数理解析研究所講究録
第 815 巻 1992 年 89-96

ON FUNCTIONAL MODULUS OF FIRST ORDER ORDINARY DIFFERENTIAL EQUATIONS

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ABSTRACT. A characterization of functional modulus associated with some normal forms of completely integrable first order ordinary differential equations is given under the original equivalence relation.

1. INTRODUCTION

In the recent article [H-I-I-Y] it has been studied local classifications of first order ordinary differential equations with complete integral by the equivalence relation under the group of point transformations in the sense of Sophus Lie. As the result of the classification, some normal forms are parametrized by C^∞function-germs which is called 'functional moduli'(see, p.2). Furthermore, they gave a characterization of the functional modulus relative to the strict equivalence(see, p.2). However, the strict equivalence relation is away from the original one. In this paper we give a characterization of the functional modulus relative to the original equivalence relation.

Now we formulate our theorem. Let $\pi\colon PT^*\mathbf{R}^2 \to \mathbf{R}^2$ be the projective cotangent bundle over $\mathbf{R}^2$. Then, $PT^*\mathbf{R}^2$ has the natural contact structure. For any $z \in PT^*\mathbf{R}^2$ there is a local coordinate system (x, y, p) around z such that $\pi(x, y, p) = (x, y)$ and the contact structure is given by the 1-form $\omega = dy - pdx$. Let (μ, g) be a pair of a C^∞map-germ $g : (\mathbf{R}^2, 0) \to (\mathbf{R}^2, 0)$ and a submersion-germ $\mu : (\mathbf{R}^2, 0) \to (\mathbf{R}, 0)$. Then the diagram

$$(\mathbf{R}, 0) \xleftarrow{\mu} (\mathbf{R}^2, 0) \xrightarrow{g} (\mathbf{R}^2, 0)$$

or briefly (μ, g) , is called *an integral diagram* if there exists an immersion-germ f : $(\mathbf{R}^2, 0) \to PT^*\mathbf{R}^2$ such that $d\mu \wedge f^*\omega = 0$, and that $g = \pi \circ f$. In this case, we say that (μ, f) is *a first order ordinary differential equation germ with complete integral* (or briefly, *differential equation germ*), and we say that the integral diagram (μ, g) is *induced by f*. Furthermore we introduce the original equivalence relation among integral diagrams. Let (μ, g) and (μ', g') be integral diagrams. Then (μ, g) and (μ', g') are *equivalent* if the diagram

$$\begin{array}{ccccc} (\mathbf{R},0) & \xleftarrow{\ \mu\ } & (\mathbf{R}^2,0) & \xrightarrow{\ g\ } & (\mathbf{R}^2,0) \\ \lambda \downarrow & & \downarrow k & & \downarrow h \\ (\mathbf{R},0) & \xleftarrow[\ \mu'\ ]{} & (\mathbf{R}^2,0) & \xrightarrow[\ g'\ ]{} & (\mathbf{R}^2,0) \end{array}$$

commutes for some C^∞diffeomorphism-germs λ, k and h. Particularly if we admit $\lambda = id_{\mathbf{R}}$, then they called *strictly equivalent.*

In [H-I-I-Y] it has been defined an equivalence relation among differential equation germs under the group of point transformations in the sense of S.Lie and shown that two differential equation germs with complete integral f and f' are equivalent if and only if induced integral diagrams $(\mu, \pi \circ f)$ and $(\mu', \pi \circ f')$ are equivalent for generic (μ, f) and (μ', f'). And they showed that generic integral diagrams (μ, g) are strictly equivalent to one of the following types:

(1) $\mu = v, g = (u, v)$,

(2) $\mu = v - \frac{1}{3}u^3, g = (u^2, v)$,

(3) $\mu = v - \frac{1}{2}u, g = (u, v^2)$,

(4) $\mu_\alpha = \frac{3}{4}u^4 + \frac{1}{2}u^2 v + \alpha \circ g, g = (u^3 + uv, v)$,
where $\alpha(x, y)$ is a C^∞function-germ on $(\mathbf{R}^2, 0)$ with $\alpha(0) = 0$, and $\frac{\partial \alpha}{\partial y}(0,0) = \pm 1$,

(5) $\mu_\alpha = v + \alpha \circ g, g = (u, v^3 + uv)$,
where $\alpha(x, y)$ is a C^∞ function-germ on $(\mathbf{R}^2, 0)$ with $\alpha(0) = 0$,

(6) $\mu_\alpha = \frac{1}{2}v^2 + \alpha \circ g, g = (u, v^3 + uv^2)$,
where $\alpha(x, y)$ is a C^∞ function-germ on $(\mathbf{R}^2, 0)$ with $\alpha(0) = 0$, and $\frac{\partial \alpha}{\partial x}(0,0) = \pm 1$.

The function-germs α which appear in normal forms of type (4),(5),(6) are called *functional moduli* of the type. The functional modulus has been characterized relative to the strict equivalence relation in [H-I-I-Y],[D1]. We obtain the following results relative to the original equivalence relation. Denote by $\mathcal{A}_y$ (resp. $\mathcal{A}_x$) as the set of functional modulus of type (4) (resp. (6)).

Theorem. **A**) *Let (μ_α, g) be an integral diagram of type (4). Then, for any $\alpha \in \mathcal{A}_y$ there exists an $\alpha' \in \mathcal{A}_y$ such that*

(i) (μ_α, g) is equivalent to $(\mu_{\alpha'}, g)$,

(ii) $\alpha'(0,y) = \frac{\partial \alpha}{\partial y}(0,0)y + \frac{1}{2}\chi_\alpha y^2$ for all $y \leq 0$,

where $\chi_\alpha = \frac{\partial^2 \alpha}{\partial y^2}(0,0)$.

B1(*resp.***B**2)) *Let (μ_α, g) be an integral diagram of type (6). Then, for any $\alpha \in \mathcal{A}_x$ there exists an $\alpha' \in \mathcal{A}_x$ such that*

(i) (μ_α, g) is equivalent to $(\mu_{\alpha'}, g)$,

(ii) $\alpha' = \delta x + \frac{1}{2}\chi_\alpha y^2$ on D_1(resp. D_2),

where $\chi_\alpha = \frac{\partial^2 \alpha}{\partial x^2}(0,0), \delta = \pm 1, D_1 = \{(x,y) | y = 0\}$ and $D_2 = \{(x,y) | 27y = 4x^3\}$.

The theorem is an analogy of Dufour's result on the normal form of type (5) in [D2]. Dufour also have shown uniqueness of functional modulus. However, our types (4),(6) are so complicated that we can not obtain the uniqueness result in this paper.

Hereafter, we assume that all mappings and diffeomorphisms are of class C^∞.

The author would like to express sincere gratitude to Professors I.Nakai, S.Izumiya, G.Ishikawa and Doctor T.Ohmoto for their valuable advices and constant encouragements.

2. The proof of theorem

The proof of our theorem are based on following two propositions.

Proposition2.1 (**Takens' Theorem.** [T]) *Let $\psi\colon \mathbf{R}, 0 \to \mathbf{R}, 0$ be a diffeomorphism such that ψ^2 has the form $\psi^2(x) = x + x^k F(x)$ with $F(0) \neq 0$ and $k \geq 2$. Then there is an* *orientation preserving diffeomorphism* $\lambda\colon \mathbf{R}, 0 \to \mathbf{R}, 0$ *such that, in some neighbourhood of* $0 \in \mathbf{R}$,

$$\lambda \circ \psi \circ \lambda^{-1}(x) = \pm x + \delta x^k + \chi x^{2k-1}$$

where $\delta = \pm 1$ *and* $\chi \in \mathbf{R}$.

The following is implicitly proved in [H-I-I-Y].

Proposition2.2. *Let* (μ', g') *be an integral diagram which is equivalent to* (μ_α, g) *of type (4)(resp. (6)) for some* $\alpha \in \mathcal{A}_y$ *(resp.* $\mathcal{A}_x$*). Then* (μ', g') *is strictly equivalent to* $(\mu_{\alpha'}, g)$ *of type (4)(resp. (6)) for some* $\alpha' \in \mathcal{A}_y$ *(resp.* $\mathcal{A}_x$*).*

For each case A, B1, B2 we will define a map-germ $\gamma_\alpha\colon (\mathbf{R}, 0) \to (\mathbf{R}^2, 0)$, as follows. In the case A: Put $\Delta = \{(x,y) | 27x^2 + 4y^3 < 0\}$. Note that $\Delta = \{(x,y) | \sharp(g^{-1}(x,y)) = 3\}$. Let $(u_1, y), (u_2, y)$ and (u_3, y) be the preimages of (x,y) by g for each $(x,y) \in \Delta$ near $(0,0)$, where $u_j = u_j(x,y) (j = 1,2,3)$ are three real roots of the equation $U^3 + yU - x = 0$ and ordered by $u_1 < u_2 < u_3$. For each $\alpha \in \mathcal{A}_y$ set $c_j(x,y) = \mu_\alpha(u_j, y)$. We see clearly $c_1(0,y) = c_3(0,y) = \frac{1}{4}y^2 + \alpha(0,y), c_2(0,y) = \alpha(0,y)$ for any $(0,y) \in \Delta$. We set

$$\gamma_\alpha(y) = (\alpha(0,y), \frac{1}{4}y^2 + \alpha(0,y))$$

for each $\alpha \in \mathcal{A}_y$.

In the case B1(resp. B2): Note that $\sharp(g^{-1}(x,y)) = 2$ for any $(x,y) \in D_1$(resp. D_2). Let $(x, v_1), (x, v_2)$ be the preimages by g for each $(x,y) \in D_1$ (resp. D_2), where $v_j = v_j(x,y) (j = 1,2)$, are three real roots of $V^3 + xV^2 - y = 0$ (v_1 is the multiple root). For any $(x,y) \in D_1$(resp.D_2), set $c_j(x,y) = \mu_\alpha(x, v_j)$. We see clearly $c_1(x,y) = \alpha(x,y), c_2(x,y) = \frac{1}{2}x^2 + \alpha(x,y)$. (resp.$c_1(x,y) = \frac{2}{9}x^2 + \alpha(x,y), c_2(x,y) = \frac{1}{18}x^2 + \alpha(x,y)$). We set

$$\gamma_\alpha(x) = (c_1(x,0), c_2(x,0))$$

$$(\text{resp.}\gamma_\alpha(x) = (c_1(x, \frac{4}{27}x^3), c_2(x, \frac{4}{27}x^3))$$

for each $\alpha \in \mathcal{A}_x$.

Lemma2.3. *Let* $\theta : (\mathbf{R}, 0) \to \mathbf{R}$ *be a function-germ such that*

$$\theta(0) = \theta'(0) = \theta''(0) = \theta'''(0) = 0. \tag{2.1}$$

In the case A, for any $\alpha \in \mathcal{A}_y$ *there exists an* $\alpha' \in \mathcal{A}_y$ *such that*

i) (μ_α, g) *and* $(\mu_{\alpha'}, g)$ *are equivalent,*

ii) $-y^2 + \alpha'(0,y)^2 - \chi_\alpha \alpha'(0,y)^3 + \theta(\alpha'(0,y)) = 0$ *for all* $y \leq 0$,

where $\chi_\alpha = \frac{\partial^2 \alpha}{\partial y^2}(0,0)$.

In the case B1(resp.B2), for any $\alpha \in \mathcal{A}_x$ *there exists an* $\alpha' \in \mathcal{A}_x$ *such that*

i) (μ_α, g) *and* $(\mu_{\alpha'}, g)$ *are equivalent,*

ii) $-x^2 + \alpha'(x,0)^2 - \chi_\alpha \alpha'(x,0)^3 + \theta(\alpha'(x,0)) = 0$ *on* D_1

(resp. $-x^2 + c^2 - (\frac{1}{9} + \chi_\alpha)c^3 + \theta(c) = 0$ *on* D_2*)*

where $c = \frac{1}{18}x^2 + \alpha'(x, \frac{4}{27}x^3), \chi_\alpha = \frac{\partial^2 \alpha}{\partial x^2}(0,0)$.

Proof. Since the modulus $\alpha \in \mathcal{A}_y$(resp. $\mathcal{A}_x$) has the condition $\frac{\partial \alpha}{\partial y}(0,0) = \pm 1$ (resp. $\frac{\partial \alpha}{\partial x}(0,0) = \pm 1$), by the implicit function theorem, there exists the function-germ $\psi_\alpha : (\mathbf{R}, 0) \to (\mathbf{R}, 0)$ such that

$$Im\gamma_\alpha = graph\psi_\alpha$$

in each case. By direct calculations, we see respectively in the case A, B1, B2

$$\psi_\alpha(c) = c + \frac{1}{4}c^2 - \frac{1}{4}\chi_\alpha c^3 + o(|c|^3),$$

$$\psi_\alpha(c) = c + \frac{1}{2}c^2 - \frac{1}{2}\chi_\alpha c^3 + o(|c|^3),$$

$$\psi_\alpha(c) = c + \frac{1}{6}c^2 - \frac{1}{6}(\frac{1}{9} + \chi_\alpha)c^3 + o(|c|^3).$$

In the case A, B1, B2 respectively, define the function-germ $\overline{\psi_\alpha} : (\mathbf{R}, 0) \to (\mathbf{R}, 0)$ by

$$\overline{\psi_\alpha}(c) = c + \frac{1}{4}c^2 - \frac{1}{4}\chi_\alpha c^3 + \frac{1}{4}\theta(c),$$

$$\overline{\psi_\alpha}(c) = c + \frac{1}{2}c^2 - \frac{1}{2}\chi_\alpha c^3 + \frac{1}{2}\theta(c),$$

$$\overline{\psi_\alpha}(c) = c + \frac{1}{6}c^2 - \frac{1}{6}(\frac{1}{9} + \chi_\alpha)c^3 + \frac{1}{6}\theta(c).$$

Then, by the Takens' theorem there exists an orientation preserving diffeomorphism-germ λ: $(\mathbf{R}, 0) \to (\mathbf{R}, 0)$ such that

$$\overline{\psi_\alpha} = \lambda \circ \psi_\alpha \circ \lambda^{-1}$$

in each case.

Since $(\lambda \circ \mu_\alpha, g)$ is equivalent to (μ_α, g), by Proposition2.2 there exists a functional moduli $\alpha' \in \mathcal{A}_y$(resp. $\mathcal{A}_x$) in the case A(resp. B1,B2) such that the following diagram commute, hence (μ_α, g) and $(\mu_{\alpha'}, g)$ are equivalent:

$$\begin{array}{ccccc}
(\mathbf{R}, 0) & \xleftarrow{\mu_\alpha} & (\mathbf{R}^2, 0) & \xrightarrow{g} & (\mathbf{R}^2, 0) \\
\lambda \downarrow & & \| & & \| \\
(\mathbf{R}, 0) & \xleftarrow[\lambda \circ \mu_\alpha]{} & (\mathbf{R}^2, 0) & \xrightarrow[g]{} & (\mathbf{R}^2, 0) \\
\| & & \downarrow h & & \downarrow h \\
(\mathbf{R}, 0) & \xleftarrow[\mu_{\alpha'}]{} & (\mathbf{R}^2, 0) & \xrightarrow[g]{} & (\mathbf{R}^2, 0)
\end{array}$$

In the case A, since the set $\{(0, y) | y \leq 0\}$ is preserved by h and λ is orientation preserving, the above commutative diagram implies

$$\lambda \times \lambda(Im\gamma_\alpha|_{y\leq 0}) = graph\overline{\psi_\alpha} \cap \{\delta c \leq 0\} \tag{2.2}$$
$$= Im(\gamma_{\alpha'}|_{y\leq 0})$$

where $\delta = \frac{\partial \alpha}{\partial y}(0, 0) = \pm 1$. In the case B1(resp. B2) since a discriminant set D_1(resp. D_2) is preserved by h, the above commutative diagram implies

$$\lambda \times \lambda(Im\gamma_\alpha) = graph\overline{\psi_\alpha}$$
$$= Im\gamma_{\alpha'}.$$

Therefore by definition of $\overline{\psi_\alpha}, \gamma_\alpha$, we have the equation in Lemma2.3(ii) in each case. This completes the proof.

Remark2.4. In the case A, from (2.2), if $\delta = 1$, then $c = \alpha'(0,y) \leq 0$, thus $\frac{\partial \alpha'}{\partial y}(0,0) = 1$. Similarly, if $\delta = -1$ then $\frac{\partial \alpha'}{\partial y}(0,0) = -1$. That is $\frac{\partial \alpha}{\partial y}(0,0) = \frac{\partial \alpha'}{\partial y}(0,0) = \pm 1$.

The functional moduli α' in Lemma2.3 depends on θ and α. Now, by means of special choice of θ, we normalize α' such that $\alpha'(0,y)$(resp.$\alpha'(x,0), \alpha'(x, \frac{4}{27}x^3)$) is the polynomial as low degree as possible in the case A(resp. B1,B2). Note that the degree of $\alpha'(0,y)$(resp.$\alpha'(x,0), \alpha'(x, \frac{4}{27}x^3)$) is more than one because of the condition of the modulus $\frac{\partial \alpha}{\partial y}(0,0) = \pm 1$(resp. $\frac{\partial \alpha}{\partial x}(0,0) = \pm 1$). In the case of $\chi_\alpha = \frac{\partial^2 \alpha}{\partial y^2}(0,0) = 0$(resp. $\chi_\alpha = \frac{\partial^2 \alpha}{\partial x^2}(0,0) = 0$), if we set $\theta = 0$, then $\alpha'(0,y) = \pm y$(resp. $\alpha'(x,0) = \pm x, \alpha'(x, \frac{4}{27}x^3) = \pm x$). In the case of $\chi_\alpha = \frac{\partial^2 \alpha}{\partial y^2}(0,0) \neq 0$(resp. $\chi_\alpha = \frac{\partial^2 \alpha}{\partial x^2}(0,0) \neq 0$), we can not have the normalization to degree one because of the condition $\theta'''(0) = 0$. Thus we consider the normalization to degree two. In fact it is possible, as follows. For any $\chi \in \mathbf{R}$, we define the function-germ $\theta_\chi : (\mathbf{R}, 0) \to (\mathbf{R}, 0)$ by

$$\theta_\chi = \begin{cases} 0 & \text{if} \quad \chi = 0 \\ a \circ \xi & \text{if} \quad \chi \neq 0 \end{cases}$$

where $\xi(t) = \frac{-1+\sqrt{1+2\chi t}}{\chi}, a(t) = \frac{5}{4}\chi^2 t^4 + \frac{3}{4}\chi^3 t^5 + \frac{1}{8}\chi^4 t^6$.

Then the θ_χ satisfy the condition(2.1) in Lemma2.3. Moreover we define the function-germ $h_\chi : (\mathbf{R} \times \mathbf{R}, (0,0)) \to (\mathbf{R}, 0)$ by

$$h_\chi(t,c) = -t^2 + c^2 - \chi c^3 + \theta_\chi(c)$$

for any $\chi \in \mathbf{R}$. By the definition of θ_χ, it can be directly showen that

$$h_\chi(\pm t, t + \frac{\chi}{2}t^2) = 0$$

for all $t \in (\mathbf{R}, 0)$. Hence we can easily have the following:

Lemma2.5. *If* $(t,c) \in {h_\chi}^{-1}(0), c = \pm t + \frac{\chi}{2}t^2$ *for any* $\chi \in \mathbf{R}$.

For any $\alpha \in \mathcal{A}_y$(resp. $\mathcal{A}_x$), set $\chi = \frac{\partial^2 \alpha}{\partial y^2}(0,0)$ (resp. $\chi = \frac{\partial^2 \alpha}{\partial x^2}(0,0), \frac{1}{9} + \frac{\partial^2 \alpha}{\partial x^2}(0,0)$). Then, by Lemma2.3, Lemma2.5(and Remark2.4 in the case A), we obtain Theorem A), B1), B2).

Remark. χ_α is invariant of type (4),(6) relative to the equivalence.

Proof. Let $(\mu_\alpha, g), (\mu_{\alpha'}, g)$ be integral diagrams of type(4)(resp. (6)). If (μ_α, g) and $(\mu_{\alpha'}, g)$ are equivalent, then it follows

$$\lambda \times \lambda(Im\gamma_\alpha|_{y\le 0}) = Im\gamma_{\alpha'}|_{y\le 0} \quad (\text{resp. } \lambda \times \lambda(Im\gamma_\alpha) = Im\gamma_{\alpha'})$$

for some diffeomorphism-germ λ: $(\mathbf{R}, 0) \to (\mathbf{R}, 0)$. Hence ψ_α and $\psi_{\alpha'}$ are conjugate, that is

$$\psi_\alpha(c) = \lambda^{-1} \circ \psi_{\alpha'} \circ \lambda(c)$$

for any $\delta c \le 0, \delta = \frac{\partial \alpha}{\partial y}(0,0) = \pm 1$ (resp. for any $c \in (\mathbf{R}, 0)$). By directly calculation, we see that the third coefficient of the Taylor expansion at the origin for ψ_α is invariant under the conjugate. Therefore we obtain the remark.

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