Simple construction of parameter map germ and its applications

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In this note, we shall construct a simple parameter map germ \((\mathbb{R}^r, 0) \rightarrow (\mathbb{R}^p, 0)\) under the assumption that there is an \(A\)-morphism (resp. topological \(A\)-morphism) from a given deformation \(\Psi: (\mathbb{R}^n \times \mathbb{R}^r, (0, 0)) \rightarrow (\mathbb{R}^p, 0)\) of a given map germ \(f: (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0)\) to the trivial deformation \(f: (\mathbb{R}^n \times \mathbb{R}^r, (0, 0)) \rightarrow (\mathbb{R}^p, 0)\).

This parameter map germ induces a \(K\)-morphism (resp. topological \(K\)-morphism) from \(\Psi\) to the graph deformation of \(f\).

By this construction, we can prove the following:

**Theorem D ([M2]):** Let \(f, g: (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0)\) be two \(C^\infty\) stable map germs. Suppose there exist a \(C^\infty\) diffeomorphic germ \(s: (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)\) and a \(C^\infty\) map germ \(M: (\mathbb{R}^n, 0) \rightarrow (GL(p, \mathbb{R}), M(0))\) such that \(f(x) = M(x)(g \circ s)(x)\). Then \(f\) and \(g\) are right-left equivalent.

Though our method seems to be close to Martinet's one ([Mr]), we can treat also map germs which are not necessarily \(C^\infty\) stable.

**Theorem E ([FF]):** Let \(f, g: (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0)\) be two MT stable map germs. Suppose there exist a \(C^\infty\) diffeomorphic germ \(s: (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)\) and a \(C^\infty\) map germ \(M: (\mathbb{R}^n, 0) \rightarrow (GL(p, \mathbb{R}), M(0))\) such that \(f(x) = M(x)(g \circ s)(x)\). Then \(f\) and \(g\) are topologically right-left equivalent.

**Theorem A:** Let \(f, g: (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0)\) be two \(C^\infty\) map germs. Suppose there exist a \(C^\infty\) diffeomorphic germ \(s: (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)\) and a \(C^\infty\) map germ

\[M(x) = (m_1(z), \ldots, m_p(z)): (\mathbb{R}^n, 0) \rightarrow (GL(p, \mathbb{R}), M(0))\]

such that \(f(x) = M(x)(g \circ s)(x)\). Suppose furthermore there exists a positive integer \(k\) such that

\[m_i(x) - m_i(0) \in m_x^{k}E_x \subset tf(m_xE_x^n) + \omega f(m_xE_y^n)\]

for any \(i\) \((1 \leq i \leq p)\). Then \(f\) and \(g\) are right-left equivalent.

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As a corollary of theorem A, we get

**Corollary A**: Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ be a $C^\infty$ map germ. Suppose there exist positive integers $k, l$ such that

\[ m_x^k E^p_x \subset tf(m_x^k E^n_x) + \omega f(m_y E^p_y) \]

and

\[ m_x^l E^p_x \subset tf(m_x^{2k} E^n_x) + f^*(m_y^k E^p_y). \]

Then $f$ is $(l - 1)$-determined with respect to right-left equivalence.

Corollary A induces the following Gaffney type estimate of the order of determinacy (c.f. [G]).

**Corollary B**: Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ be a $C^\infty$ map germ. Suppose there exist positive integers $k, l$ such that

\[ m_x^k E^p_x \subset tf(m_x^k E^n_x) + \omega f(m_y E^p_y) \]

and

\[ m_x^l E^p_x \subset tf(m_x^2 E^n_x) + f^* m_y^k E^p_x. \]

Then $f$ is $(k + l - 1)$-determined with respect to right-left equivalence.

Corollary B induces the following du Plessis-Wall's estimate of the order of determinacy.

**Corollary C ([dP,W])**: Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ be a $C^\infty$ map germ. Suppose there exist a positive integer $k$ such that

\[ m_x^k E^p_x \subset tf(m_x^k E^n_x) + \omega f(m_y E^p_y). \]

Then $f$ is $(2k - 1)$-determined with respect to right-left equivalence.

In [W], we can find an estimate of the order of topological determinacy of an MT stable map germ (corollary D below) which is due to T. Gaffney, but without proof. By using of our method, we can give a proof of his estimate.

**Corollary D (Gaffney)**: Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ be an MT stable map germ. Suppose there exist a positive integer $k$ such that

\[ m_x^k E^p_x \subset tf(m_x^k E^n_x) + f^*(m_y^2 E^p_x). \]

Then $f$ is $k$-determined with respect to topologically right-left equivalence.

For details on these corollaries, refer to [N].

This note is organized in the following way. In §1 and §2, we give several preparations for the proofs of theorem A, a generalized version of Mather's
classification theorem (theorem D in §5) and the theorem of Fukuda-Fukuda (theorem E in §6). §3 treats algebraic argument which we need for the proof of theorem A. Theorem A will be proved in §4. A generalized version of Mather's classification theorem will be proved in §5. In §6, an alternative proof of the theorem of Fukuda-Fukuda will be given.

The results in this paper are all valid in the complex analytic category as well except example (1.5.2).

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§1. $\mathcal{K}$-MORPHISM FROM A GIVEN DEFORMATION TO THE GRAPH DEFORMATION

Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ be a $C^\infty$ map germ and $\Psi_i : (\mathbb{R}^n \times \mathbb{R}^{r(i)}, (0,0)) \to (\mathbb{R}^p, 0)$ be a $C^\infty$ deformation of $f$ (i.e. $\Psi_i(z, 0) = f(z)$) $(i = 1, 2)$.

DEFINITION (1.1). We say if there exist $C^\infty$ (resp. continuous) map germs $h : (\mathbb{R}^n \times \mathbb{R}^{r(1)}, (0,0)) \to (\mathbb{R}^n \times \mathbb{R}^{r(2)}, (0,0))$, $H : (\mathbb{R}^n \times \mathbb{R}^{r(1)} \times \mathbb{R}^{p}, (0,0,0)) \to (\mathbb{R}^n \times \mathbb{R}^{r(2)} \times \mathbb{R}^{p}, (0,0,0))$ and $\phi : (\mathbb{R}^{r(1)}, 0) \to (\mathbb{R}^{r(2)}, 0)$ such that the following conditions (1.1.1), (1.1.2), (1.1.3) and (1.1.4) hold, then $\{h, H, \phi\}$ is a $\mathcal{K}$-morphism (resp. topological $\mathcal{K}$-morphism) from $\Psi_1$ to $\Psi_2$.

(1.1.1) the restrictions $h|_{\mathbb{R}^n \times \{\lambda\}}$ and $H|_{\mathbb{R}^n \times \{\lambda\} \times \mathbb{R}^p}$ are $C^\infty$ diffeomorphic (resp. homeomorphic) for any $\lambda \in \mathbb{R}^{r(1)}$,

(1.1.2) $H(\mathbb{R}^n \times \mathbb{R}^{r(1)} \times \{0\}) \subset \mathbb{R}^n \times \mathbb{R}^{r(2)} \times \{0\}$,

(1.1.3) the following diagram commutes:

\[
\begin{array}{ccc}
(\mathbb{R}^n \times \mathbb{R}^{r(1)} \times \mathbb{R}^p, (0,0,0)) & \xrightarrow{\pi_{*, \lambda}} & (\mathbb{R}^n \times \mathbb{R}^{r(1)}, (0,0)) \\
H \downarrow & & h \downarrow & & \phi \downarrow \\
(\mathbb{R}^n \times \mathbb{R}^{r(2)} \times \mathbb{R}^p, (0,0,0)) & \xrightarrow{\pi_{*, \lambda}} & (\mathbb{R}^n \times \mathbb{R}^{r(2)}, (0,0)) \\
& & \pi_{\lambda} & & \\
& & (\mathbb{R}^{r(2)}, 0),
\end{array}
\]
the following diagram commutes:

\[
\begin{array}{ccc}
(R^n \times R^{r(1)}, (0, 0)) & \xrightarrow{(\pi_{x, \lambda}, \Psi_1)} & (R^n \times R^{(1)} \times R^p, (0, 0, 0)) \\
\downarrow h & & \downarrow H \\
(R^n \times R^{r(2)}, (0, 0)) & \xrightarrow{(\pi_{x, \lambda}, \Psi_2)} & (R^n \times R^{(2)} \times R^p, (0, 0, 0)).
\end{array}
\]

Here \(\pi_{x, \lambda}, \pi_{\lambda}\) mean the canonical projection to \(R^n \times R^{r(i)}, R^{(i)}\) respectively. We remark that the conditions (1.1.1), (1.1.2) and (1.1.3) in the definition (1.1) imply \(H(R^n \times R^{r(1)} \times (R^p - \{0\}) \subset R^n \times R^{r(2)} \times (R^p - \{0\})\); and the condition (1.1.4) implies \(H(\text{graph}(\Psi_1)) \subset \text{graph}(\Psi_2)\).

**Definition (1.2).** We say if there exist \(C^\infty\) (resp. continuous) map germs \(h : (R^n \times R^{r(1)}, (0, 0)) \to (R^n \times R^{r(2)}, (0, 0)), H : (R^p \times R^{r(1)}, (0, 0)) \to (R^p \times R^{r(2)}, (0, 0))\) and \(\phi : (R^{r(1)}, 0) \to (R^{r(2)}, 0)\) such that the following conditions (1.2.1) and (1.2.2) hold, then \(\{h, H, \phi\}\) is a \(A\)-morphism (resp. topological \(A\)-morphism) from \(\Psi_1\) to \(\Psi_2\).

\[(1.2.1)\] the restrictions \(h|_{R^n \times \{\lambda\}}\) and \(H|_{R^p \times \{\lambda\}}\) are \(C^\infty\) diffeomorphic (resp. homeomorphic) for any \(\lambda \in R^{r(1)},\)

\[(1.2.2)\] the following diagram commutes:

\[
\begin{array}{ccc}
(R^n \times R^{r(1)}, (0, 0)) & \xrightarrow{(\Psi_1, \pi_\lambda)} & (R^p \times R^{r(1)}, (0, 0)) & \xrightarrow{\pi_\lambda} & (R^{r(1)}, 0) \\
\downarrow h & & \downarrow H & & \downarrow \phi \\
(R^n \times R^{r(2)}, (0, 0)) & \xrightarrow{(\Psi_2, \pi_\lambda)} & (R^p \times R^{r(2)}, (0, 0)) & \xrightarrow{\pi_\lambda} & (R^{r(2)}, 0).
\end{array}
\]

Let \(G\) be \(K\) or \(A\). A \(G\)-morphism (resp. topological \(G\)-morphism) \(\{h, H, \phi\}\) from \(\Psi_1\) to \(\Psi_2\) is said to be equivalent (resp. topologically equivalent) if \(\phi\) is \(C^\infty\)-diffeomorphic (resp. homeomorphic). Definitions of \(G\)-morphism and equivalent \(G\)-morphism are equivalent to those of Martinet's definitions ([Mr]); and definitions of topological \(G\)-morphism and topologically equivalent topological \(G\)-morphism are topological analogues of these. If there exists an equivalent \(A\)-morphism (resp. topologically equivalent topological \(A\)-morphism) from a given deformation \(\Psi : (R^n \times R^r, (0, 0)) \to (R^p, 0)\) to the trivial deformation \(f : (R^n \times R^r, (0, 0)) \to (R^p, 0)\), then we say \(\Psi\) has a triviality (resp. topological triviality).
In this chapter, we show if there is a $\mathcal{A}$-morphism (resp. topological $\mathcal{A}$-morphism) from a given deformation $\Psi : (\mathbb{R}^n \times \mathbb{R}^r, (0, 0)) \rightarrow (\mathbb{R}^p, 0)$ to the trivial deformation $f : (\mathbb{R}^n \times \mathbb{R}^r, (0, 0)) \rightarrow (\mathbb{R}^p, 0)$, then we can directly construct a $\mathcal{K}$-morphism (resp. topological $\mathcal{K}$-morphism) from $\Psi$ to the graph deformation.

Now suppose there exist $C^\infty$ (resp. continuous) map germs $h : (\mathbb{R}^n \times \mathbb{R}^r, (0, 0)) \rightarrow (\mathbb{R}^n \times \mathbb{R}^r, (0, 0))$, $H : (\mathbb{R}^p \times \mathbb{R}^r, (0, 0)) \rightarrow (\mathbb{R}^p \times \mathbb{R}^r, (0, 0))$ and $\phi : (\mathbb{R}^r, 0) \rightarrow (\mathbb{R}^r, 0)$ such that the following (1.3.1) and (1.3.2) hold:

(1.3.1) the restrictions $h|_{\mathbb{R}^n \times \{\lambda\}}$ and $H|_{\mathbb{R}^p \times \{\lambda\}}$
are $C^\infty$ diffeomorphic (resp. homeomorphic)
for any $\lambda \in \mathbb{R}^r$,

(1.3.2) the following diagram commutes:

\[
\begin{array}{ccc}
(\mathbb{R}^n \times \mathbb{R}^r, (0, 0)) & \xrightarrow{(\Psi, \pi_{\lambda})} & (\mathbb{R}^p \times \mathbb{R}^r, (0, 0)) \\
\downarrow h & & \downarrow H \\
(\mathbb{R}^n \times \mathbb{R}^r, (0, 0)) & \xrightarrow{(f, \pi_{\lambda})} & (\mathbb{R}^p \times \mathbb{R}^r, (0, 0)) \\
\end{array}
\]

By (1.3.2), we can write

\[h = (h_1, \phi) \quad \text{and} \quad H = (H_1, \phi).\]

Then, set $\phi'_H : (\mathbb{R}^r, 0) \rightarrow (\mathbb{R}^p, 0)$ as

\[\phi'_H(\lambda) = H_1(0, \lambda).\]

Also, set $h' : (\mathbb{R}^n \times \mathbb{R}^r, (0, 0)) \rightarrow (\mathbb{R}^n \times \mathbb{R}^p, (0, 0))$ as

\[h'(z, \lambda) = (h_1(z, \lambda), \phi'_H(\lambda))\]

and set $H' : (\mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R}^p, (0, 0, 0)) \rightarrow (\mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^p, (0, 0, 0))$ as

\[H'(z, \lambda, y) = (h'(z, \lambda), H_1(y, \lambda) - H_1(0, \lambda)).\]

Then we have

(1.4.0) $h'$ and $H'$ are $C^\infty$ (resp. continuous) map germs,

(1.4.1) the restrictions $h'|_{\mathbb{R}^n \times \{\lambda\}}$ and $H'|_{\mathbb{R}^n \times \{\lambda\} \times \mathbb{R}^p}$
are $C^\infty$ diffeomorphic (resp. homeomorphic)
for any $\lambda \in \mathbb{R}^r$,.
(1.4.2) \[ H'(\mathbb{R}^n \times \mathbb{R}^r \times \{0\}) \subset \mathbb{R}^n \times \mathbb{R}^p \times \{0\}, \]

(1.4.3) the following diagram commutes:

\[
\begin{array}{ccc}
\mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R}^p, (0,0,0) & \xrightarrow{\pi_{*,\lambda}} & (\mathbb{R}^n \times \mathbb{R}^r, (0,0)) \\
H' & \downarrow & \phi_H' \\
(\mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R}^p, (0,0,0)) & \xrightarrow{\pi_{*,\varphi}} & (\mathbb{R}^n \times \mathbb{R}^r, (0,0)) \xrightarrow{\pi \varphi} (\mathbb{R}^r, 0).
\end{array}
\]

Next, we set \( F : (\mathbb{R}^n \times \mathbb{R}^p, (0,0)) \to (\mathbb{R}^p, 0) \) as \( F(z, y) = f(z) - y \).

We call \( F : (\mathbb{R}^n \times \mathbb{R}^p, (0,0)) \to (\mathbb{R}^p, 0) \) the graph deformation of \( f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0) \).

Then, we can see

\[
F(h'(z, \lambda)) = F(h_1(z, \lambda), \phi_H'(\lambda)) = f(h_1(z, \lambda)) - \phi_H'(\lambda) = H_1(\Psi(z, \lambda), \lambda) - \phi_H'(\lambda) = H_1(\Psi(z, \lambda), \lambda) - H_1(0, \lambda).
\]

Hence, we have

(1.4.4) the following diagram also commutes:

\[
\begin{array}{ccc}
\mathbb{R}^n \times \mathbb{R}^r, (0,0) & \xrightarrow{\pi_{*,\lambda,\Psi}} & (\mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R}^p, (0,0,0)) \\
h' & \downarrow & \phi_H' \\
(\mathbb{R}^n \times \mathbb{R}^p, (0,0)) & \xrightarrow{\pi_{*,\varphi}} & (\mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^p, (0,0,0)).
\end{array}
\]

Therefore, \( \{h', H', \phi_H'\} \) is a \( \mathcal{K} \)-morphism (resp. topological \( \mathcal{K} \)-morphism) from the given deformation \( \Psi \) to the graph deformation \( F \).

In particular, by (1.4.2) and (1.4.4) we have

(1.4.5) \( h'(\Psi^{-1}(0)) \subset F^{-1}(0) \).

Furthermore, by (1.4.1) - (1.4.4) and the remark after definition (1.1) we have

(1.4.6) \( h'(\mathbb{R}^n \times \mathbb{R}^r - \Psi^{-1}(0)) \subset \mathbb{R}^n \times \mathbb{R}^p - F^{-1}(0) \).
For the proofs of theorems A, D, E, we need only the properties (1.4.1), (1.4.5) and (1.4.6) (see § 2).

Example (1.5): For any $C^\infty$ map germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$, let

(1) \[ \Psi_1 : (\mathbb{R}^n \times \mathbb{R}^p, (0, 0)) \to (\mathbb{R}^p, 0) \] be its $C^\infty$ deformation of the form $\Psi_1(z, \lambda) = f(z) + \lambda$. Then, $\{h(z, \lambda) = (z, \lambda), H(y, \lambda) = (y - \lambda, \lambda) \}$ gives a trivialisation of $\Psi_1$. In this case, $\phi'_H(\lambda) = -\lambda$, $h'(z, \lambda) = (z, -\lambda)$ and $H'(z, \lambda, y) = (z, -\lambda, y)$ as we expect. Of course, $\{h', H', \phi'_H\}$ is an equivalent $K$-morphism from $\Psi_1$ to $F$.

(2) \[ \Psi_2 : (\mathbb{R}^n \times \mathbb{R}^p, (0, 0)) \to (\mathbb{R}^p, 0) \] be the deformation of $f$ of the form $\Psi_2(z, \lambda) = f(z) - \lambda^3$; where $\lambda^3 = (\lambda_1^3, \ldots, \lambda_p^3)$. Then $\{h(z, \lambda) = (z, \lambda), H(y, \lambda) = (y + \lambda^3, \lambda) \}$ gives a topological triviality of $\Psi_2$. In this case, $\phi'_H(\lambda) = \lambda^3$, $h'(z, \lambda) = (z, \lambda^3)$ and $H'(z, \lambda, y) = (z, \lambda^3, y)$. We see $\{h', H', \phi'_H\}$ is a topologically equivalent topological $K$-morphism from $\Psi_2$ to $F$.

Definition (1.6). Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ be a $C^\infty$ map germ and let $\Psi : (\mathbb{R}^n \times \mathbb{R}^t, (0, 0)) \to (\mathbb{R}^p, 0)$ be a $C^\infty$ deformation of $f$. We say $\Psi$ is $K$-versal (resp. topologically $K$-versal) if for any $C^\infty$ deformation $\tilde{\Psi} : (\mathbb{R}^n \times \mathbb{R}^t, (0, 0)) \to (\mathbb{R}^p, 0)$ of $f$ there exist $C^\infty$ (resp. continuous) map germs $h : (\mathbb{R}^n \times \mathbb{R}^t, (0, 0)) \to (\mathbb{R}^n \times \mathbb{R}^t, (0, 0))$, $H : (\mathbb{R}^n \times \mathbb{R}^t \times \mathbb{R}^p, (0, 0)) \to (\mathbb{R}^n \times \mathbb{R}^t \times \mathbb{R}^p, (0, 0))$ and $\phi : (\mathbb{R}^t, 0) \to (\mathbb{R}^p, 0)$ which give a $K$-morphism (resp. topological $K$-morphism) from $\Psi$ to $\tilde{\Psi}$.

We can define $\mathcal{G}$-versality and topological $\mathcal{G}$-versality similarly. Let $\mathcal{G}$ be $K$ or $A$. The definition of $\mathcal{G}$-versality is equivalent to that of Martinet's definitions ([Mr]); and the definition of topological $\mathcal{G}$-versality is its topological analogue.

Since any $C^\infty$ stable map germ is, when viewed as a $C^\infty$ deformation of itself, $A$-versal: i.e. any $C^\infty$ deformation $\Psi$ of a $C^\infty$ stable map germ has a triviality; by the above argument we see

Theorem B ([Martinet ([Mr])]. For any $C^\infty$ stable map germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$, its graph deformation $F(z, y) = f(z) - y$ is $K$-versal.

There are several definitions for topological stable map germs (for instance, [dW]). However, it is well-known that for any MT-stable map germ (map germ multi-transversal to Thom-Mather canonical stratification) any $C^\infty$ deformation of it has a topological triviality (see [M3] or [GWdL]). Hence, again by the above argument, we see

Theorem C. For any MT-stable map germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$, its graph deformation $F(z, y) = f(z) - y$ is topologically $K$-versal.

§2. Special case of §1

In this chapter, we review a part of Martinet's argument in [Mr]. Let $f, g$:
$(\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0)$ be $C^\infty$ map germs. Suppose there exist a $C^\infty$ diffeomorphic (resp. homeomorphic) map germ $s : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$ and a $C^\infty$ map germ

$$M(x) = (m_1(x), \ldots, m_p(x)) : (\mathbb{R}^n, 0) \rightarrow (GL(p, \mathbb{R}), M(0))$$

such that $f(z) = M(x)(g \circ s)(z)$.

We set a $C^\infty$ map germ $\Phi : (\mathbb{R}^n \times \mathbb{R}^p, (0,0)) \rightarrow (\mathbb{R}^p, 0)$ as

$$\Phi(x, y) = M(x)((g \circ s)(x) - y) = f(x) - M(x)y.$$ 

Hereafter, we concentrate on studying deformations of this type. Hence, in particular, we assume $r = p$. We treat two kinds of $p$-dimensional euclidean space $\mathbb{R}^p$. When we are considering $\mathbb{R}^p$ as the target space, we write it $\mathbb{R}^p_\lambda$. When we are considering $\mathbb{R}^p$ as the parameter space, we write it $\mathbb{R}^p_y$.

Now suppose there exist $C^\infty$-diffeomorphic (resp. homeomorphic) map germs $h : (\mathbb{R}^n \times \mathbb{R}^p_\lambda, (0,0)) \rightarrow (\mathbb{R}^n \times \mathbb{R}^p_\lambda, (0,0))$, $H : (\mathbb{R}^p_y \times \mathbb{R}^p_\lambda, (0,0)) \rightarrow (\mathbb{R}^p_y \times \mathbb{R}^p_\lambda, (0,0))$ and $\phi : (\mathbb{R}^p_\lambda, 0) \rightarrow (\mathbb{R}^p_\lambda, 0)$ such that the following diagram commutes:

$$
\begin{array}{ccc}
(R^n \times R^p_\lambda, (0,0)) & \xrightarrow{(E, \pi_\lambda)} & (R^p_y \times R^p_\lambda, (0,0)) & \xrightarrow{\pi_\lambda} & (R^p_\lambda, 0) \\
\downarrow h & & \downarrow H & & \downarrow \phi \\
(R^n \times R^p_\lambda, (0,0)) & \xrightarrow{(I, \pi_\lambda)} & (R^p_y \times R^p_\lambda, (0,0)) & \xrightarrow{\pi_\lambda} & (R^p_\lambda, 0)
\end{array}
$$

In §1, we defined $C^\infty$ (resp. continuous) map germs

$$\phi'_{H} : (R^p_\lambda, 0) \rightarrow (R^p_y, 0)$$

$$h' : (R^n \times R^p_\lambda, (0,0)) \rightarrow (R^n \times R^p_y, (0,0))$$

$$H' : (R^n \times R^p_\lambda \times R^p_y, (0,0,0)) \rightarrow (R^n \times R^p_y \times R^p_y, (0,0,0))$$

and we saw $\{h', H', \phi'_{H}\}$ is a $\mathcal{K}$-morphism from $\Phi$ to $F$. By (1.4.5) in §1 and by the form of $\Phi$, we have

\[(2.1) \quad f(h_1(z, (g \circ s)(x)) = \phi'_{H}((g \circ s)(x))\]

as germs at the origin.

We would like to show the following map germ (2.2) is $C^\infty$ diffeomorphic (resp. homeomorphic) if we assume $\phi'_{H}$ is $C^\infty$ diffeomorphic (resp. homeomorphic).

\[(2.2) \quad z \mapsto h_1(z, (g \circ s)(x))\]
The map germ (2.2) can be decomposed as follows.

\[(2.3) \quad z \mapsto (z, (g \circ s)(z)) \mapsto h'(z, (g \circ s)(z)) \mapsto h_1(z, (g \circ s)(z)).\]

The first map germ of (2.3) is trivially $C^\infty$ diffeomorphic. If we assume $\phi'_H$ is $C^\infty$ diffeomorphic (resp. homeomorphic), then by (1.4.1) in §1 $h' = (h_1, \phi'_H)$ is $C^\infty$ diffeomorphic (resp. homeomorphic). Thus, the second map germ of (2.3) is $C^\infty$ diffeomorphic (resp. homeomorphic). Furthermore, in the case that we assume $\phi'_H$ is $C^\infty$ diffeomorphic (resp. homeomorphic), by (1.4.5) and (1.4.6) in §1 we have

\[(2.4) \quad h'(\Phi^{-1}(0)) = F^{-1}(0).\]

By the form of $\Phi$ and $F$, (2.4) means

\[(2.5) \quad \text{the germ of the set } \{h'(z, (g \circ s)(z))|z \in \mathbb{R}^n\} = \text{the germ of } F^{-1}(0) = \text{graph}(f).\]

By (2.5) and by the form of $h' = (h_1, \phi'_H)$, the last map germ of (2.3) is also $C^\infty$ diffeomorphic.

Therefore, we see

**Lemma (2.6).** If $\Phi$ has a triviality (resp. topological triviality) and $\phi'_H$ is $C^\infty$ diffeomorphic (resp. homeomorphic), then $f$ and $g$ are right-left equivalent (resp. topologically right-left equivalent).

§3. Module

Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ be a $C^\infty$ map germ and let

\[M(z) = (m_1(z), \ldots, m_p(z)) : (\mathbb{R}^n, 0) \to (GL(p, \mathbb{R}), M(0))\]

be also a $C^\infty$ map germ. Let $\Phi : (\mathbb{R}^n \times \mathbb{R}^p, (0, 0)) \to (\mathbb{R}^p, 0)$ be the $C^\infty$ deformation of $f$ having the following form:

\[\Phi(z, \lambda) = f(z) - M(z)\lambda.\]

In this chapter, we prove the following lemma.

**Lemma (3.1).** Suppose there exists a positive integer $k$ such that

\[m_t(z) - m_t(0) \in m^k_t \mathcal{E}_\mathcal{P} \subset tf(m_t \mathcal{E}_\mathcal{P}) + \omega f(m_t \mathcal{E}_\mathcal{P})\]
for any \(i\ (1 \leq i \leq p)\). Then \(m_i(x) - m_i(0)\) is included in 
\[t\Phi_x(m_{x,x,\lambda}E^n_x) + \omega(\Phi, \pi_\lambda)(m_{y,y,\lambda}E^p_{y,\lambda})\]
for any \(i\ (1 \leq i \leq p)\).

**Proof of Lemma (3.1):** Since we assumed 
\[m^k_xE_x^p \subset tf(m_{x,x}^n) + \omega f(m_{y,y}^p),\]
by Malgrange preparation theorem we have
\[(3.2)\]
\[m^k_xE^p_{x,\lambda} \subset tf(m_{x,x,\lambda}^n) + \omega(f, \pi_\lambda)(m_{y,y,\lambda}^p).\]
We set \(\Phi : (\mathbb{R}^n \times \mathbb{R}^p_\lambda, (0,0)) \to (\mathbb{R}^p_\lambda, 0)\) as
\[
\Phi(z, \lambda) = \Phi(z, \lambda) + M(0)\lambda
= f(z) - (M(z) - M(0))\lambda.
\]
Since we assumed 
\[m_i(x) - m_i(0) \in m^k_xE_x^p\]
for any \(i\ (1 \leq i \leq p)\), the difference
\[
\Phi(z, \lambda) - f(z) = (M(z) - M(0))\lambda = \sum_{i=1}^{p} \lambda_i(m_i(z) - m_i(0))
\]
is included in 
\[\pi_\lambda^*m_{x,\lambda}^kE_{x,\lambda}^p \subset (\Phi, \pi_\lambda)^*m_{y,\lambda}^pE_{x,\lambda}^p.\]
Hence, we can approximate (3.2) as follows.
\[(3.3)\]
\[m^k_xE^p_{x,\lambda} \subset t\Phi_x(m_{x,x,\lambda}E^n_x) + \omega(\Phi, \pi_\lambda)(m_{y,y,\lambda}E^p_{y,\lambda}) + (\Phi, \pi_\lambda)^*m_{y,\lambda}^pE_{x,\lambda}^p,\]
We set 
\[C = E_{x,\lambda}^p/t\Phi_x(m_{x,x,\lambda}E^n_x),\]
\[A = \text{image of } \omega(\Phi, \pi_\lambda)(m_{y,y,\lambda}E^p_{y,\lambda}) \text{ by the canonical projection to } C,\]
\[B = m^k_x.C.\]
Then, by (3.3) we have
\[(3.4)\]
\[B \subset A + (\Phi, \pi_\lambda)^*m_{y,\lambda}B.\]
Since
\[
\dim_k B/(\bar{\Phi}, \pi_{\lambda})^* m_{y, \lambda} B = \dim_k m_{x}^k \mathcal{E}_{l}^{p}/m_{l}^k (t f(m_{x} \mathcal{E}_{x}^{p}) + f^* m_{y} \mathcal{E}_{l}^{p}) < \infty,
\]
by Malgrange preparation theorem we see $B$ is finitely generated $\mathcal{E}_{y, \lambda}$-module via $(\bar{\Phi}, \pi_{\lambda})$. Hence, by Nakayama's lemma (3.4) implies

(3.5) \quad B \subset A

From the form $\bar{\Phi}(\bar{z}, \lambda) = \Phi(\bar{z}, \lambda) + M(0)\lambda$, we see

(3.6) \quad t \bar{\Phi}_x(m_{x, \lambda} \mathcal{E}_{x, \lambda}^n) + \omega(\bar{\Phi}, \pi_{\lambda})(m_{y, \lambda} \mathcal{E}_{y, \lambda}^p) = t \Phi_x(m_{x, \lambda} \mathcal{E}_{x, \lambda}^n) + \omega(\Phi, \pi_{\lambda})(m_{y, \lambda} \mathcal{E}_{y, \lambda}^p)

(3.5) and (3.6) yields

\[
m_i(z) - m_i(0) \in m_{x}^k \mathcal{E}_{x, \lambda} \subset t \Phi_x(m_{x, \lambda} \mathcal{E}_{x, \lambda}^n) + \omega(\Phi, \pi_{\lambda})(m_{y, \lambda} \mathcal{E}_{y, \lambda}^p)
\]

for any $i (1 \leq i \leq p)$. \hfill \Box

§4. PROOF OF THEOREM A

Let $\Phi : (\mathbb{R}^n \times \mathbb{R}_{\lambda}^p, (0, 0)) \to (\mathbb{R}_{\nu}^p, 0)$ be the $C^\infty$ deformation of $f$ having the following form:

$\Phi(\bar{z}, \lambda) = f(\bar{z}) - M(\bar{z})\lambda$.

Since

\[
\frac{\partial \Phi}{\partial \lambda_i} = - m_i(\bar{z})
\]

for any $i (1 \leq i \leq p)$, by lemma (3.1) we can choose germs of $C^\infty$ vector fields $\xi_i \in \mathcal{E}_{x, \lambda}^n$ and $\eta_i \in \mathcal{E}_{y, \lambda}^p$

such that

(4.1) \quad \frac{\partial \Phi}{\partial \lambda_i} = \xi_i(\Phi) - \eta_i \circ (\Phi, \pi_{\lambda})

(4.2) \quad \frac{\partial \Phi}{\partial \lambda_i}(0) = \eta_i(0, 0)

for any $i (1 \leq i \leq p)$. 
By (4.1), integrating germs of $C^\infty$ vector fields

$$
\xi_1 + \partial/\partial\lambda_1, \ldots, \xi_p + \partial/\partial\lambda_p
$$

and

$$
\eta_1 + \partial/\partial\lambda_1, \ldots, \eta_p + \partial/\partial\lambda_p
$$
yields $C^\infty$ diffeomorphic map germs

$$
h^{-1} : (\mathbb{R}^n \times \mathbb{R}^p_\lambda, (0,0)) \to (\mathbb{R}^n \times \mathbb{R}^p_\lambda, (0,0))
$$

and

$$
H^{-1} : (\mathbb{R}^p_y \times \mathbb{R}^p_\lambda, (0,0)) \to (\mathbb{R}^p_y \times \mathbb{R}^p_\lambda, (0,0))
$$
such that the following diagram commutes.

Consider the inverse map germ $H$ of $H^{-1}$ and

$$
\phi_H'(\lambda) : (\mathbb{R}^p_\lambda, 0) \to (\mathbb{R}^p_y, 0)
$$

associated with $H$.

Let $\Theta_i(t; y)$ be the integral curve of $\eta_i$ starting from $y$ and of time $t$. Then we can get the image $y(\lambda_1, \ldots, \lambda_p) = \phi_H'(\lambda_1, \ldots, \lambda_p)$ of $\lambda = (\lambda_1, \ldots, \lambda_p)$ by $\phi_H'$ as the unique solution of the integral equation

(4.3) $$
\Theta_1(\lambda_1; \Theta_2(\lambda_2; \ldots; \Theta_p(\lambda_p; y(\lambda_1, \ldots, \lambda_p))) \ldots) = 0.
$$

We differentiate (4.3) with respect to $\lambda_i$. Then we get

(4.4) $$
\eta_i(\Theta_{i+1}(\lambda_{i+1}; \ldots; \Theta_p(\lambda_p; y)) \ldots) \\
+ (d\Theta_1)_y \ldots (d\Theta_p)_y \partial y(\lambda_1, \ldots, \lambda_p)/\partial\lambda_i = 0
$$

for any $i$ ($1 \leq i \leq p$).

Taking values at $\lambda = 0$ in (4.4), we get

$$
\frac{\partial\phi_H'}{\partial\lambda_i}(0) = \frac{\partial y}{\partial\lambda_i}(0)
$$

$$
= -\eta_i(0,0) \quad (y(0, \ldots, 0) = 0)
$$

$$
= -\frac{\partial\Phi}{\partial\lambda_i}(0) \quad (4.2)
$$

$$
= m_i(0). \quad (\frac{\partial\Phi}{\partial\lambda_i} = -m_i)$$
Since \((m_1(0), \ldots, m_p(0)) = M(0)\) is in \(GL(p, \mathbb{R})\), \(\phi_H'\) is \(C^\infty\) diffeomorphic. Hence, by lemma (2.3), \(f\) and \(g\) are right-left equivalent. \(\blacksquare\)

§5. AN ALTERNATIVE PROOF OF MATHER'S CLASSIFICATION THEOREM

In this chapter, we give a proof of the following theorem D which is a generalized version of Mather's classification theorem.

**Definition (5.1).** Let \(X\) be a Banach space. We say a \(C^\infty\) map germ \(f : (X, 0) \rightarrow (\mathbb{R}^p, 0)\) is \(C^\infty\) stable if for any finite dimensional \(C^\infty\) deformation \(\Phi : (X \times \mathbb{R}', (0, 0)) \rightarrow (\mathbb{R}^p, 0)\) of \(f\) there exist \(C^\infty\) diffeomorphic map germs

\[
\begin{align*}
  h &: (X \times \mathbb{R}', (0, 0)) \rightarrow (X \times \mathbb{R}', (0, 0)) \\
  H &: (\mathbb{R}^p \times \mathbb{R}', (0, 0)) \rightarrow (\mathbb{R}^p \times \mathbb{R}', (0, 0)) \\
  \phi &: (\mathbb{R}', 0) \rightarrow (\mathbb{R}', 0)
\end{align*}
\]

such that the following diagram commutes:

\[
\begin{array}{ccc}
(X \times \mathbb{R}', (0, 0)) & \xrightarrow{(\Phi, \pi_\lambda)} & (\mathbb{R}^p \times \mathbb{R}', (0, 0)) \\
\downarrow h & & \downarrow H \\
(X \times \mathbb{R}', (0, 0)) & \xrightarrow{(f, \pi_\lambda)} & (\mathbb{R}^p \times \mathbb{R}', (0, 0)) \\
\downarrow & & \downarrow \phi \\
& & (\mathbb{R}', 0)
\end{array}
\]

**Theorem D.** Let \(X\) be a Banach space. Let \(f, g : (X, 0) \rightarrow (\mathbb{R}^p, 0)\) be two \(C^\infty\) stable map germs. Suppose there exists a \(C^\infty\) diffeomorphic germ \(s : (X, 0) \rightarrow (X, 0)\) and a \(C^\infty\) map germ

\[
M(z) = (m_1(z), \ldots, m_p(z)) : (X, 0) \rightarrow (GL(p, \mathbb{R}), M(0))
\]

such that \(f(z) = M(z)(g \circ s)(z)\). Then \(f\) and \(g\) are isomorphic.

Mather's classification theorem ([M2]) is the case when \(X\) is finite dimensional.

**Proof:** Let \(M_p(\mathbb{R})\) be the set of all \((p \times p)\) matrices of real elements and let \(E_p\) be the \((p \times p)\) unit matrix. For any fixed matrix \(A = (a_1, \ldots, a_p) \in M_p(\mathbb{R})\), define a map germ

\[
\Phi_A : (X \times \mathbb{R}^p \times M_p(\mathbb{R}), (0, 0, E_p)) \rightarrow (\mathbb{R}^p, 0)
\]

as

\[
\Phi_A(z, \lambda, B) = f(z) - (A + M(z)B)\lambda.
\]
Then $\Phi_A$ is a finite dimensional $C^\infty$ deformation of $f$. Since $f$ is $C^\infty$ stable, for any $i \ (1 \leq i \leq p)$ and $A = O$ (zero matrix) we see

$$\sum_{j=1}^{p} b_{ji} m_j(x) = 0 + \sum_{j=1}^{p} b_{ji} m_j(x) = \frac{\partial \Phi_0}{\partial \lambda_i}$$

is included in the set

$$t(\Phi_0)_x(\mathcal{E}^n_{x,\lambda,B} + \omega(\Phi_0, \pi_\lambda, \pi_B)(\mathcal{E}_y, \lambda,B)).$$

Here we set $B = [b_{ij}]_{1 \leq i, j \leq p}$. Since we see trivially

$$t(\Phi_0)_x = t(\Phi_A)_x$$

and

$$\omega(\Phi_0, \pi_\lambda, \pi_B)(\mathcal{E}^p_{y,\lambda,B}) = \omega(\Phi_A, \pi_\lambda, \pi_B)(\mathcal{E}^p_{y,\lambda,B})$$

for any fixed $A \in M_p(\mathbb{R})$, we can choose germs of $C^\infty$ vector fields

$$\xi_i \in \mathcal{E}^n_{x,\lambda,B} \quad \text{and} \quad \eta_{i,A} \in \mathcal{E}^p_{y,\lambda,B}$$

such that

$$-\frac{\partial \Phi_A}{\partial \lambda_i} = (a_i + \sum_{j=1}^{p} b_{ji} m_j(x))$$

$$= \xi_i(\Phi_A) - \eta_{i,A} \circ (\Phi_A, \pi_\lambda, \pi_B)$$

for any $i \ (1 \leq i \leq p)$. Since $f$ is $C^\infty$ stable, we can choose germs of $C^\infty$ vector fields

$$\xi_{jk,A} \in \mathcal{E}^n_{x,\lambda,B} \quad \text{and} \quad \eta_{jk,A} \in \mathcal{E}^p_{y,\lambda,B}$$

such that

$$-\frac{\partial \Phi_A}{\partial b_{jk}} = \lambda_k m_j(x))$$

$$= \xi_{jk,A}(\Phi_A) - \eta_{jk,A} \circ (\Phi_A, \pi_\lambda, \pi_B)$$

for any $j, k \ (1 \leq j, k \leq p)$ and any $A$ of $M_p(\mathbb{R})$.

By integrating

$$\eta_{1,A} + \partial/\partial \lambda_1, \ldots, \eta_{p,A} + \partial/\partial \lambda_p,$$

$$\eta_{11,A} + \partial/\partial b_{11}, \ldots, \eta_{pp,A} + \partial/\partial b_{pp},$$
we get a $C^\infty$ diffeomorphic germ

$$H_A^{-1} : (\mathbb{R}_y^p \times \mathbb{R}_\lambda^p \times M_p(\mathbb{R}), (0, 0, E_p)) \to (\mathbb{R}_y^p \times \mathbb{R}_\lambda^p \times M_p(\mathbb{R}), (0, 0, E_p)).$$

We consider the map germ

$$\phi_{H_A}^I : (\mathbb{R}_y^p \times M_p(\mathbb{R}), (0, E_p)) \to (\mathbb{R}_y^p, 0)$$

associated with $H_A$ and its restriction

$$\phi_{H_A}^I|_{\mathbb{R}_y^p \times \{B\}}$$

for $B$ sufficiently near the zero matrix.

Let $\Theta_{i,A}(t; y)$ (resp. $\Theta_{jk,A}(t; y)$) be the integral curve of $\eta_{i,A}$ (resp. $\eta_{jk,A}$) starting from $y$ and of time $t$ for any $i, j, k$ (1 $\leq i, j, k \leq p$). Then $\Theta_{i,A}(\lambda_1, \ldots, \lambda_p, b_{11}, \ldots, b_{pp}) = y$, where $y$ is the unique solution of the following integral equation

$$\Theta_{1,A}(\lambda_1; \ldots; \Theta_{p,A}(\lambda_p; \ldots; \Theta_{11,A}(b_{11}; \ldots; \Theta_{pp,A}(b_{pp}; y)) \ldots) = 0.$$

We differentiate this equation with respect to $\lambda_i$. Then we have

$$\frac{\partial \phi_{H_A}^I}{\partial \lambda_i}(0, E_p) = \frac{\partial y}{\partial \lambda_i}(0, E_p)$$

for any $i$ (1 $\leq i \leq p$).

Taking values at $\lambda = 0$ and $B = E_p$ in (5.2), we get

$$\frac{\delta \phi_{H_A}^I}{\delta \lambda_i}(0, E_p) = \frac{\partial y}{\partial \lambda_i}(0, E_p)$$

for any $i$ (1 $\leq i \leq p$). From (5.3) and since $\xi_i$ is $C^\infty$ with respect to $B = [b_{ij}]$, we have

**Lemma (5.4).** There exists an open dense subset $\mathcal{U}$ of $M_p(\mathbb{R})$ such that for any $A$ of $\mathcal{U}$ there exists a neighborhood $\mathcal{V}_A$ of $E_p$ in $M_p(\mathbb{R})$ such that the germ of the restriction

$$\phi_{H_A}^I|_{\mathbb{R}_y^p \times \{B\}} : (\mathbb{R}_y^p \times \{B\}, (0, B)) \to (\mathbb{R}_y^p, 0)$$

is $C^\infty$ diffeomorphic for any $B$ of $\mathcal{V}_A$.

Therefore, by lemmata (2.6) and (5.4), we have
**Lemma (5.5).** If we choose \((p \times p)\) matrix \(A\) of \(\mathcal{U}\) sufficiently near the zero matrix, then \(f(x)\) and \(g_{A,B}(x) = (A + M(z)B)^{-1}f(z)\) are right-left equivalent for any \(B\) of \(\mathcal{V}_{A}\).

Next, we take a matrix \(A_0\) of \(\mathcal{U}\) sufficiently near the zero matrix and fix it. We set
\[
M(z)^{-1}A_0 = N_{A_0}(z) = (n_1(z), \ldots, n_p(z)).
\]
For any fixed \(B\) of \(\mathcal{V}_{A_0}\), we define the \(C^\infty\) map germ
\[
\tilde{\Phi}_{A_0,B} : (X \times \mathbb{R}^p, (0,0)) \to (\mathbb{R}^p, 0)
\]
as
\[
\tilde{\Phi}_{A_0,B}(x, \lambda) = (N_{A_0}(x) + B)(g_{A_0,B}(x) - \lambda).
\]
Then, since
\[
(g \circ s)(x) = M(z)^{-1}(A_0 + M(z)B)(A_0 + M(z)B)^{-1}f(z)
\]
\[
= M(z)^{-1}(A_0 + M(z)B)g_{A_0,B}(x)
\]
we see \(\tilde{\Phi}_{A_0,B}(x, \lambda) = (g \circ s)(x) - (N_{A_0}(x) + B)\lambda\) is a \(C^\infty\) deformation of \((g \circ s)\).
Since \((g \circ s)\) is \(C^\infty\) stable, for any \(i\) \((1 \leq i \leq p)\) and \(B = E_p\) we see
\[
\frac{\partial \tilde{\Phi}_{A_0,E_p}}{\partial \lambda_i} \in t(\tilde{\Phi}_{A_0,E_p})_{x}(\mathcal{E}^n_{x,\lambda}) + \omega(\tilde{\Phi}_{A_0,E_p}, \pi_\lambda)(\mathcal{E}^p_{y,\lambda}).
\]
Since
\[
t(\tilde{\Phi}_{A_0,E_p})_{x} = t(\tilde{\Phi}_{A,B})_{x}
\]
and
\[
\omega(\tilde{\Phi}_{A_0,E_p}, \pi_\lambda)(\mathcal{E}^p_{y,\lambda}) = \omega(\tilde{\Phi}_{A,B}, \pi_\lambda)(\mathcal{E}^p_{y,\lambda})
\]
for any \(A \in \mathcal{U}\) and \(B \in \mathcal{V}_{A}\), we can choose germs of \(C^\infty\) vector fields
\[
\tilde{\xi}_i \in \mathcal{E}^n_{x,\lambda} \quad \text{and} \quad \tilde{\eta}_{i,B} \in \mathcal{E}^p_{y,\lambda}
\]
such that
\[
-\frac{\partial \tilde{\Phi}_{A_0,B}}{\partial \lambda_i} = \tilde{\xi}_i(\tilde{\Phi}_{A_0,B}) - \tilde{\eta}_{i,B} \circ (\tilde{\Phi}_{A_0,B}, \pi_\lambda)
\]
for any \(i\) \((1 \leq i \leq p)\).

By integrating
\[
\tilde{\eta}_{i,B} + \partial / \partial \lambda_1, \ldots, \tilde{\eta}_{p,B} + \partial / \partial \lambda_p,
\]
we get a $C^\infty$ diffeomorphic germ

$$H_{A_0,B}^{-1} : (\mathbb{R}_y^p \times \mathbb{R}_\lambda^p, (0,0)) \rightarrow (\mathbb{R}_y^p \times \mathbb{R}_\lambda^p, (0,0)).$$

We consider the map germ

$$\phi'_{H_{A_0,B}} : (\mathbb{R}_\lambda^p, 0) \rightarrow (\mathbb{R}_y^p, 0)$$

associated with $H_{A_0,B}$. We see

$$\frac{\partial \phi'_{H_{A_0,B}}}{\partial \lambda_i}(0) = -\eta_{i,B}(0,0)$$

for any $i$ $(1 \leq i \leq p)$. By (5.6), we can choose a matrix $B$ of $\mathcal{V}_{A_0}$ with the property that $\phi'_{H_{A_0,B}}$ is $C^\infty$ diffeomorphic. Thus, by lemma (2.6), we have

**Lemma (5.7).** We can choose a matrix $B$ of $\mathcal{V}_{A_0}$ with the property that $(g \circ s)$ and $g_{A_0,B}$ are right-left equivalent.

Lemmas (5.5) and (5.7) concludes that $f$ and $g$ are isomorphic.

§6. AN ALTERNATIVE PROOF OF FUKUDA-FUKUDA’S THEOREM

In this chapter, we give a proof of the following theorem.

**Theorem E ([FF]).** Let $f, g : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0)$ be two MT-stable map germs. Suppose there exists a $C^\infty$ diffeomorphic germ $s : (\mathbb{R}^n, 0) \rightarrow (X, 0)$ and a $C^\infty$ map germ

$$M(x) = (m_1(x), \ldots, m_p(x)) : (\mathbb{R}^n, 0) \rightarrow (GL(p, \mathbb{R}), M(0))$$

such that $f(x) = M(x)(g \circ s)(x)$. Then $f$ and $g$ are topologically isomorphic.

For the definition of MT-stable map germs, refer to [M3] or [GWdL]. For our proof of theorem E, we use only the following fact on MT-stable map germs.
FACT (6.1). Let $f : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0)$ be an MT-stable map germ. Then for any $C^\infty$ deformation $\Phi : (\mathbb{R}^n \times \mathbb{R}^r, (0, 0)) \rightarrow (\mathbb{R}^p, 0)$ of $f$ there exist Whitney stratifications $S$ of $\mathbb{R}^n \times \mathbb{R}^r$ and $T$ of $\mathbb{R}^p \times \mathbb{R}^r$ such that the germ of the sequence

$$(\mathbb{R}^n \times \mathbb{R}^r, (0, 0)) \xrightarrow{(\Phi, \pi_\lambda)} (\mathbb{R}^p \times \mathbb{R}^r, (0, 0)) \xrightarrow{\pi_\lambda} (\mathbb{R}^r, 0)$$

is a Thom sequence with respect to $S, T$ and $\{\mathbb{R}^r\}$.

PROOF OF THEOREM E: As in §5, for any fixed matrix $A = (a_1, \ldots, a_p) \in M_p(\mathbb{R})$, define a map germ

$$(X \times \mathbb{R}_{\lambda}^n \times M_p(\mathbb{R}), (0, 0, E_p)) \rightarrow (\mathbb{R}_{y}^p, 0)$$

as

$$\Psi_A : (X \times \mathbb{R}_{\lambda}^n \times M_p(\mathbb{R}), (0, 0, E_p)) \rightarrow (\mathbb{R}_{y}^p, 0)$$

Then $\Psi_A$ is a $C^\infty$ deformation of $f$. Since $f$ is MT-stable, by (6.1), there exist Whitney stratifications $S$ of $\mathbb{R}^n \times \mathbb{R}_{\lambda}^n \times M_p(\mathbb{R})$ and $T$ of $\mathbb{R}_{y}^p \times \mathbb{R}_{\lambda}^n \times M_p(\mathbb{R})$ such that the germ of the sequence

$$(\mathbb{R}^n \times \mathbb{R}_{\lambda}^n \times M_p(\mathbb{R}), (0, 0, E_p))$$

$$\downarrow^{(\Psi_0, \pi_\lambda, B)}$$

$$(\mathbb{R}_{y}^p \times \mathbb{R}_{\lambda}^n \times M_p(\mathbb{R}), (0, 0, E_p))$$

$$\downarrow^{\pi_\lambda, B}$$

$$(\mathbb{R}_{y}^p \times M_p(\mathbb{R}), (0, E_p))$$

is a Thom sequence with respect to $S, T$ and $\{\mathbb{R}_{\lambda}^n \times M_p(\mathbb{R})\}$.

For any stratum $T$ of $T$ and any $A$ of $M_p(\mathbb{R})$, we set

$$T_A = \{(y, \lambda, B) - (A\lambda, \lambda, B) \in \mathbb{R}_{y}^p \times \mathbb{R}_{\lambda}^n \times M_p(\mathbb{R}) | (y, \lambda, B) \in T\}$$

and

$$T_A = \{T_A\}.$$ 

Then, since

$$\Psi_A(z, \lambda, B) = \Psi_0(z, \lambda, B) - A\lambda$$

$$= \Psi_0(z, \lambda, B) + \text{(family of parallel translation of } \mathbb{R}_{y}^p\text{)}$$

we see
**Lemma (6.2).** For any matrix $A$ of $M_p(R)$, the germ of the sequence

$$(\mathbb{R}^n \times \mathbb{R}_n^p \times M_p(R), (0, 0, E_p))$$

is a Thom sequence with respect to $S, T_A$ and $\{R^p \times M_p(R)\}$.

By (6.2) we see

**Lemma (6.3).** There exists an open dense subset $\mathcal{U}$ of $M_p(R)$ such that for any $A$ of $\mathcal{U}$ there exists a neighborhood $\mathcal{V}_A$ of $E_p$ in $M_p(R)$ such that the subset $\{0\} \times R^p_\lambda \times \{B\}$ near $\{0\} \times \{0\} \times \{0\}$ for any $B$ of $\mathcal{V}_A$ and any $T_A$ of $T_A$.

By lifting vector fields $\partial/\partial \lambda_1, \ldots, \partial/\partial \lambda_p, \partial/\partial b_{11}, \ldots, \partial/\partial b_{pp}$, we get germs of vector fields

$$\mathcal{T}_{A} = \{0\} \times \mathbb{R}_\lambda \times \{B\} \times \{0\} \times \{0\} \times \{0\}.$$

We consider the map germ

$$\phi'_{H_A} : (R^p_\lambda \times M_p(R), (0, 0, E_p)) \rightarrow (R^p_\lambda \times M_p(R), (0, 0, E_p)).$$

associated with $H_A$ and its restriction

$$\phi'_{H_A} | \mathcal{T}_{A}$$

for $B$ sufficiently near the zero matrix.

Let $\Theta_{i,A}(t;y)$ (resp. $\Theta_{j,k,A}(t;y)$) be the integral curve of $\eta_{i,A}$ (resp. $\eta_{j,k,A}$) starting from $y$ and of time $t$ for any $i, j, k$ (1 $\leq i, j, k \leq p$). Then
\( \phi'_{H_A}(\lambda_1, \ldots, \lambda_p, b_{11}, \ldots, b_{pp}) \) can be given as the unique solution of the following integral equation

\[
\Theta_{1,A}(\lambda_1; \ldots; \Theta_{p,A}(\lambda_p; \Theta_{11,A}(b_{11}; \ldots(\Theta_{pp,A}(b_{pp}; \phi_{H_4}'(\lambda_1, \ldots, b_{pp})) \ldots)) = 0.
\]

Since the germs of vector fields

\[
\eta_{1,A} + \partial/\partial \lambda_1, \ldots, \eta_{p,A} + \partial/\partial \lambda_p,
\eta_{11,A} + \partial/\partial b_{11}, \ldots, \eta_{pp,A} + \partial/\partial b_{pp},
\]

are controlled, by lemma (6.3), we see

**Lemma (6.4).** For any \( A \) of \( \mathcal{U} \) and any \( B \) of \( \mathcal{V}_A \), the germ of the restriction

\[
\phi'_{H_A}|_{\mathbb{R}_{\lambda}^p \times \{B\}} : (\mathbb{R}_{\lambda}^p \times \{B\}, (0, B)) \rightarrow (\mathbb{R}_{y}^p, 0)
\]

is injective.

Since \( \phi'_{H_A}|_{\mathbb{R}_{\lambda}^p \times \{B\}} \) is continuous, injectivity means being homeomorphic. Therefore, by lemma (2.6) we have

**Lemma (6.5).** If we choose \((p \times p)\) matrix \( A \) of \( \mathcal{U} \) sufficiently near the zero matrix, then \( f(x) \) and \( g_{A,B}(x) = (A + M(x)B)^{-1}f(x) \) are topologically right-left equivalent for any \( B \) of \( \mathcal{V}_A \).

Next, we take a matrix \( A_0 \) of \( \mathcal{U} \) sufficiently near the zero matrix and fix it. We set

\[
M(x)^{-1}A_0 = N_{A_0}(x) = (n_1(x), \ldots, n_p(x)).
\]

For any fixed \( B \) of \( \mathcal{V}_{A_0} \), we define the \( C^\infty \) map germ

\[
\tilde{\Psi}_{A_0,B} : (\mathbb{R}_{\lambda}^n \times (0, B)) \rightarrow (\mathbb{R}_{y}^n, 0)
\]

as

\[
\tilde{\Psi}_{A_0,B}(x, \lambda) = (N_{A_0}(x) + B)(g_{A_0,B}(x) - \lambda).
\]

Then, since

\[
(g \circ s)(x) = M(x)^{-1}(A_0 + M(x)B)(A_0 + M(x)B)^{-1}f(x)
\]

\[
= M(x)^{-1}(A_0 + M(x)B)g_{A_0,B}(x)
\]

\[
= (N_{A_0}(x) + B)g_{A_0,B}(x);
\]

we see \( \tilde{\Psi}_{A_0,B}(x, \lambda) = (g \circ s)(x) - (N_{A_0}(x) + B)\lambda \) is a \( C^\infty \) deformation of \( (g \circ s) \). Since \( g : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0) \) is MT-stable and \( s : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0) \) is \( C^\infty \)
diffeomorphic, \((g \circ s)(x) = M(x)^{-1}f(x)\) is MT-stable. Thus, by (6.1), there exist Whitney stratifications \(\tilde{S}\) of \(\mathbb{R}^n \times \mathbb{R}^p\), \(\tilde{T}\) of \(\mathbb{R}^p \times \mathbb{R}^p\) such that the germ of the sequence

\[
(\mathbb{R}^n \times \mathbb{R}^p, (0,0)) \xrightarrow{\tilde{\Psi}_{A_0,0,\pi_\lambda}} (\mathbb{R}^p \times \mathbb{R}^p, (0,0)) \xrightarrow{\pi_\lambda} (\mathbb{R}^p, 0)
\]

is a Thom sequence with respect to \(\tilde{S}, \tilde{T}\) and \(\{\mathbb{R}^p\}\).

For any stratum \(\tilde{T}\) of \(\tilde{T}\) and any \(B\) of \(M_p(\mathbb{R})\), we set

\[
\tilde{T}_B = \{(y, \lambda) - (B\lambda, \lambda) \in \mathbb{R}^p \times \mathbb{R}^p | (y, \lambda) \in \tilde{T}\}
\]

and

\[
\tilde{T}_B = \{\tilde{T}_B\}.
\]

Then, since

\[
\tilde{\Psi}_{A_0,B}(x, \lambda) = \tilde{\Psi}_{A_0,0}(x, \lambda) - B\lambda
\]

we see

**Lemma (6.6).** For any matrix \(B\) of \(M_p(\mathbb{R})\), the germ of the sequence

\[
(\mathbb{R}^n \times \mathbb{R}^p, (0,0)) \xrightarrow{\tilde{\Psi}_{A_0,B,\pi_\lambda}} (\mathbb{R}^p \times \mathbb{R}^p, (0,0)) \xrightarrow{\pi_\lambda} (\mathbb{R}^p, 0)
\]

is a Thom sequence with respect to \(\tilde{S}, \tilde{T}_B\) and \(\{\mathbb{R}^p\}\).

By lemma (6.6) and by using the same argument as before, we can choose a matrix \(B\) of \(\mathcal{V}_{A_0}\) with the property that \(\phi_{H_{A_0,B}}'\) is homeomorphic. Thus, by lemma (2.6), we have

**Lemma (6.7).** We can choose a matrix \(B\) of \(\mathcal{V}_{A_0}\) with the property that \((g \circ s)\) and \(g_{A_0,B}\) are topologically right-left equivalent.

Lemmatea (6.6) and (6.7) concludes that \(f\) and \(g\) are topologically isomorphic.

**References**


