehat{T(\Gamma)}$.

$$\begin{array}{ccl} \tilde{\theta} : \widetilde{T(\Gamma)} & \longrightarrow & \widehat{T(\Gamma)} \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ \alpha & \mapsto & (\log|\tau_g(\alpha)|)_{[g]\in\mathcal{S}} \end{array}$$

2) $\tilde{\theta}$ is $\mathrm{Out}^+(\Gamma)$-equivariant. //

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