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Simple construction of parameter map germ and its applications

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In this note, we shall construct a simple parameter map germ $(\mathbb{R}^r, 0) \to (\mathbb{R}^p, 0)$ under the assumption that there is an $\mathcal{A}$-morphism (resp. topological $\mathcal{A}$-morphism) from a given deformation $\Psi : (\mathbb{R}^n \times \mathbb{R}^r, (0,0) \to (\mathbb{R}^p, 0)$ of a given map germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ to the trivial deformation $f : (\mathbb{R}^n \times \mathbb{R}^r, (0,0)) \to (\mathbb{R}^p, 0)$.

This parameter map germ induces a $\mathcal{K}$-morphism (resp. topological $\mathcal{K}$-morphism) from Ψ to the graph deformation of f.

By this construction, we can prove the following:

THEOREM D ([**M2**]): Let $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ be two C^∞ stable map germs. Suppose there exist a C^∞ diffeomorphic germ $s : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ and a C^∞ map germ $M : (\mathbb{R}^n, 0) \to (GL(p, \mathbb{R}), M(0))$ such that $f(x) = M(x)(g \circ s)(x)$. Then f and g are right-left equivalent.

Though our method seems to be close to Martinet's one ([**Mr**]), we can treat also map germs which are not necessarily C^∞ stable.

THEOREM E ([**FF**]): Let $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ be two MT stable map germs. Suppose there exist a C^∞ diffeomorphic germ $s : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ and a C^∞ map germ $M : (\mathbb{R}^n, 0) \to (GL(p, \mathbb{R}), M(0))$ such that $f(x) = M(x)(g \circ s)(x)$. Then f and g are topologically right-left equivalent.

THEOREM A: Let $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ be two C^∞ map germs. Suppose there exist a C^∞ diffeomorphic germ $s : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ and a C^∞ map germ

$$M(x) = (\mathbf{m}_1(x), \ldots, \mathbf{m}_p(x)) : (\mathbb{R}^n, 0) \to (GL(p, \mathbb{R}), M(0))$$

such that $f(x) = M(x)(g \circ s)(x)$. Suppose furthermore there exists a positive integer k such that

$$\mathbf{m}_i(x) - \mathbf{m}_i(0) \in \mathfrak{m}_x^k \mathcal{E}_x^p \subset tf(\mathfrak{m}_x \mathcal{E}_x^n) + \omega f(\mathfrak{m}_y \mathcal{E}_y^p)$$

for any i $(1 \leq i \leq p)$. Then f and g are right-left equivalent.

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As a corollary of theorem A, we get

COROLLARY A: Let $f:(\mathbb{R}^n,0)\to(\mathbb{R}^p,0)$ be a C^∞ map germ. Suppose there exist positive integers k,l such that

$$\mathfrak{m}_x^k\mathcal{E}_x^p\subset tf(\mathfrak{m}_x\mathcal{E}_x^n)+\omega f(\mathfrak{m}_y\mathcal{E}_y^p)$$

and

$$\mathfrak{m}_x^l\mathcal{E}_x^p\subset tf(\mathfrak{m}_x^2\mathcal{E}_x^n)+f^*\mathfrak{m}_y\mathfrak{m}_x^k\mathcal{E}_x^p.$$

Then f is $(l-1)$-determined with respect to right-left equivalence.

Corollary A induces the following Gaffney type estimate of the order of determinacy (c.f. [**G**]).

COROLLARY B: Let $f:(\mathbb{R}^n,0)\to(\mathbb{R}^p,0)$ be a C^∞ map germ. Suppose there exist positive integers k,l such that

$$\mathfrak{m}_x^k\mathcal{E}_x^p\subset tf(\mathfrak{m}_x\mathcal{E}_x^n)+\omega f(\mathfrak{m}_y\mathcal{E}_y^p)$$

and

$$\mathfrak{m}_x^l\mathcal{E}_x^p\subset tf(\mathfrak{m}_x\mathcal{E}_x^n)+f^*\mathfrak{m}_y\mathcal{E}_x^p.$$

Then f is $(k+l-1)$-determined with respect to right-left equivalence.

Corollary B induces the following du Plessis-Wall's estimate of the order of determinacy.

COROLLARY C ([**dP,W**]): Let $f:(\mathbb{R}^n,0)\to(\mathbb{R}^p,0)$ be a C^∞ map germ. Suppose there exist a positive integer k such that

$$\mathfrak{m}_x^k\mathcal{E}_x^p\subset tf(\mathfrak{m}_x\mathcal{E}_x^n)+\omega f(\mathfrak{m}_y\mathcal{E}_y^p).$$

Then f is $(2k-1)$-determined with respect to right-left equivalence.

In [**W**], we can find an estimate of the order of topological determinacy of an MT stable map germ (corollary D bellow) which is due to T. Gaffney, but without proof. By using of our method, we can give a proof of his estimate.

COROLLARY D (GAFFNEY): Let $f:(\mathbb{R}^n,0)\to(\mathbb{R}^p,0)$ be an MT stable map germ. Suppose there exist a positive integer k such that

$$\mathfrak{m}_x^k\mathcal{E}_x^p\subset tf(\mathfrak{m}_x\mathcal{E}_x^n)+f^*(\mathfrak{m}_y^2)\mathcal{E}_x^p.$$

Then f is k-determined with respect to topologically right-left equivalence.

For details on these corollaries, refer to [**N**].

This note is organized in the following way. In §1 and §2, we give several preparations for the proofs of theorem A, a generalized version of Mather's classification theorem (theorem D in §5) and the theorem of Fukuda-Fukuda (theorem E in §6). §3 treats algebraic argument which we need for the proof of theorem A. Theorem A will be proved in §4. A generalized version of Mather's classification theorem will be proved in §5. In §6, an alternative proof of the theorem of Fukuda-Fukuda will be given.

The results in this paper are all valid in the complex analytic category as well except example (1.5.2).

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§1. $\mathcal{K}$-MORPHISM FROM A GIVEN DEFORMATION TO THE GRAPH DEFORMATION

Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ be a C^∞ map germ and $\Psi_i : (\mathbb{R}^n \times \mathbb{R}^{r(i)}, (0,0)) \to (\mathbb{R}^p, 0)$ be a C^∞ deformation of f (i.e. $\Psi_i(x, 0) = f(x)$) (i = 1, 2).

DEFINITION (1.1). *We say if there exist C^∞ (resp. continuous) map germs $h : (\mathbb{R}^n \times \mathbb{R}^{r(1)}, (0,0)) \to (\mathbb{R}^n \times \mathbb{R}^{r(2)}, (0,0)), H : (\mathbb{R}^n \times \mathbb{R}^{r(1)} \times \mathbb{R}^p, (0,0,0)) \to (\mathbb{R}^n \times \mathbb{R}^{r(2)} \times \mathbb{R}^p, (0,0,0))$ and $\phi : (\mathbb{R}^{r(1)}, 0) \to (\mathbb{R}^{r(2)}, 0)$ such that the following conditions (1.1.1), (1.1.2), (1.1.3) and (1.1.4) hold, then $\{h, H, \phi\}$ is a $\mathcal{K}$-morphism (resp. topological $\mathcal{K}$-morphism) from Ψ_1 to Ψ_2.*

(1.1.1) the restrictions $h|_{\mathbb{R}^n \times \{\lambda\}}$ and $H|_{\mathbb{R}^n \times \{\lambda\} \times \mathbb{R}^p}$ are C^∞ diffeomorphic (resp. homeomorphic) for any $\lambda \in \mathbb{R}^{r(1)}$,

(1.1.2) $H(\mathbb{R}^n \times \mathbb{R}^{r(1)} \times \{0\}) \subset \mathbb{R}^n \times \mathbb{R}^{r(2)} \times \{0\}$,

(1.1.3) the following diagram commutes:

$$\begin{array}{ccccc}
(\mathbb{R}^n \times \mathbb{R}^{r(1)} \times \mathbb{R}^p, (0,0,0)) & \xrightarrow{\pi_{x,\lambda}} & (\mathbb{R}^n \times \mathbb{R}^{r(1)}, (0,0)) & \xrightarrow{\pi_\lambda} & (\mathbb{R}^{r(1)}, 0) \\
{\scriptstyle H}\downarrow & & {\scriptstyle h}\downarrow & & {\scriptstyle \phi}\downarrow \\
(\mathbb{R}^n \times \mathbb{R}^{r(2)} \times \mathbb{R}^p, (0,0,0)) & \xrightarrow{\pi_{x,\lambda}} & (\mathbb{R}^n \times \mathbb{R}^{r(2)}, (0,0)) & \xrightarrow{\pi_\lambda} & (\mathbb{R}^{r(2)}, 0),
\end{array}$$

(1.1.4) the following diagram commutes:

$$\begin{array}{ccc}
(\mathbb{R}^n \times \mathbb{R}^{r(1)}, (0,0)) & \xrightarrow{(\pi_x,\lambda,\Psi_1)} & (\mathbb{R}^n \times \mathbb{R}^{r(1)} \times \mathbb{R}^p, (0,0,0)) \\
{\scriptstyle h}\downarrow & & {\scriptstyle H}\downarrow \\
(\mathbb{R}^n \times \mathbb{R}^{r(2)}, (0,0)) & \xrightarrow{(\pi_x,\lambda,\Psi_2)} & (\mathbb{R}^n \times \mathbb{R}^{r(2)} \times \mathbb{R}^p, (0,0,0)).
\end{array}$$

Here $\pi_{x,\lambda}$, π_λ mean the canonical projection to $\mathbb{R}^n \times \mathbb{R}^{r(i)}$, $\mathbb{R}^{r(i)}$ respectively. We remark that the conditions (1.1.1), (1.1.2) and (1.1.3) in the definition (1.1) imply $H(\mathbb{R}^n \times \mathbb{R}^{r(1)} \times (\mathbb{R}^p - \{0\})) \subset \mathbb{R}^n \times \mathbb{R}^{r(2)} \times (\mathbb{R}^p - \{0\})$; and the condition (1.1.4) implies $H(graph(\Psi_1)) \subset graph(\Psi_2)$.

DEFINITION (1.2). *We say if there exist C^∞ (resp. continuous) map germs $h : (\mathbb{R}^n \times \mathbb{R}^{r(1)}, (0,0)) \to (\mathbb{R}^n \times \mathbb{R}^{r(2)}, (0,0)), H : (\mathbb{R}^p \times \mathbb{R}^{r(1)}, (0,0)) \to (\mathbb{R}^p \times \mathbb{R}^{r(2)}, (0,0))$ and $\phi : (\mathbb{R}^{r(1)}, 0) \to (\mathbb{R}^{r(2)}, 0)$ such that the following conditions (1.2.1) and (1.2.2) hold, then $\{h, H, \phi\}$ is a $\mathcal{A}$-morphism (resp. topological $\mathcal{A}$-morphism) from Ψ_1 to Ψ_2.*

(1.2.1) the restrictions $h|_{\mathbb{R}^n \times \{\lambda\}}$ and $H|_{\mathbb{R}^p \times \{\lambda\}}$
are C^∞ diffeomorphic (resp. homeomorphic)
for any $\lambda \in \mathbb{R}^{r(1)}$,

(1.2.2) the following diagram commutes:

$$\begin{array}{ccccc}
(\mathbb{R}^n \times \mathbb{R}^{r(1)}, (0,0)) & \xrightarrow{(\Psi_1,\pi_\lambda)} & (\mathbb{R}^p \times \mathbb{R}^{r(1)}, (0,0)) & \xrightarrow{\pi_\lambda} & (\mathbb{R}^{r(1)}, 0) \\
{\scriptstyle h}\downarrow & & {\scriptstyle H}\downarrow & & {\scriptstyle \phi}\downarrow \\
(\mathbb{R}^n \times \mathbb{R}^{r(2)}, (0,0)) & \xrightarrow{(\Psi_2,\pi_\lambda)} & (\mathbb{R}^p \times \mathbb{R}^{r(2)}, (0,0)) & \xrightarrow{\pi_\lambda} & (\mathbb{R}^{r(2)}, 0).
\end{array}$$

Let $\mathcal{G}$ be $\mathcal{K}$ or $\mathcal{A}$. A $\mathcal{G}$-morphism (resp. topological $\mathcal{G}$-morphism) $\{h, H, \phi\}$ from Ψ_1 toΨ_2 is said to be *equivalent* (resp. *topologically equivalent*) if ϕ is C^∞-diffeomorphic (resp. homeomorphic). Definitions of $\mathcal{G}$-morphism and equivalent $\mathcal{G}$-morphism are equivalent to those of Martinet's definitions ([**Mr**]); and definitions of topological $\mathcal{G}$-morphism and topologically equivalent topological $\mathcal{G}$-morphism are topological analogues of these. If there exists an equivalent $\mathcal{A}$-morphism (resp. topologically equivalent topological $\mathcal{A}$-morphism) from a given deformation $\Psi : (\mathbb{R}^n \times \mathbb{R}^r, (0,0)) \to (\mathbb{R}^p, 0)$ to the trivial deformation $f : (\mathbb{R}^n \times \mathbb{R}^r, (0,0)) \to (\mathbb{R}^p, 0)$, then we say Ψ has a *triviality* (resp. *topological triviality*).

In this chapter, we show if there is a $\mathcal{A}$-morphism (resp. topological $\mathcal{A}$-morphism) from a given deformation $\Psi : (\mathbb{R}^n \times \mathbb{R}^r, (0,0)) \to (\mathbb{R}^p, 0)$ to the trivial deformation $f : (\mathbb{R}^n \times \mathbb{R}^r, (0,0)) \to (\mathbb{R}^p, 0)$, then we can directly construct a $\mathcal{K}$-morphism (resp. topological $\mathcal{K}$-morphism) from Ψ to the graph deformation.

Now suppose there exist C^∞ (resp. continuous) map germs $h : (\mathbb{R}^n \times \mathbb{R}^r, (0,0)) \to (\mathbb{R}^n \times \mathbb{R}^r, (0,0)), H : (\mathbb{R}^p \times \mathbb{R}^r, (0,0)) \to (\mathbb{R}^p \times \mathbb{R}^r, (0,0))$ and $\phi : (\mathbb{R}^r, 0) \to (\mathbb{R}^r, 0)$ such that the following (1.3.1) and (1.3.2) hold:

(1.3.1) the restrictions $h|_{\mathbb{R}^n \times \{\lambda\}}$ and $H|_{\mathbb{R}^p \times \{\lambda\}}$ are C^∞ diffeomorphic (resp. homeomorphic) for any $\lambda \in \mathbb{R}^r$,

(1.3.2) the following diagram commutes:

$$\begin{array}{ccccc}
(\mathbb{R}^n \times \mathbb{R}^r, (0,0)) & \xrightarrow{(\Psi, \pi_\lambda)} & (\mathbb{R}^p \times \mathbb{R}^r, (0,0)) & \xrightarrow{\pi_\lambda} & (\mathbb{R}^r, 0) \\
{\scriptstyle h}\downarrow & & {\scriptstyle H}\downarrow & & {\scriptstyle \phi}\downarrow \\
(\mathbb{R}^n \times \mathbb{R}^r, (0,0)) & \xrightarrow{(f, \pi_\lambda)} & (\mathbb{R}^p \times \mathbb{R}^r, (0,0)) & \xrightarrow{\pi_\lambda} & (\mathbb{R}^r, 0).
\end{array}$$

By (1.3.2), we can write

$$h = (h_1, \phi) \quad \text{and} \quad H = (H_1, \phi).$$

Then, set $\phi'_H : (\mathbb{R}^r, 0) \to (\mathbb{R}^p, 0)$ as

$$\phi'_H(\lambda) = H_1(0, \lambda).$$

Also, set $h' : (\mathbb{R}^n \times \mathbb{R}^r, (0,0)) \to (\mathbb{R}^n \times \mathbb{R}^p, (0,0))$ as

$$h'(x, \lambda) = (h_1(x, \lambda), \phi'_H(\lambda))$$

and set $H' : (\mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R}^p, (0,0,0)) \to (\mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^p, (0,0,0))$ as

$$H'(x, \lambda, y) = (h'(x, \lambda), H_1(y, \lambda) - H_1(0, \lambda)).$$

Then we have

(1.4.0) h' and H' are C^∞ (resp. continuous) map germs,

(1.4.1) the restrictions $h'|_{\mathbb{R}^n \times \{\lambda\}}$ and $H'|_{\mathbb{R}^n \times \{\lambda\} \times \mathbb{R}^p}$ are C^∞ diffeomorphic (resp. homeomorphic) for any $\lambda \in \mathbb{R}^r$, (1.4.2) $$H'(\mathbb{R}^n \times \mathbb{R}^r \times \{0\}) \subset \mathbb{R}^n \times \mathbb{R}^p \times \{0\},$$

(1.4.3) the following diagram commutes:

$$\begin{array}{ccccc}
(\mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R}^p, (0,0,0)) & \xrightarrow{\pi_{x,\lambda}} & (\mathbb{R}^n \times \mathbb{R}^r, (0,0)) & \xrightarrow{\pi_\lambda} & (\mathbb{R}^r, 0) \\
{\scriptstyle H'}\downarrow & & {\scriptstyle h'}\downarrow & & {\scriptstyle \phi'_H}\downarrow \\
(\mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^p, (0,0,0)) & \xrightarrow{\pi_{x,y}} & (\mathbb{R}^n \times \mathbb{R}^p, (0,0)) & \xrightarrow{\pi_y} & (\mathbb{R}^p, 0).
\end{array}$$

Next, we set $F : (\mathbb{R}^n \times \mathbb{R}^p, (0,0)) \to (\mathbb{R}^p, 0)$ as

$$F(x, y) = f(x) - y.$$

We call $F : (\mathbb{R}^n \times \mathbb{R}^p, (0,0)) \to (\mathbb{R}^p, 0)$ *the graph deformation* of $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$.

Then, we can see

$$\begin{aligned}
F(h'(x,\lambda)) &= F(h_1(x,\lambda), \phi'_H(\lambda)) && \text{(definition of } h'\text{)} \\
&= f(h_1(x,\lambda)) - \phi'_H(\lambda) && \text{(definition of } F\text{)} \\
&= H_1(\Psi(x,\lambda),\lambda) - \phi'_H(\lambda) && \text{(1.3.2)} \\
&= H_1(\Psi(x,\lambda),\lambda) - H_1(0,\lambda) && \text{(definition of } \phi'_H\text{).}
\end{aligned}$$

Hence, we have

(1.4.4) the following diagram also commutes:

$$\begin{array}{ccc}
(\mathbb{R}^n \times \mathbb{R}^r, (0,0)) & \xrightarrow{(\pi_{x,\lambda}, \Psi)} & (\mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R}^p, (0,0,0)) \\
{\scriptstyle h'}\downarrow & & {\scriptstyle H'}\downarrow \\
(\mathbb{R}^n \times \mathbb{R}^p, (0,0)) & \xrightarrow{(\pi_{x,y}, F)} & (\mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^p, (0,0,0)).
\end{array}$$

Therefore, $\{h', H', \phi'_H\}$ is a $\mathcal{K}$-morphism (resp. topological $\mathcal{K}$-morphism) from the given deformation Ψ to the graph deformation F.

In particular, by (1.4.2) and (1.4.4) we have

(1.4.5) $$h'(\Psi^{-1}(0)) \subset F^{-1}(0).$$

Furthermore, by (1.4.1) - (1.4.4) and the remark after definition (1.1) we have

(1.4.6) $$h'(\mathbb{R}^n \times \mathbb{R}^r - \Psi^{-1}(0)) \subset \mathbb{R}^n \times \mathbb{R}^p - F^{-1}(0).$$

For the proofs of theorems A, D, E, we need only the properties (1.4.1), (1.4.5) and (1.4.6) (see §. 2).

EXAMPLE (1.5): For any C^∞ map germ $f:(\mathbb{R}^n,0)\to(\mathbb{R}^p,0)$,
(1) let $\Psi_1:(\mathbb{R}^n\times\mathbb{R}^p,(0,0))\to(\mathbb{R}^p,0)$ be its C^∞ deformation of the form $\Psi_1(x,\lambda)=f(x)+\lambda$. Then , $\{h(x,\lambda)=(x,\lambda), H(y,\lambda)=(y-\lambda,\lambda)$ and $\phi(\lambda)=\lambda\}$ gives a triviality of Ψ_1. In this case, $\phi'_H(\lambda)=-\lambda$, $h'(x,\lambda)=(x,-\lambda)$ and $H'(x,\lambda,y)=(x,-\lambda,y)$ as we expect. Of course, $\{h', H', \phi'_H\}$ is an equivalent $\mathcal{K}$-morphism from Ψ_1 to F .
(2) let $\Psi_2:(\mathbb{R}^n\times\mathbb{R}^p,(0,0))\to(\mathbb{R}^p,0)$ be the deformation of f of the form $\Psi_2(x,\lambda)=f(x)-\lambda^3$; where $\lambda^3=(\lambda_1^3,\dots,\lambda_p^3)$. Then $\{h(x,\lambda)=(x,\lambda)$, $H(y,\lambda)=(y+\lambda^3,\lambda)$ and $\phi(\lambda)=\lambda\}$ gives a topological triviality of Ψ_2. In this case, $\phi'_H(\lambda)=\lambda^3$, $h'(x,\lambda)=(x,\lambda^3)$ and $H'(x,\lambda,y)=(x,\lambda^3,y)$. We see $\{h', H', \phi'_H\}$ is a topologically equivalent topological $\mathcal{K}$-morphism from Ψ_2 to F.

DEFINITION (1.6). *Let $f:(\mathbb{R}^n,0)\to(\mathbb{R}^p,0)$ be a C^∞ map germ and let $\Psi:(\mathbb{R}^n\times\mathbb{R}^r,(0,0))\to(\mathbb{R}^p,0)$ be a C^∞ deformation of f. We say Ψ is $\mathcal{K}$-versal (resp. topologically $\mathcal{K}$-versal) if for any C^∞ deformation $\widetilde{\Psi}:(\mathbb{R}^n\times\mathbb{R}^t,(0,0))\to(\mathbb{R}^p,0)$ of f there exist C^∞ (resp. continuous) map germs $h:(\mathbb{R}^n\times\mathbb{R}^t,(0,0))\to(\mathbb{R}^n\times\mathbb{R}^r,(0,0))$, $H:(\mathbb{R}^n\times\mathbb{R}^t\times\mathbb{R}^p,(0,0))\to(\mathbb{R}^n\times\mathbb{R}^r\times\mathbb{R}^p,(0,0))$ and $\phi:(\mathbb{R}^t,0)\to(\mathbb{R}^r,0)$ which give a $\mathcal{K}$-morphism (resp. topological $\mathcal{K}$-morphism) from $\widetilde{\Psi}$ to Ψ.*

We can define $\mathcal{A}$-versality and topological $\mathcal{A}$-versality similarly. Let $\mathcal{G}$ be $\mathcal{K}$ or $\mathcal{A}$. The definition of $\mathcal{G}$-versality is equivalent to that of Martinet's definitions ([**Mr**]); and the definition of topological $\mathcal{G}$-versality is its topological analogue.

Since any C^∞ stable map germ is ,when viewed as a C^∞ deformation of itself, $\mathcal{A}$-versal: i.e. any C^∞ deformation Ψ of a C^∞ stabe map germ has a triviality; by the above argument we see

THEOREM B (MARTINET([**Mr**]). *For any C^∞ stable map germ $f:(\mathbb{R}^n,0)\to(\mathbb{R}^p,0)$, its graph deformation $F(x,y)=f(x)-y$ is $\mathcal{K}$-versal.*

There are several definitions for topological stable map germs (for instance, [**dW**]). However, it is well-known that for any MT-stable map germ (map germ multi-transversal to Thom-Mather canonical stratification) any C^∞ deformation of it has a topological triviality (see [**M3**] or [**GWdL**]). Hence, again by the above argument, we see

THEOREM C. *For any MT-stable map germ $f:(\mathbb{R}^n,0)\to(\mathbb{R}^p,0)$, its graph deformation $F(x,y)=f(x)-y$ is topologically $\mathcal{K}$-versal.*

§2. SPECIAL CASE OF §1

In this chapter, we review a part of Martinet's argument in [**Mr**]. Let f,g :

$(\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ be C^∞ map germs. Suppose there exist a C^∞ diffeomorphic (resp. homeomorphic) map germ $s : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ and a C^∞ map germ

$$M(x) = (\mathbf{m}_1(x), \ldots, \mathbf{m}_p(x)) : (\mathbb{R}^n, 0) \to (GL(p, \mathbb{R}), M(0))$$

such that $f(x) = M(x)(g \circ s)(x)$.

We set a C^∞ map germ $\Phi : (\mathbb{R}^n \times \mathbb{R}^p, (0,0)) \to (\mathbb{R}^p, 0)$ as

$$\begin{aligned}\Phi(x, y) &= M(x)((g \circ s)(x) - y) \\ &= f(x) - M(x)y.\end{aligned}$$

Hereafter, we concentrate on studying deformatins of this type. Hence, in particular, we assume $r = p$. We treat two kinds of p-dimensional euclidean space $\mathbb{R}^p$. When we are considering $\mathbb{R}^p$ as the target space, we write it $\mathbb{R}^p_y$. When we are considering $\mathbb{R}^p$ as the parameter space, we write it $\mathbb{R}^p_\lambda$.

Now suppose there exist C^∞-diffeomorphic (resp. homeomorphic) map germs $h : (\mathbb{R}^n \times \mathbb{R}^p_\lambda, (0,0)) \to (\mathbb{R}^n \times \mathbb{R}^p_\lambda, (0,0)), H : (\mathbb{R}^p_y \times \mathbb{R}^p_\lambda, (0,0)) \to (\mathbb{R}^p_y \times \mathbb{R}^p_\lambda, (0,0))$ and $\phi : (\mathbb{R}^p_\lambda, 0) \to (\mathbb{R}^p_\lambda, 0)$ such that the following diagram commutes:

$$\begin{array}{ccccc}
(\mathbb{R}^n \times \mathbb{R}^p_\lambda, (0,0)) & \xrightarrow{(\Phi, \pi_\lambda)} & (\mathbb{R}^p_y \times \mathbb{R}^p_\lambda, (0,0)) & \xrightarrow{\pi_\lambda} & (\mathbb{R}^p_\lambda, 0) \\
{\scriptstyle h}\downarrow & & {\scriptstyle H}\downarrow & & {\scriptstyle \phi}\downarrow \\
(\mathbb{R}^n \times \mathbb{R}^p_\lambda, (0,0)) & \xrightarrow{(f, \pi_\lambda)} & (\mathbb{R}^p_y \times \mathbb{R}^p_\lambda, (0,0)) & \xrightarrow{\pi_\lambda} & (\mathbb{R}^p_\lambda, 0)
\end{array}$$

In §1, we defined C^∞ (resp. continuous) map germs

$$\begin{aligned}\phi'_H &: (\mathbb{R}^p_\lambda, 0) \to (\mathbb{R}^p_y, 0) \\ h' &: (\mathbb{R}^n \times \mathbb{R}^p_\lambda, (0,0)) \to (\mathbb{R}^n \times \mathbb{R}^p_y, (0,0)) \\ H' &: (\mathbb{R}^n \times \mathbb{R}^p_\lambda \times \mathbb{R}^p_y, (0,0,0)) \to (\mathbb{R}^n \times \mathbb{R}^p_y \times \mathbb{R}^p_y, (0,0,0))\end{aligned}$$

and we saw $\{h', H', \phi'_H\}$ is a $\mathcal{K}$-morphism from Φ to F. By (1.4.5) in §1 and by the form of Φ, we have

$$f(h_1(x, (g \circ s)(x)) = \phi'_H((g \circ s)(x)) \tag{2.1}$$

as germs at the origin.

We would like to show the following map germ (2.2) is C^∞ diffeomorphic (resp. homeomorphic) if we assume ϕ'_H is C^∞ diffeomorphic (resp. homeomorphic).

$$x \longmapsto h_1(x, (g \circ s)(x)) \tag{2.2}$$

The map germ (2.2) can be decomposed as follows.

$$x \mapsto (x, (g \circ s)(x)) \mapsto h'(x, (g \circ s)(x)) \mapsto h_1(x, (g \circ s)(x)). \tag{2.3}$$

The first map germ of (2.3) is trivially C^∞ diffeomorphic. If we assume ϕ'_H is C^∞ diffeomorphic (resp. homeomorphic), then by (1.4.1) in §1 $h' = (h_1, \phi'_H)$ is C^∞ diffeomorphic (resp. homeomorphic). Thus, the second map germ of (2.3) is C^∞ diffeomorphic (resp. homeomorphic). Furthermore, in the case that we assume ϕ'_H is C^∞ diffeomorphic (resp. homeomorphic), by (1.4.5) and (1.4.6) in §1 we have

$$h'(\Phi^{-1}(0)) = F^{-1}(0). \tag{2.4}$$

By the form of Φ and F, (2.4) means

$$\begin{aligned} &\text{the germ of the set } \{h'(x, (g \circ s)(x)) | x \in \mathbb{R}^n\} \\ &= \text{the germ of } F^{-1}(0) \\ &= \text{graph}(f). \end{aligned} \tag{2.5}$$

By (2.5) and by the form of $h' = (h_1, \phi'_H)$, the last map germ of (2.3) is also C^∞ diffeomorphic.

Therefore, we see

LEMMA (2.6). *If Φ has a triviality (resp. topological triviality) and ϕ'_H is C^∞ diffeomorphic (resp. homeomorphic), then f and g are right-left equivalent (resp. topologically right-left equivalent).*

§3. MODULE

Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ be a C^∞ map germ and let

$$M(x) = (\mathbf{m}_1(x), \ldots, \mathbf{m}_p(x)) : (\mathbb{R}^n, 0) \to (GL(p, \mathbb{R}), M(0))$$

be also a C^∞ map germ. Let $\Phi : (\mathbb{R}^n \times \mathbb{R}^p_\lambda, (0,0)) \to (\mathbb{R}^p_y, 0)$ be the C^∞ deformation of f having the following form:

$$\Phi(x, \lambda) = f(x) - M(x)\lambda.$$

In this chapter, we prove the following lemma.

LEMMA (3.1). *Suppose there exists a positive integer k such that*

$$\mathbf{m}_i(x) - \mathbf{m}_i(0) \in \mathfrak{m}_x^k \mathcal{E}_x^p \subset tf(\mathfrak{m}_x \mathcal{E}_x^n) + \omega f(\mathfrak{m}_y \mathcal{E}_y^p)$$

for any i $(1 \le i \le p)$. Then $\mathbf{m}_i(x) - \mathbf{m}_i(0)$ is included in

$$t\Phi_x(\mathfrak{m}_{x,\lambda}\mathcal{E}^n_{x,\lambda}) + \omega(\Phi, \pi_\lambda)(\mathfrak{m}_{y,\lambda}\mathcal{E}^p_{y,\lambda})$$

for any i $(1 \le i \le p)$.

PROOF OF LEMMA (3.1): Since we assumed

$$\mathfrak{m}^k_x\mathcal{E}^p_x \subset tf(\mathfrak{m}_x\mathcal{E}^n_x) + \omega f(\mathfrak{m}_y\mathcal{E}^p_y),$$

by Malgrange preparation theorem we have

$$\mathfrak{m}^k_x\mathcal{E}^p_{x,\lambda} \subset tf(\mathfrak{m}_x\mathcal{E}^n_{x,\lambda}) + \omega(f, \pi_\lambda)(\mathfrak{m}_y\mathcal{E}^p_{y,\lambda}). \tag{3.2}$$

We set $\widetilde{\Phi} : (\mathbb{R}^n \times \mathbb{R}^p_\lambda, (0,0)) \to (\mathbb{R}^p_y, 0)$ as

$$\begin{aligned}\widetilde{\Phi}(x,\lambda) &= \Phi(x,\lambda) + M(0)\lambda \\ &= f(x) - (M(x) - M(0))\lambda.\end{aligned}$$

Since we assumed

$$\mathbf{m}_i(x) - \mathbf{m}_i(0) \in \mathfrak{m}^k_x\mathcal{E}^p_x$$

for any i $(1 \le i \le p)$, the difference

$$\widetilde{\Phi}(x,\lambda) - f(x) = (M(x) - M(0))\lambda = \sum_{i=1}^{p} \lambda_i(\mathbf{m}_i(x) - \mathbf{m}_i(0))$$

is included in

$$\pi^*_\lambda \mathfrak{m}_\lambda \mathfrak{m}^k_x\mathcal{E}^p_{x,\lambda} \subset (\widetilde{\Phi}, \pi_\lambda)^* \mathfrak{m}_{y,\lambda}\mathfrak{m}^k_x\mathcal{E}^p_{x,\lambda}.$$

Hence, we can approximate (3.2) as follows.

(3.3)

$$\begin{aligned}\mathfrak{m}^k_x\mathcal{E}^p_{x,\lambda} &\subset t\widetilde{\Phi}_x(\mathfrak{m}_x\mathcal{E}^n_{x,\lambda}) + \omega(\widetilde{\Phi}, \pi_\lambda)(\mathfrak{m}_y\mathcal{E}^p_{y,\lambda}) + (\widetilde{\Phi}, \pi_\lambda)^* \mathfrak{m}_{y,\lambda}\mathfrak{m}^k_x\mathcal{E}^p_{x,\lambda} \\ &\subset t\widetilde{\Phi}_x(\mathfrak{m}_{x,\lambda}\mathcal{E}^n_{x,\lambda}) + \omega(\widetilde{\Phi}, \pi_\lambda)(\mathfrak{m}_{y,\lambda}\mathcal{E}^p_{y,\lambda}) + (\widetilde{\Phi}, \pi_\lambda)^* \mathfrak{m}_{y,\lambda}\mathfrak{m}^k_x\mathcal{E}^p_{x,\lambda}.\end{aligned}$$

We set

$C = \mathcal{E}^p_{x,\lambda}/t\widetilde{\Phi}_x(\mathfrak{m}_{x,\lambda}\mathcal{E}^n_{x,\lambda})$,
$A =$ image of $\omega(\widetilde{\Phi}, \pi_\lambda)(\mathfrak{m}_{y,\lambda}\mathcal{E}^p_{y,\lambda})$ by the canonical projection to C,
$B = \mathfrak{m}^k_x.C$.

Then, by (3.3) we have

$$B \subset A + (\widetilde{\Phi}, \pi_\lambda)^* \mathfrak{m}_{y,\lambda}B. \tag{3.4}$$

Since

$$\begin{aligned}&\dim_{\mathbb{R}} B/(\widetilde{\Phi},\pi_\lambda)^* \mathfrak{m}_{y,\lambda} B\\ &=\dim_{\mathbb{R}} \mathfrak{m}_x^k \mathcal{E}_x^p/\mathfrak{m}_x^k(tf(\mathfrak{m}_x\mathcal{E}_x^n)+f^*\mathfrak{m}_y\mathcal{E}_x^p)<\infty,\end{aligned}$$

by Malgrange preparation theorem we see B is finitely generated $\mathcal{E}_{y,\lambda}$-module via $(\widetilde{\Phi},\pi_\lambda)$. Hence, by Nakayama's lemma (3.4) implies

$$B \subset A \tag{3.5}$$

From the form $\widetilde{\Phi}(x,\lambda)=\Phi(x,\lambda)+M(0)\lambda$, we see

$$\begin{aligned}&t\widetilde{\Phi}_x(\mathfrak{m}_{x,\lambda}\mathcal{E}_{x,\lambda}^n)+\omega(\widetilde{\Phi},\pi_\lambda)(\mathfrak{m}_{y,\lambda}\mathcal{E}_{y,\lambda}^p)\\ &=t\Phi_x(\mathfrak{m}_{x,\lambda}\mathcal{E}_{x,\lambda}^n)+\omega(\Phi,\pi_\lambda)(\mathfrak{m}_{y,\lambda}\mathcal{E}_{y,\lambda}^p)\end{aligned} \tag{3.6}$$

(3.5) and (3.6) yields

$$\mathbf{m}_i(x)-\mathbf{m}_i(0)\in \mathfrak{m}_x^k\mathcal{E}_{x,\lambda}^p \subset t\Phi_x(\mathfrak{m}_{x,\lambda}\mathcal{E}_{x,\lambda}^n)+\omega(\Phi,\pi_\lambda)(\mathfrak{m}_{y,\lambda}\mathcal{E}_{y,\lambda}^p)$$

for any i $(1\le i\le p)$. ∎

§4. Proof of theorem A

Let $\Phi:(\mathbb{R}^n\times\mathbb{R}_\lambda^p,(0,0))\to(\mathbb{R}_y^p,0)$ be the C^∞ deformation of f having the following form:

$$\Phi(x,\lambda)=f(x)-M(x)\lambda.$$

Since

$$\frac{\partial\Phi}{\partial\lambda_i}=-\mathbf{m}_i(x)$$

for any i $(1\le i\le p)$, by lemma (3.1) we can choose germs of C^∞ vector fields

$$\xi_i\in\mathcal{E}_{x,\lambda}^n \qquad \text{and} \qquad \eta_i\in\mathcal{E}_{y,\lambda}^p$$

such that

$$-\frac{\partial\Phi}{\partial\lambda_i}=\xi_i(\Phi)-\eta_i\circ(\Phi,\pi_\lambda) \tag{4.1}$$

$$\frac{\partial\Phi}{\partial\lambda_i}(0)=\eta_i(0,0) \tag{4.2}$$

for any i $(1\le i\le p)$.

By (4.1), integrating germs of C^∞ vector fields

$$\xi_1 + \partial/\partial\lambda_1, \ldots, \xi_p + \partial/\partial\lambda_p$$

and

$$\eta_1 + \partial/\partial\lambda_1, \ldots, \eta_p + \partial/\partial\lambda_p$$

yields C^∞ diffeomorphic map germs

$$h^{-1} : (\mathbb{R}^n \times \mathbb{R}^p_\lambda, (0,0)) \to (\mathbb{R}^n \times \mathbb{R}^p_\lambda, (0,0))$$

and

$$H^{-1} : (\mathbb{R}^p_y \times \mathbb{R}^p_\lambda, (0,0)) \to (\mathbb{R}^p_y \times \mathbb{R}^p_\lambda, (0,0))$$

such that the following diagram commutes.

$$\begin{array}{ccccc}
(\mathbb{R}^n \times \mathbb{R}^p_\lambda, (0,0)) & \xrightarrow{(\Phi,\pi_\lambda)} & (\mathbb{R}^p_y \times \mathbb{R}^p_\lambda, (0,0)) & \xrightarrow{\pi_\lambda} & (\mathbb{R}^p_\lambda, 0) \\
{\scriptstyle h^{-1}}\uparrow & & {\scriptstyle H^{-1}}\uparrow & & \| \\
(\mathbb{R}^n \times \mathbb{R}^p_\lambda, (0,0)) & \xrightarrow{(f,\pi_\lambda)} & (\mathbb{R}^p_y \times \mathbb{R}^p_\lambda, (0,0)) & \xrightarrow{\pi_\lambda} & (\mathbb{R}^p_\lambda, 0)
\end{array}$$

Consider the inverse map germ H of H^{-1} and

$$\phi'_H : (\mathbb{R}^p_\lambda, 0) \to (\mathbb{R}^p_y, 0)$$

associated with H.

Let $\Theta_i(t; \mathbf{y})$ be the integral curve of η_i starting from $\mathbf{y}$ and of time t. Then we can get the image $\mathbf{y}(\lambda_1, \ldots, \lambda_p) = \phi'_H(\lambda_1, \ldots, \lambda_p)$ of $\lambda = (\lambda_1, \ldots, \lambda_p)$ by ϕ'_H as the unique solution of the integral equation

$$\Theta_1(\lambda_1; \Theta_2(\lambda_2; \ldots; \Theta_p(\lambda_p; \mathbf{y}(\lambda_1, \ldots, \lambda_p))\ldots) = \mathbf{0}. \tag{4.3}$$

We differentiate (4.3) with respect to λ_i. Then we get

$$\begin{aligned} &\eta_i(\Theta_{i+1}(\lambda_{i+1}; \ldots; \Theta_p(\lambda_p; \mathbf{y}))\ldots) \\ &+(d\Theta_1)_y \ldots (d\Theta_p)_y \partial\mathbf{y}(\lambda_1, \ldots, \lambda_p)/\partial\lambda_i = \mathbf{0} \end{aligned} \tag{4.4}$$

for any i $(1 \le i \le p)$.

Taking values at $\lambda = 0$ in (4.4), we get

$$\begin{aligned} \frac{\partial\phi'_H}{\partial\lambda_i}(0) &= \frac{\partial\mathbf{y}}{\partial\lambda_i}(0) \\ &= -\eta_i(0,0) \qquad (\mathbf{y}(0,\ldots,0) = \mathbf{0}) \\ &= -\frac{\partial\Phi}{\partial\lambda_i}(0) \qquad (4.2) \\ &= \mathbf{m}_i(0). \qquad (\frac{\partial\Phi}{\partial\lambda_i} = -\mathbf{m}_i) \end{aligned}$$

Since $(\mathbf{m}_1(0), \dots, \mathbf{m}_p(0)) = M(0)$ is in $GL(p, \mathbb{R})$, ϕ'_H is C^∞ diffeomorphic. Hence, by lemma (2.3), f and g are right-left equivalent. ∎

§5. An alternative proof of Mather's classification theorem

In this chapter, we give a proof of the following theorem D which is a generalized version of Mather's classification theorem.

Definition (5.1). *Let X be a Banach space. We say a C^∞ map germ $f : (X, 0) \to (\mathbb{R}^p, 0)$ is C^∞ stable if for any finite dimensional C^∞ deformation $\Phi : (X \times \mathbb{R}^r, (0,0)) \to (\mathbb{R}^p, 0)$ of f there exist C^∞ diffeomorphic map germs*

$$h : (X \times \mathbb{R}^r, (0,0)) \to (X \times \mathbb{R}^r, (0,0))$$
$$H : (\mathbb{R}^p \times \mathbb{R}^r, (0,0)) \to (\mathbb{R}^p \times \mathbb{R}^r, (0,0))$$
$$\phi : (\mathbb{R}^r, 0) \to (\mathbb{R}^r, 0)$$

such that the following diagram commutes:

$$\begin{array}{ccccc} (X \times \mathbb{R}^r, (0,0)) & \xrightarrow{(\Phi, \pi_\lambda)} & (\mathbb{R}^p \times \mathbb{R}^r, (0,0)) & \xrightarrow{\pi_\lambda} & (\mathbb{R}^r, 0) \\ {\scriptstyle h}\downarrow & & {\scriptstyle H}\downarrow & & {\scriptstyle \phi}\downarrow \\ (X \times \mathbb{R}^r, (0,0)) & \xrightarrow{(f, \pi_\lambda)} & (\mathbb{R}^p \times \mathbb{R}^r, (0,0)) & \xrightarrow{\pi_\lambda} & (\mathbb{R}^r, 0). \end{array}$$

Theorem D. *Let X be a Banach space. Let $f, g : (X, 0) \to (\mathbb{R}^p, 0)$ be two C^∞ stable map germs. Suppose there exists a C^∞ diffeomorphic germ $s : (X, 0) \to (X, 0)$ and a C^∞ map germ*

$$M(x) = (\mathbf{m}_1(x), \dots, \mathbf{m}_p(x)) : (X, 0) \to (GL(p, \mathbb{R}), M(0))$$

such that $f(x) = M(x)(g \circ s)(x)$. Then f and g are isomorphic.

Mather's classification theorem ([**M2**]) is the case when X is finite dimensional.

Proof: Let $M_p(\mathbb{R})$ be the set of all $(p \times p)$ matrices of real elements and let E_p be the $(p \times p)$ unit matrix. For any fixed matrix $A = (\mathbf{a}_1, \dots, \mathbf{a}_p) \in M_p(\mathbb{R})$, define a map germ

$$\Phi_A : (X \times \mathbb{R}^p_\lambda \times M_p(\mathbb{R}), (0, 0, E_p)) \to (\mathbb{R}^p_y, 0)$$

as

$$\Phi_A(x, \lambda, B) = f(x) - (A + M(x)B)\lambda.$$

Then Φ_A is a finite dimensional C^∞ deformation of f. Since f is C^∞ stable, for any i $(1 \le i \le p)$ and $A = O$ (zero matrix) we see

$$\sum_{j=1}^{p} b_{ji}\mathbf{m}_j(x) = 0 + \sum_{j=1}^{p} b_{ji}\mathbf{m}_j(x) = \frac{\partial \Phi_0}{\partial \lambda_i}$$

is included in the set

$$t(\Phi_0)_x(\mathcal{E}^n_{x,\lambda,B} + \omega(\Phi_0, \pi_\lambda, \pi_B)(\mathcal{E}^p_{y,\lambda,B}).$$

Here we set $B = [b_{ij}]_{1 \le i,j \le p}$. Since we see trivially

$$t(\Phi_0)_x = t(\Phi_A)_x$$

and

$$\omega(\Phi_0, \pi_\lambda, \pi_B)(\mathcal{E}^p_{y,\lambda,B}) = \omega(\Phi_A, \pi_\lambda, \pi_B)(\mathcal{E}^p_{y,\lambda,B})$$

for any fixed $A \in M_p(\mathbb{R})$, we can choose germs of C^∞ vector fields

$$\xi_i \in \mathcal{E}^n_{x,\lambda,B} \qquad \text{and} \qquad \eta_{i,A} \in \mathcal{E}^p_{y,\lambda,B}$$

such that

$$\begin{aligned} -\frac{\partial \Phi_A}{\partial \lambda_i} &= (\mathbf{a}_i + \sum_{j=1}^{p} b_{ji}\mathbf{m}_j(x)) \\ &= \xi_i(\Phi_A) - \eta_{i,A} \circ (\Phi_A, \pi_\lambda, \pi_B) \end{aligned}$$

for any i $(1 \le i \le p)$. Since f is C^∞ stable, we can choose germs of C^∞ vector fields

$$\xi_{jk,A} \in \mathcal{E}^n_{x,\lambda,B} \qquad \text{and} \qquad \eta_{jk,A} \in \mathcal{E}^p_{y,\lambda,B}$$

such that

$$\begin{aligned} -\frac{\partial \Phi_A}{\partial b_{jk}} &= \lambda_k \mathbf{m}_j(x)) \\ &= \xi_{jk,A}(\Phi_A) - \eta_{jk,A} \circ (\Phi_A, \pi_\lambda, \pi_B) \end{aligned}$$

for any j, k $(1 \le j, k \le p)$ and any A of $M_p(\mathbb{R})$.

By integrating

$$\begin{gathered} \eta_{1,A} + \partial/\partial\lambda_1, \ldots, \eta_{p,A} + \partial/\partial\lambda_p, \\ \eta_{11,A} + \partial/\partial b_{11}, \ldots, \eta_{pp,A} + \partial/\partial b_{pp}, \end{gathered}$$

we get a C^∞ diffeomorphic germ

$$H_A^{-1} : (\mathbb{R}_y^p \times \mathbb{R}_\lambda^p \times M_p(\mathbb{R}), (0,0,E_p)) \to (\mathbb{R}_y^p \times \mathbb{R}_\lambda^p \times M_p(\mathbb{R}), (0,0,E_p)).$$

We consider the map germ

$$\phi'_{H_A} : (\mathbb{R}_\lambda^p \times M_p(\mathbb{R}), (0, E_p)) \to (\mathbb{R}_y^p, 0)$$

associated with H_A and its restriction

$$\phi'_{H_A}|_{\mathbb{R}_\lambda^p \times \{B\}}$$

for B sufficiently near the zero matrix.

Let $\Theta_{i,A}(t;\mathbf{y})$ (resp. $\Theta_{jk,A}(t;\mathbf{y})$) be the integral curve of $\eta_{i,A}$ (resp. $\eta_{jk,A}$) starting from $\mathbf{y}$ and of time t for any i,j,k $(1 \le i,j,k \le p)$. Then $\phi'_{H_A}(\lambda_1,\ldots,\lambda_p,b_{11},\ldots,b_{pp}) = \mathbf{y}$, where $\mathbf{y}$ is the unique solution of the following integral equation

$$\Theta_{1,A}(\lambda_1;\ldots;\Theta_{p,A}(\lambda_p;\Theta_{11,A}(b_{11};\ldots(\Theta_{pp,A}(b_{pp};\mathbf{y}(\lambda_1,\ldots,b_{11},\ldots,b_{pp}))\ldots) = \mathbf{0}.$$

We differentiate this equation with respect to λ_i. Then we have

$$\begin{aligned}(5.2)\qquad &\eta_{i,A}(\Theta_{i+1,A}(\lambda_{i+1};\ldots;\Theta_{pp,A}(b_{pp};\mathbf{y}))\ldots)\\ &+(d\Theta_{1,A})_y\ldots(d\Theta_{pp,A})_y\partial\mathbf{y}(\lambda_1,\ldots,\lambda_p,b_{11}\ldots,b_{pp})/\partial\lambda_i = \mathbf{0}\end{aligned}$$

for any i $(1 \le i \le p)$.
Taking values at $\lambda = 0$ and $B = E_p$ in (5.2), we get

$$\begin{aligned}(5.3)\qquad \frac{\partial\phi'_{H_A}}{\partial\lambda_i}(0,E_p) &= \frac{\partial\mathbf{y}}{\partial\lambda_i}(0,E_p)\\ &= -\eta_{i,A}(0,0,E_p)\\ &= -\frac{\partial\Phi_A}{\partial\lambda_i}(0,0,E_p) - d(\Phi_A)_x(\xi_i(0,0,E_p))\\ &= -\frac{\partial\Phi_A}{\partial\lambda_i}(0,0,E_p) - df_0(\xi_i(0,0,E_p))\\ &= \mathbf{a}_i + \mathbf{m}_i(0) - df_0(\xi_i(0,0,E_p))\end{aligned}$$

for any i $(1 \le i \le p)$. From (5.3) and since ξ_i is C^∞ with respect to $B = [b_{ij}]$, we have

LEMMA (5.4). *There exists an open dense subset $\mathcal{U}$ of $M_p(\mathbb{R})$ such that for any A of $\mathcal{U}$ there exists a neighborhood $\mathcal{V}_A$ of E_p in $M_p(\mathbb{R})$ such that the germ of the restriction*

$$\phi'_{H_A}|_{\mathbb{R}_\lambda^p \times \{B\}} : (\mathbb{R}_\lambda^p \times \{B\}, (0,B)) \to (\mathbb{R}_y^p, 0)$$

is C^∞ diffeomorphic for any B of $\mathcal{V}_A$.

Therefore, by lemmata (2.6) and (5.4), we have

Lemma (5.5). *If we choose $(p \times p)$ matrix A of $\mathcal{U}$ sufficiently near the zero matrix, then $f(x)$ and $g_{A,B}(x) = (A + M(x)B)^{-1}f(x)$ are right-left equivalent for any B of $\mathcal{V}_A$.*

Next, we take a matrix A_0 of $\mathcal{U}$ sufficiently near the zero matrix and fix it. We set

$$M(x)^{-1}A_0 = N_{A_0}(x) = (\mathbf{n}_1(x), \ldots, \mathbf{n}_p(x)).$$

For any fixed B of $\mathcal{V}_{A_0}$, we define the C^∞ map germ

$$\widetilde{\Phi}_{A_0,B} : (X \times \mathbb{R}^p_\lambda, (0,0)) \to (\mathbb{R}^p_y, 0)$$

as

$$\widetilde{\Phi}_{A_0,B}(x,\lambda) = (N_{A_0}(x) + B)(g_{A_0,B}(x) - \lambda).$$

Then, since

$$\begin{aligned}(g \circ s)(x) &= M(x)^{-1}(A_0 + M(x)B)(A_0 + M(x)B)^{-1}f(x)\\ &= M(x)^{-1}(A_0 + M(x)B)g_{A_0,B}(x)\\ &= (N_{A_0}(x) + B)g_{A_0,B}(x);\end{aligned}$$

we see $\widetilde{\Phi}_{A_0,B}(x,\lambda) = (g \circ s)(x) - (N_{A_0}(x) + B)\lambda$ is a C^∞ deformation of $(g \circ s)$. Since $(g \circ s)$ is C^∞ stable, for any i $(1 \le i \le p)$ and $B = E_p$ we see

$$\frac{\partial \widetilde{\Phi}_{A_0,E_p}}{\partial \lambda_i} \in t(\widetilde{\Phi}_{A_0,E_p})_x(\mathcal{E}^n_{x,\lambda}) + \omega(\widetilde{\Phi}_{A_0,E_p}, \pi_\lambda)(\mathcal{E}^p_{y,\lambda}).$$

Since

$$t(\widetilde{\Phi}_{A,E_p})_x = t(\widetilde{\Phi}_{A,B})_x$$

and

$$\omega(\widetilde{\Phi}_{A,E_p}, \pi_\lambda)(\mathcal{E}^p_{y,\lambda}) = \omega(\widetilde{\Phi}_{A,B}, \pi_\lambda)(\mathcal{E}^p_{y,\lambda})$$

for any $A \in \mathcal{U}$ and $B \in \mathcal{V}_A$, we can choose germs of C^∞ vector fields

$$\widetilde{\xi}_i \in \mathcal{E}^n_{x,\lambda} \qquad \text{and} \qquad \widetilde{\eta}_{i,B} \in \mathcal{E}^p_{y,\lambda}$$

such that

$$-\frac{\partial \widetilde{\Phi}_{A_0,B}}{\partial \lambda_i} = \widetilde{\xi}_i(\widetilde{\Phi}_{A_0,B}) - \widetilde{\eta}_{i,B} \circ (\widetilde{\Phi}_{A_0,B}, \pi_\lambda)$$

for any i $(1 \le i \le p)$.

By integrating

$$\widetilde{\eta}_{1,B} + \partial/\partial\lambda_1, \ldots, \widetilde{\eta}_{p,B} + \partial/\partial\lambda_p,$$

we get a C^∞ diffeomorphic germ

$$H^{-1}_{A_0,B} : (\mathbb{R}^p_y \times \mathbb{R}^p_\lambda, (0,0)) \to (\mathbb{R}^p_y \times \mathbb{R}^p_\lambda, (0,0)).$$

We consider the map germ

$$\phi'_{H_{A_0,B}} : (\mathbb{R}^p_\lambda, 0) \to (\mathbb{R}^p_y, 0)$$

associated with $H_{A_0,B}$. We see

$$\begin{aligned}
(5.6) \qquad \frac{\partial \phi'_{H_{A_0,B}}}{\partial \lambda_i}(0) &= -\tilde{\eta}_{i,B}(0,0) \\
&= -\frac{\partial \tilde{\Phi}_{A_0,B}}{\partial \lambda_i}(0) - d(\tilde{\Phi}_{A_0,B})_x(\tilde{\xi}_i(0,0)) \\
&= -\frac{\partial \tilde{\Phi}_{A_0,B}}{\partial \lambda_i}(0) - d(g \circ s)_0(\tilde{\xi}_i(0,0)) \\
&= \mathbf{n}_i(0) + \mathbf{b}_i - d(g \circ s)_0(\tilde{\xi}_i(0,0))
\end{aligned}$$

for any i $(1 \leq i \leq p)$. By (5.6), we can choose a matrix B of $\mathcal{V}_{A_0}$ with the property that $\phi'_{H_{A_0,B}}$ is C^∞ diffeomorphic. Thus, by lemma (2.6), we have

LEMMA (5.7). *We can choose a matrix B of $\mathcal{V}_{A_0}$ with the property that $(g \circ s)$ and $g_{A_0,B}$ are right-left equivalent.*

Lemmata (5.5) and (5.7) concludes that f and g are isomorphic. ∎

§6. AN ALTERNATIVE PROOF OF FUKUDA-FUKUDA'S THEOREM

In this chapter, we give a proof of the following theorem.

THEOREM E ([**FF**]). *Let $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ be two MT-stable map germs. Suppose there exists a C^∞ diffeomorphic germ $s : (\mathbb{R}^n, 0) \to (X, 0)$ and a C^∞ map germ*

$$M(x) = (\mathbf{m}_1(x), \dots, \mathbf{m}_p(x)) : (\mathbb{R}^n, 0) \to (GL(p, \mathbb{R}), M(0))$$

such that $f(x) = M(x)(g \circ s)(x)$. Then f and g are topologically isomorphic.

For the definition of MT-stable map germs, refer to [**M3**] or [**GWdL**]. For our proof of theorem E, we use only the following fact on MT-stable map germs.

FACT (6.1). *Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ be an MT-stable map germ. Then for any C^∞ deformation $\Phi : (\mathbb{R}^n \times \mathbb{R}^r, (0,0)) \to (\mathbb{R}^p, 0)$ of f there exist Whitney stratifications $\mathcal{S}$ of $\mathbb{R}^n \times \mathbb{R}^r$ and $\mathcal{T}$ of $\mathbb{R}^p \times \mathbb{R}^r$ such that the germ of the sequence*

$$(\mathbb{R}^n \times \mathbb{R}^r, (0,0)) \xrightarrow{(\Phi, \pi_\lambda)} (\mathbb{R}^p \times \mathbb{R}^r, (0,0)) \xrightarrow{\pi_\lambda} (\mathbb{R}^r, 0)$$

is a Thom sequence with respect to $\mathcal{S}, \mathcal{T}$ and $\{\mathbb{R}^r\}$.

PROOF OF THEOREM E: As in §5, for any fixed matrix $A = (\mathbf{a}_1, \dots, \mathbf{a}_p) \in M_p(\mathbb{R})$, define a map germ

$$\Psi_A : (X \times \mathbb{R}^p_\lambda \times M_p(\mathbb{R}), (0, 0, E_p)) \to (\mathbb{R}^p_y, 0)$$

as

$$\Psi_A(x, \lambda, B) = f(x) - (A + M(x)B)\lambda.$$

Then Ψ_A is a C^∞ deformation of f. Since f is MT-stable, by (6.1), there exist Whitney stratifications $\mathcal{S}$ of $\mathbb{R}^n \times \mathbb{R}^p_\lambda \times M_p(\mathbb{R})$ and $\mathcal{T}$ of $\mathbb{R}^p_y \times \mathbb{R}^p_\lambda \times M_p(\mathbb{R})$ such that the germ of the sequence

$$\begin{array}{c} (\mathbb{R}^n \times \mathbb{R}^p_\lambda \times M_p(\mathbb{R}), (0, 0, E_p)) \\ \big\downarrow {\scriptstyle (\Psi_0, \pi_{\lambda, B})} \\ (\mathbb{R}^p_y \times \mathbb{R}^p_\lambda \times M(\mathbb{R}), (0, 0, E_p)) \\ \big\downarrow {\scriptstyle \pi_{\lambda, B}} \\ (\mathbb{R}^p_\lambda \times M_p(\mathbb{R}), (0, E_p)) \end{array}$$

is a Thom sequence with respect to $\mathcal{S}, \mathcal{T}$ and $\{\mathbb{R}^p_\lambda \times M_p(\mathbb{R})\}$.

For any stratum T of $\mathcal{T}$ and any A of $M_p(\mathbb{R})$, we set

$$T_A = \{(y, \lambda, B) - (A\lambda, \lambda, B) \in \mathbb{R}^p_y \times \mathbb{R}^p_l ambda \times M_p(\mathbb{R}) | (y, \lambda, B) \in T\}$$

and

$$\mathcal{T}_A = \{T_A\}.$$

Then, since

$$\begin{aligned} \Psi_A(x, \lambda, B) &= \Psi_0(x, \lambda, B) - A\lambda \\ &= \Psi_0(x, \lambda, B) + (\text{family of parallel translation of } \mathbb{R}^p_y) \end{aligned}$$

we see

LEMMA (6.2). *For any matrix A of $M_p(\mathbb{R})$, the germ of the sequence*

$$\begin{array}{c}(\mathbb{R}^n \times \mathbb{R}^p_\lambda \times M_p(\mathbb{R}), (0,0,E_p)) \\ \big\downarrow {\scriptstyle (\Psi_A, \pi_{\lambda,B})} \\ (\mathbb{R}^p_y \times \mathbb{R}^p_\lambda \times M(\mathbb{R}), (0,0,E_p)) \\ \big\downarrow {\scriptstyle \pi_{\lambda,B}} \\ (\mathbb{R}^p_\lambda \times M_p(\mathbb{R}), (0,E_p))\end{array}$$

is a Thom sequence with respect to $\mathcal{S}, \mathcal{T}_A$ and $\{\mathbb{R}^p_\lambda \times M_p(\mathbb{R})\}$.

By (6.2) we see

LEMMA (6.3). *There exists an open dense subset $\mathcal{U}$ of $M_p(\mathbb{R})$ such that for any A of $\mathcal{U}$ there exists a neighborhood $\mathcal{V}_A$ of E_p in $M_p(\mathbb{R})$ such that the subset $\{0\} \times \mathbb{R}^p_\lambda \times \{B\} (\subset \mathbb{R}^p_y \times \mathbb{R}^p_\lambda \times M_p(\mathbb{R}))$ is transversal to the intersection $T_A \cap (\mathbb{R}^p_y \times \mathbb{R}^p_\lambda \times \{B\})$ near $\{0\} \times \{0\} \times \{0\}$ for any B of $\mathcal{V}_A$ and any T_A of $\mathcal{T}_A$.*

By lifting vector fields $\partial/\partial\lambda_1, \ldots, \partial/\partial\lambda_p, \partial/\partial b_{11}, \ldots, \partial/\partial b_{pp}$, we get germs of vector fields

$$\begin{gathered}\eta_{1,A} + \partial/\partial\lambda_1, \ldots, \eta_{p,A} + \partial/\partial\lambda_p, \\ \eta_{11,A} + \partial/\partial b_{11}, \ldots, \eta_{pp,A} + \partial/\partial b_{pp},\end{gathered}$$

which are stratified with respect to the stratification $\mathcal{T}_A$ and satisfy the control conditions. By integrating these stratified vector fields, we get a homeomorphic germ

$$H_A^{-1} : (\mathbb{R}^p_y \times \mathbb{R}^p_\lambda \times M_p(\mathbb{R}), (0,0,E_p)) \to (\mathbb{R}^p_y \times \mathbb{R}^p_\lambda \times M_p(\mathbb{R}), (0,0,E_p)).$$

We consider the map germ

$$\phi'_{H_A} : (\mathbb{R}^p_\lambda \times M_p(\mathbb{R}), (0,E_p)) \to (\mathbb{R}^p_y, 0)$$

associated with H_A and its restriction

$$\phi'_{H_A}|_{\mathbb{R}^p_\lambda \times \{B\}}$$

for B sufficiently near the zero matrix.

Let $\Theta_{i,A}(t; \mathbf{y})$ (resp. $\Theta_{jk,A}(t; \mathbf{y})$) be the integral curve of $\eta_{i,A}$ (resp. $\eta_{jk,A}$) starting from $\mathbf{y}$ and of time t for any i,j,k $(1 \le i,j,k \le p)$. Then

$\phi'_{H_A}(\lambda_1,\dots,\lambda_p,b_{11},\dots,b_{pp})$ can be given as the unique solution of the following integral equation

$$\Theta_{1,A}(\lambda_1;\dots;\Theta_{p,A}(\lambda_p;\Theta_{11,A}(b_{11};\dots(\Theta_{pp,A}(b_{pp};\phi'_{H_A}(\lambda_1,\dots,b_{pp}))\dots)=0.$$

Since the germs of vector fields

$$\eta_{1,A}+\partial/\partial\lambda_1,\dots,\eta_{p,A}+\partial/\partial\lambda_p,$$
$$\eta_{11,A}+\partial/\partial b_{11},\dots,\eta_{pp,A}+\partial/\partial b_{pp},$$

are controlled, by lemma (6.3), we see

LEMMA (6.4). *For any A of $\mathcal{U}$ and any B of $\mathcal{V}_A$, the germ of the restriction*

$$\phi'_{H_A}|_{\mathbb{R}^p_\lambda\times\{B\}}:(\mathbb{R}^p_\lambda\times\{B\},(0,B))\to(\mathbb{R}^p_y,0)$$

is injective.

Since $\phi'_{H_A}|_{\mathbb{R}^p_\lambda\times\{B\}}$ is continuous, injectivity means being homeomorphic. Therefore, by lemma (2.6) we have

LEMMA (6.5). *If we choose $(p\times p)$ matrix A of $\mathcal{U}$ sufficiently near the zero matrix, then $f(x)$ and $g_{A,B}(x)=(A+M(x)B)^{-1}f(x)$ are topologically right-left equivalent for any B of $\mathcal{V}_A$.*

Next, we take a matrix A_0 of $\mathcal{U}$ sufficiently near the zero matrix and fix it. We set

$$M(x)^{-1}A_0=N_{A_0}(x)=(\mathbf{n}_1(x),\dots,\mathbf{n}_p(x)).$$

For any fixed B of $\mathcal{V}_{A_0}$, we define the C^∞ map germ

$$\widetilde{\Psi}_{A_0,B}:(\mathbb{R}^n\times\mathbb{R}^p_\lambda,(0,0))\to(\mathbb{R}^p_y,0)$$

as

$$\widetilde{\Psi}_{A_0,B}(x,\lambda)=(N_{A_0}(x)+B)(g_{A_0,B}(x)-\lambda).$$

Then, since

$$\begin{aligned}(g\circ s)(x)&=M(x)^{-1}(A_0+M(x)B)(A_0+M(x)B)^{-1}f(x)\\&=M(x)^{-1}(A_0+M(x)B)g_{A_0,B}(x)\\&=(N_{A_0}(x)+B)g_{A_0,B}(x);\end{aligned}$$

we see $\widetilde{\Psi}_{A_0,B}(x,\lambda)=(g\circ s)(x)-(N_{A_0}(x)+B)\lambda$ is a C^∞ deformation of $(g\circ s)$. Since $g:(\mathbb{R}^n,0)\to(\mathbb{R}^p,0)$ is MT-stable and $s:(\mathbb{R}^n,0)\to(\mathbb{R}^n,0)$ is C^∞

diffeomorphic, $(g \circ s)(x) = M(x)^{-1}f(x)$ is MT-stable. Thus, by (6.1), there exist Whitney stratifications $\widetilde{\mathcal{S}}$ of $\mathbb{R}^n \times \mathbb{R}^p_\lambda$, $\widetilde{\mathcal{T}}$ of $\mathbb{R}^p_y \times \mathbb{R}^p_\lambda$ such that the germ of the sequence

$$(\mathbb{R}^n \times \mathbb{R}^p_\lambda, (0,0)) \xrightarrow{(\widetilde{\Psi}_{A_0,0}, \pi_\lambda)} (\mathbb{R}^p_y \times \mathbb{R}^p_\lambda, (0,0)) \xrightarrow{\pi_\lambda} (\mathbb{R}^p_\lambda, 0)$$

is a Thom sequence with respect to $\widetilde{\mathcal{S}}, \widetilde{\mathcal{T}}$ and $\{\mathbb{R}^p_\lambda\}$.

For any stratum $\widetilde{T}$ of $\widetilde{\mathcal{T}}$ and any B of $M_p(\mathbb{R})$, we set

$$\widetilde{T}_B = \{(y,\lambda) - (B\lambda, \lambda) \in \mathbb{R}^p_y \times \mathbb{R}^p_\lambda | (y,\lambda) \in \widetilde{T}\}$$

and

$$\widetilde{\mathcal{T}}_B = \{\widetilde{T}_B\}.$$

Then, since

$$\widetilde{\Psi}_{A_0,B}(x,\lambda) = \widetilde{\Psi}_{A_0,0}(x,\lambda) - B\lambda$$

we see

LEMMA (6.6). *For any matrix B of $M_p(\mathbb{R})$, the germ of the sequence*

$$(\mathbb{R}^n \times \mathbb{R}^p_\lambda, (0,0)) \xrightarrow{(\widetilde{\Psi}_{A_0,B}, \pi_\lambda)} (\mathbb{R}^p_y \times \mathbb{R}^p_\lambda, (0,0)) \xrightarrow{\pi_\lambda} (\mathbb{R}^p_\lambda, 0)$$

is a Thom sequence with respect to $\widetilde{\mathcal{S}}, \widetilde{\mathcal{T}}_B$ and $\{\mathbb{R}^p_\lambda\}$.

By lemma (6.6) and by using the same argument as before, we can choose a matrix B of $\mathcal{V}_{A_0}$ with the property that $\phi'_{H_{A_0,B}}$ is homeomorphic. Thus, by lemma (2.6), we have

LEMMA (6.7). *We can choose a matrix B of $\mathcal{V}_{A_0}$ with the property that $(g \circ s)$ and $g_{A_0,B}$ are topologically right-left equivalent.*

Lemmata (6.6) and (6.7) concludes that f and g are topologically isomorphic. ∎

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