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Kyoto University
Stable mappings with trivial monodromies
and application to inactive log-transformations

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0. Introduction.

(0.0) In this article we state two main results, based on the same idea. In the first one (Theorem B), we classify the diffeo-types of closed, oriented, smooth four-manifolds which are domains of certain stable maps $f : M \to \mathbb{R}^2$. In the second one (Theorem C), we give examples of such maps $h : M^4 \to S^2$ as follows; if the two log-transformations at distinct torus fibres of $h$ preserve the homeo-type of the domain, then they preserve the diffeo-type also, for any pair of multiplicities. Such log-transformations are called inactive by Viro [Vi]. It is a remarkable contrast that the same log-transformations often provide exotic manifolds for some pairs of multiplicities ([FM],[Theorem 1, Vi]). Our examples contain the Viro’s and certain types of good torus fibrations of Y. Matsumoto [Mt].

To prove these results, we use a result (Theorem A) on removing a certain connected component of the singular set $S(f)$ or $S(h)$ by performing a surgery. By this, Theorem B and C are reduced to the cases that $S(f)$ and $S(h)$ consist of two, and one connected components, respectively. To prove Theorem C in the reduced case, we shall construct a stable map $g$ from the manifold obtained from $M$ by the two log-transformations, into $\mathbb{R}^2$, and to $g$ we shall apply Theorem B.

(0.1) Here we recall some definitions and notations given in [Kb]. Let $f : M^4 \to P^2$ be a stable map where $P^2 = \mathbb{R}^2$ or $S^2$. By $S(f)$, we mean the singular set of $f$, the set of points in $M$ where the Jacobian is not surjective. Note that $S(f)$ consists of smooth closed curves in $M$. Let $q_f : M \to W_f$ be the quotient map which collapses each connected component of $f$-fibres to a point. In the case that the Euler number $\chi(M)$ is even, we call a stable map $f : M \to \mathbb{R}^2$ simple if (1) $W_f$ is homeomorphic to $D^2$, a closed 2-disc, and (2) $q_f|S(f)$ is an embedding. The symbol $g_f \leq 1$ means
that each regular fibre of $q_f$ is a torus $T^2$ or a sphere $S^2$. Let $R$ be a connected component of $W_f \setminus q_f(S(f))$. We say $R$ is a 0-region if the regular fibre over a point in $R$ is a sphere, and a 1-region if it is a torus. For a simple map $f : M \to \mathbb{R}^2$ with $g_f \leq 1$, we say $f$ is configuration trivial if there is no 1-region inside of any 1-region.

1. Simple maps with trivial monodromies.

(1.1) Let $f : M \to \mathbb{R}^2$ be a simple map with $g_f \leq 1$. Note that $\partial W_f \subset q_f(S(f))$ and that the $q_f$-preimage of a thin collar neighbourhood of $\partial W_f$ is a trivial $D^3$-bundle over $\partial W_f$. For other connected components $C_i$ of $q_f(S(f))$, the $q_f$-preimages are $T'$-bundles over $C_i$'s where $T'$ is a solid torus with an open 3-disc removed. Therefore we can regard $M$ as some $T^2$-bundles, $S^2$-bundles, $T'$-bundles, and a $D^3 \times S^1$ pasted together along their boundaries. We call the isomorphism on $H_1(\partial T' T'; \mathbb{Z})$ induced by $C_i$ the monodromy of $q_f$ over $C_i$ where $\partial T'$ is the torus in $\partial T'$. The monodromies over $C_i$'s determine the bundle structures of the rest (see Proposition 3.2, [Kb]).

(1.2) By changing the glueings, compatible with the restrictions of $q_f$ to each boundary, one obtains another simple stable map $f' : M' \to \mathbb{R}^2$ with $g_f \leq 1$. Now assume that all monodromies are trivial. Then such glueings are ample, hence there are many right-equivalent classes of the pair $(M, f)$ with $f$ a simple map of $g_f \leq 1$. To the contrast, we get the following result, which states that the diffeo-types of the domains are strictly restricted.

**Theorem B.** Let $f : M \to \mathbb{R}^2$ be a simple map with $g_f \leq 1$. Assume that $f$ is configuration trivial and has trivial monodromies. Then $M$ is diffeomorphic to either (a) $L(a_1) \# \ldots \# L(a_n) \# l(S^3 \times S^1) \# m(S^2 \times S^2)$ or, (b) $L(a_1) \# \ldots \# L(a_n) \# l(S^3 \times S^1) \# m(S^2 \times S^2)$. Here $l, m$ are non-negative integers, and for an integer $a_i$, $L(a_i)$ means the Pao's manifold defined in [Pa].

Conversely, the manifolds listed above admit such maps $f$'s.

**Remark** ([Pa]). $L(1) = S^4$ and $L(0) = S^3 \times S^2 \times S^2$. 
2. Inactive log-transformations.

(2.1) Let $h : M \to S^2$ be a stable map with the following conditions.

1. $\operatorname{Im}(h) = S^2$.
2. Each regular fibre of $h$ is a torus or a sphere.
   (Note that (1) and (2) implies $q_h = h$)
3. $S(h)$ is not empty and $h|S(h)$ is an embedding.
4. $h$ has a unique 1-region $R$.
5. The monodromies over $h(S(h))$ are trivial.

In addition, assume that (6) $h$ has a smooth cross-section, in the case of $\chi(M) = 2$.

(2.2) In the following, we shall define a $C^\infty$-log-transformation. For a pair of co-prime integers $(p, r)$ with $p \neq 0$, let $\Pi_{p, r} : S^1 \times S^1 \times D^2 \to D^2$ be a map defined by $\Pi_{p, r}(x, z, w) = z^p \cdot w^r$ where the second factor $S^1$ is regarded as $\{z \in \mathbb{C} \mid |z| = 1\}$, and the third factor $D^2$ is regarded as $\{w \in \mathbb{C} \mid |w| \leq 1\}$. Note that $\Pi_{p, r}$ has a multiple torus fibre over $0 \in D^2$ with multiplicity $|p|$.

Let $T = h^{-1}(a)$ be a regular torus fibre of $h$ and $a \in D$ a closed 2-disc in $S^2 \setminus h(S(h))$. Take an essential closed curve $d$ in $T$, co-prime integers $p, q$, and a matrix

$$A = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in SL(2, \mathbb{Z}).$$

We define a pair $(M', h')$ as,

$$(M', h') = (\overline{M \setminus h^{-1}(D)}, h|) \cup \varphi (S^1 \times S^1 \times D^2, \Pi_{p, r})$$

where $\varphi : \partial \overline{M \setminus h^{-1}(D)} \to \partial (S^1 \times S^1 \times D^2)$ is a diffeomorphism given in (2.3).

Definition. The log-transformation along $T$, of type $(p, q)$, with direction $d$ is the operation that changes $(M, h)$ to $(M', h')$. We call $p$ the multiplicity, and $q$ the sub-multiplicity.

(2.3) To describe the glueing $\varphi$, we fix essential simple closed curves $c', d'$ and $e$ in $h^{-1}(\partial D)$. Take a path $\lambda$ connecting a point $b \in \partial D$ and $a$. Let $d'$ be a curve in $h^{-1}(b)$ which is isotopic to $d$ in $h^{-1}(\lambda)$. Let $c'$ be a curve in $h^{-1}(b)$ with which $d'$ spuns $H_1(h^{-1}(b), \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$. Take a cross section $\tilde{D}$ of $h$ over $D$ which passes
through $c' \cap d'$. Let $e = \partial \tilde{D}$.

Then $\varphi$ is a diffeomorphism which induces an isomorphism between the first homology groups with $\mathbb{Z}$ coefficients, of the form $\varphi_* = 1 \oplus A$ where $H_1(h^{-1}(\partial D))$ and $H_1(\partial(S^1 \times S^1 \times D^2))$ are identified with $\mathbb{Z}^3$ with respect to the basis $< c', d', e >$ and the canonical basis, respectively.

Note that $(M', h')$ is independent of the choice of $r$ and $c'$, that is, for another pair $(M'', h'')$ derived from another $r$ and $c'$, one can show that $M'$ and $M''$ are diffeomorphic, and $h'$ and $h''$ are right-equivalent via the diffeomorphism.

(2.4) Let $T_a$ and $T_b$ be two torus fibres of $h$, and let $M(p_1, q_1; p_2, q_2)$ be the manifold obtained from $M$ by a log-transformation of type $(p_1, q_1)$ along $T_a$, and another one of type $(p_2, q_2)$ along $T_b$. The directions $d_1, d_2$ of the log-transformations were taken as follows.

(1) $d_1$ and $d_2$ are homotopic in $h^{-1}(\lambda)$ where $\lambda$ is a path in $\mathbb{R}$ connecting $h(T_a)$ and $h(T_b)$.

(2) $d_1$ is not a meridian with respect to some $(C, J)$, namely, does not bound a disc in $h^{-1}(J) = S^1 \times D^2 \setminus Int D^3$ where $C$ is a curve in $h(S(h))$ and $J$ a path starting $h(T_a)$ and meeting $C$ transversely at a single point.

**Theorem C.** $M(p_1, q_1; p_2, q_2)$ is diffeomorphic to either (a) $L(a) \# k(S^2 \times S^2)$, (a') $L'(a) \# k(S^2 \times S^2)$, $a$ is even, or (b) $L(a) \# k(S^2 \times S^2)$. Here $k$ is a non-negative integer and $L'(a)$ is the other Pao's manifold defined in [Pa].

**Remark ([Pa]).** $L'(a) \# S^2 \times S^2 \cong L(a) \# S^2 \times S^2, L(0) = S^3 \times S^1 \# S^2 \times S^2$, and $L'(0) = S^3 \times S^1 \# S^2 \times S^2$.

Note that for the manifolds listed above, homeo-types and diffeo-types coincide, which follows from the facts that $\pi_1(L(a)) = \pi_1(L'(a)) = \mathbb{Z}_a$, that (a) and (c) are spin and others are not spin, and that the intersection form of (a') is even ([Iw]). Note also that $M$ is diffeomorphic to one of the manifolds listed above, since $M = M(1, 0; 1, 0)$. It turns out that the log-transformations are inactive.
(2.5) It is shown that $\chi(M) = 2$, our $M$ is $S^4$. This implies that $M$ with any Euler number is simply connected. On $S^4$, one can construct $h$ directly from a simple map $g : S^4 = L(1) \to \mathbb{R}^2$ (which is the procedure converse to the one mentioned in (4.3)). It is shown that $k(S^2 \times S^2)$ and $k(S^2 \times \mathbb{S}^2)$ admit our $h$.

(2.6) With respect to Viro's inactive log-transformations, (which is performed for certain two tori in $S^2 \times S^2$ and which he defines without maps), we can show that the tori are regular fibres of our $h$. It is checked that his direction coincides with ours. Theorem C together with this gives another proof to Theorem 3 in [Vi].

3. Stable change of "twin" fibres.

(3.1) The map $h$ has a deep connection with good torus fibrations. Let $C \subset S(h)$ be a connected component, $D$ a closed 2-disc in $S^2 \backslash h(S(h))$ which contains $C$ in its interior. Let $N'$ be a fibred tubular neighbourhood of a twin fibre of multiplicities $(1,1)$, and $\varphi : N' \to D^2$ be the torus fibration (see [Iw]).

**Lemma.** (1) There is a diffeomorphism $\phi : h^{-1}(D) \to N'$.
(2) $(\varphi \circ \phi)|\partial h^{-1}(D)$ and $h|\partial h^{-1}(D)$ are right-equivalent.

By this, in a neighbourhood of a twin fibre of this type, we can replace the torus fibration with a stable map, preserving the map of the outside.

(3.2) Let $\phi : M \to S^2$ be a good torus fibration with at least one singular fibre. Assume that the singular fibres are of type $I_1^+, I_1^-$ and that the signature $\sigma(M) = 0$. Then one can deform $\varphi$ to $\varphi' : M \to S^2$, a good torus fibration with twin singular fibres of multiplicities $(1,1)$ [Mt]. Therefore we get the corollaries below.

**Corollary.** Let $\varphi : M^4 \to S^2$ be a good torus fibration mentioned above. Assume also that $\chi(M) > 2$. Then the log-transformations performed along whose two distinct regular fibres are inactive.

**Corollary.** $S^4$ admits a good torus fibration mentioned above such that the log-transformations performed along whose two distinct regular fibres are inactive.
Note that \( L(a) \) admit the map \( h \) except for the additional condition (6). Hence they admit the torus fibrations mentioned above. It is an open problem whether such torus fibrations on \( L(a) \) have an active log-transformation or not.

Torus fibrations with twin fibres are studied in [Iw]. One can see the complete list of the domain manifolds of such fibrations there.


(4.1) Here we state a result which is the base of previous two theorems. Let \( f : M^4 \rightarrow P^2 \) be a stable map into any oriented, connected surface, possibly non-compact, possibly with boundary. Let \( C \subset S(f) \) be a connected component with the four conditions.

1. \( q_f|C \) is an embedding.
2. \( q_f(C) \) separates one 1-region and one 0-region.
3. The 0-region bounded by \( q_f(C) \) is an open 2-disc.
4. The monodromy over \( q_f(C) \) is trivial.

**Theorem A.** There is an embedded 2-sphere \( S \) in \( M \) containing \( C \), and a stable map \( f' : M' \rightarrow P^2 \) such that \( (M', f') \) is obtained from \( (M, f) \) by a surgery detaching \( S \), namely, \( M' \) is obtained from \( M \) by the surgery and \( f' \) on \( M' \setminus \nu(s) \) and \( f \) on \( M \setminus \nu(S) \) coincide via the natural identification \( M' \setminus \nu(s) = M \setminus \nu(S) \), and that \( q_f(S(f')) = q'_f(S(f)) \setminus C \), where \( s \) is the attaching circle and where \( \nu(s), \nu(S) \) are tubular neighbourhoods.

(4.2) Applying this to the pair \( (M, f) \) of Theorem B, one obtains a sequence of surgeries

\[
(M, f) = (M_k, f_k) \rightarrow (M_{k-1}, f_{k-1}) \rightarrow \cdots \rightarrow (M_2, f_2).
\]

Here the index is taken to indicate the number of connected components of \( S(f_j) \). The terminal domain \( M_2 \) is seen to be \( S^3 \times S^1 \). Thus \( (M, f) \) is obtained from \( (S^3 \times S^1, f_2) \) by surgeries detaching \((k - 2)\) simple closed curves in \( S^3 \times S^1 \). This, with some detailed discussion, implies the theorem.
(4.3) Let \((M, h)\) be the pair of Theorem C. It is obtained from \((M_1, h_1)\) by a sequence of surgeries, in the same way as (4.2). Since the log-transformations are compatible with the surgeries, \((M', h')\), the pair after performing the log-transformations to \((M, h)\), is obtained from \((M'_1, h'_1)\) by a sequence of surgeries. We construct a simple map \(g' : M'_1 \to \mathbb{R}^2\) with \(g'_\leq 1\) and with trivial monodromies, which is the crucial point of the proof. Applying Theorem B, we get \(M'_1 = L(a)\). The effect of log-transformations appears as the change of the glueings of the bundle decomposition mentioned in (1, 2). Theorem C follows in the same way as Theorem B.

References


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