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<tr>
<td>Author(s)</td>
<td>HASHIMOTO, Takashi</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (1992), 816: 22-35</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1992-12</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/83110">http://hdl.handle.net/2433/83110</a></td>
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<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
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Kyoto University
Kirillov-Kostant theory and Feynman path integrals on coadjoint orbits of a certain real semisimple Lie group

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0 Introduction

Alekseev, Faddeev and Shatashvili showed in [1] that any irreducible unitary representation of compact groups can be obtained by path integrals. They computed characters of the representations. We showed in [3] that path integrals give unitary operators of the representation which is constructed by Kirillov-Kostant theory for some Lie groups.

In [4] we found that, in order to compute the path integrals with nontrivial Hamiltonians for $SU(1,1)$ and $SU(2)$ to obtain unitary operators realized by Borel-Weil theory, we have to regularize the Hamiltonian functions, and in [5] we extended the results to the case that the maximal compact subgroup $K$ of a connected semisimple Lie group $G$ has equal rank to the complex rank of $G$.

In the rest of this section we shall show how the path integral reproduces the representation constructed by Kirillov-Kostant theory in the case of $SL(2,\mathbb{R})$ with real polarization. This was done in [3].

Let

$$G = SL(2, \mathbb{R}) = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} ; ad - bc = 1 \right\}$$

$$g = sl(2, \mathbb{R}) = \left\{ X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} ; a + d = 0 \right\}$$

Since the bilinear form $\langle , \rangle$ on $g$ given by $\langle X, Y \rangle = \text{tr}XY$ is nondegenerate, the dual space $g^*$ of $g$ is identified with $g$.

For a nonzero real number $\sigma$, we put $\lambda = \left( \begin{smallmatrix} \sigma/2 & 0 \\ 0 & -\sigma/2 \end{smallmatrix} \right) \in g^*$ and put $\mathcal{H}_\lambda = L_2(\mathbb{R})$. We define a representation $(U_\lambda, \mathcal{H}_\lambda)$ of $G$ as follows:

$$U_\lambda(g)F(x) = | -cx + a |^{-(\sqrt{-1}\sigma+1)} F \left( \frac{dx - b}{-cx + a} \right)$$

for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ and $F \in \mathcal{H}_\lambda$.

We can obtain this representation by path integrals as we shall show below.
We introduce local coordinates on $G$ by
\[ g = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} \begin{pmatrix} \pm e^v & 0 \\ 0 & \pm e^{-v} \end{pmatrix}. \]

Note that such elements form an open subset of $G$ which is also dense.

Then define a 1-form $\varphi$ by
\[ \varphi = \{ \lambda, g^{-1} dg \} = \sigma(udx + dv). \]

Since $dv$ is exact 1-form, we choose $\alpha = \sigma u dx$ and put $p = \sigma u$. The $p$ is called momentum in quantum mechanics. Define a function $H(g : Y)$ for $Y \in g$, which we call Hamiltonian function, by
\[ H(g : Y) = \langle \text{Ad}^*(g) \lambda, Y \rangle \]
\[ = \begin{cases} \sigma(\sigma + 2px) & \text{if } Y = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \\ bp & \text{if } Y = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \\ -c(\sigma x + px^2) & \text{if } Y = \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}, \end{cases} \]

where $\text{Ad}^*$ denotes the coadjoint action of $G$ on $g^*$.

The path integral we should compute is given, symbolically, by
\[ \int D(x, p) \exp \left( \sqrt{-1} \int_0^T \gamma^* \alpha - H(g : Y) \, dt \right), \]

where $\gamma$ denotes the paths in the phase space given below.

We divide the time interval $[0, T]$ into $N$ small intervals $[k/N T, k/N T]$ ($k = 1, \cdots, N$) and fix $x_0(= x'), x_1, \cdots, x_{N-1}, x_N(= x'')$ and $p_0, p_1, \cdots, p_{N-1}$ arbitrarily. Then we consider the following paths:
\[ x(0) = x', \quad x(T) = x'' \]
\[ x(t) = x_{k-1} + \left( t - \frac{k - 1}{N} T \right) \left( \frac{x_k - x_{k-1}}{T/N} \right) \]
\[ p(t) = p_{k-1} \]

for $t \in [k/N T, k/N T]$.

Furthermore, corresponding to a quantization of the Hamiltonian functions, we take the following ordering of the Hamiltonians: On each interval $[k/N T, k/N T]$,
we replace $H(g : Y)$ by

$$H_k(g : Y) = \begin{cases} a(\sigma + p_{k-1}(x_k + x_{k-1})) & \text{if } Y = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \\ b p_{k-1} & \text{if } Y = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \\ -c(\sigma x_{k-1} + p_{k-1}x_k) & \text{if } Y = \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} \end{cases}$$

For each generator $Y \in \mathfrak{g}$, we compute

$$K_Y(x'', x' : T) = \lim_{N \to \infty} \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \prod_{j=1}^{N-1} dx_j \prod_{j=0}^{N-1} \frac{dp_j}{2\pi} \exp \sqrt{-1} \left\{ \sum_{k=1}^{N} p_{k-1}(x_k - x_{k-1}) - H_k(g : Y) \frac{T}{N} \right\}.$$ 

Then we have

$$\int_{\mathbb{R}} K_Y(x'', x' : T)F(x')dx' = (U_\lambda(\exp TY)F)(x'')$$

for each generator $Y \in \mathfrak{g}$ and $F \in \mathcal{H}_\lambda$.

Now we take another polarization and construct, following Kirillov-Kostant theory, another unitary representation which is known to be equivalent to the one given above.

Put $\mathcal{H}_{\tilde{\lambda}} = L^2(\mathbb{R})$. Then the representation $(U_{\tilde{\lambda}}, \mathcal{H}_{\tilde{\lambda}})$ is given by

$$U_{\tilde{\lambda}}(g)F(y) = | -by + d|^{-\sqrt{-1}e \sigma - 1} F \left( \frac{ay - c}{-by + d} \right)$$

for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ and $F \in \mathcal{H}_{\tilde{\lambda}}$.

Corresponding to the second polarization, we introduce local coordinates on $G$ by

$$g = \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} \begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \pm e^s & 0 \\ 0 & \pm e^{-s} \end{pmatrix}.$$ 

Then the 1-form $\varphi$ is, in this parametrization, given by

$$\varphi = \sigma(-wdy + ds).$$

Since $ds$ is exact 1-form, we choose $\tilde{\alpha} = -\sigma wdy$ and put $p' = \sigma w$. 
Then, proceeding analogously to the argument above, we can show that the path integrals give the kernel functions $\tilde{K}_Y(y'', y' : T)$ of the unitary operators $U_{\tilde{\lambda}}(\exp TY)$ for each generator $Y \in \mathfrak{g}$.

Now consider the difference of the two 1-forms:

$$\tilde{\alpha} - \alpha = \sigma d \log |1 - xy|.$$  

Therefore

$$\int_0^T \tilde{\gamma}^* \tilde{\alpha} - H(g : Y) dt - \int_0^T \gamma^* \alpha - H(g : Y) dt = \sigma (\log |1 - x''y''| - \log |1 - x'y'|),$$

which implies that

$$\int_0^T \tilde{\gamma}^* \tilde{\alpha} - H(g : Y) dt + \sigma \log |1 - x'y'|$$

$$= \sigma \log |1 - x''y''| + \int_0^T \gamma^* \alpha - H(g : Y) dt.$$  

Suggested by this, consider an integral operator with kernel function

$$e^{\sqrt{-1} \sigma \log |1 - xy|} = |1 - xy|^\sigma.$$  

But this operator does not commute with the unitary operators $U_{\lambda}(g)$ and $U_{\tilde{\lambda}}(g)$ ($g \in G$), so we modify the kernel function by multiplying $|1 - xy|^{-1}$. Then the following integral operator $A$ gives a formal intertwining operator between $(U_{\lambda}, \mathcal{H}_{\lambda})$ and $(U_{\tilde{\lambda}}, \mathcal{H}_{\tilde{\lambda}})$ [9][10]:

$$(AF)(y) = \int_{\mathbb{R}} |1 - xy|^{\sqrt{-1} \sigma - 1} F(x) dx$$  

for $F \in \mathcal{H}_{\lambda}$.

We shall give a slight generalization of this in the following.
1 Kirillov-Kostant theory

Let \( G \) be a linear connected noncompact semisimple Lie group, \( \mathfrak{g} \) its Lie algebra. We fix a Cartan involution \( \theta \) of \( \mathfrak{g} \) and denote the Cartan involution of \( G \) corresponding to that of \( \mathfrak{g} \), also by \( \theta \). Let

\[ \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \]

be the corresponding Cartan decomposition, \( B \) the Killing form on \( \mathfrak{g} \). Since \( B \) is nondegenerate, the dual space \( \mathfrak{g}^* \) of \( \mathfrak{g} \) is identified with \( \mathfrak{g} \) by

\[ \mathfrak{g}^* \ni \nu \mapsto X_\nu \in \mathfrak{g}, \quad (1.1) \]

where

\[ B(X_\nu, X) = \nu(X) \quad \text{for all } X \in \mathfrak{g}. \]

We also use the notation \( \langle \nu, X \rangle \) for \( \nu(X) \).

Let \( \mathfrak{a} \subset \mathfrak{p} \) be a maximal abelian subspace, \( \Sigma \) the corresponding set of nonzero restricted roots, and \( \mathfrak{m} \) the centralizer \( Z_{\mathfrak{k}}(\mathfrak{a}) \) of \( \mathfrak{a} \) in \( \mathfrak{k} \). Fix a Weyl chamber in \( \mathfrak{a} \) and let \( \Sigma^+ \) denote the corresponding set of positive restricted roots. Then we have the decomposition

\[ \mathfrak{g} = \mathfrak{g}_0 \oplus \sum_{\alpha \in \Sigma} \mathfrak{g}_\alpha, \]

where

\[ \mathfrak{g}_0 = \mathfrak{a} \oplus \mathfrak{m} \quad \text{and} \quad \mathfrak{g}_\alpha = \{ X \in \mathfrak{g} \mid [H, X] = \alpha(H)X \text{ for } H \in \mathfrak{a} \}. \]

Define

\[ \mathfrak{n} = \sum_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha \quad \text{and} \quad \rho = \frac{1}{2} \sum_{\alpha \in \Sigma^+} m_\alpha \alpha, \]

where \( m_\alpha = \dim \mathfrak{g}_\alpha \).

Let \( K, A, N \) be the analytic subgroups corresponding to \( \mathfrak{k}, \mathfrak{a}, \mathfrak{n} \), respectively, and \( M \) the centralizer \( Z_K(\mathfrak{a}) \) of \( \mathfrak{a} \) in \( K \). Then \( NMA\overline{N} \) is an open subset of \( G \) whose complement is of lower dimension and has Haar measure 0, where \( \overline{N} = \theta N \).

For any element \( \nu \in \mathfrak{a}^* \) we denote by \( H_\nu \) the element of \( \mathfrak{a} \) such that

\[ B(H, H_\nu) = \nu(H) \quad \text{for all } H \in \mathfrak{a}. \quad (1.2) \]

We extend any linear form \( \nu \) on \( \mathfrak{a} \) to a linear form on \( \mathfrak{g} \) by defining \( \nu \) to vanish on the orthogonal complement of \( \mathfrak{a} \) with respect to the Killing form.
Let \( \lambda \) be an element of \( \mathfrak{a}^* \) which corresponds to a regular element of \( \mathfrak{a} \) by (1.2). We denote the coadjoint action of \( G \) on \( \mathfrak{g}^* \) by \( \text{Ad}^* \). Then it is easy to see that the isotropy subgroup

\[
G_\lambda = \{ g \in G ; \text{Ad}^*(g)\lambda = \lambda \}
\]

at \( \lambda \) equals \( MA \), and its Lie algebra \( \mathfrak{g}_\lambda \) equals \( \mathfrak{m} \oplus \mathfrak{a} \). As a real polarization we take \( \mathfrak{s}_- = \mathfrak{m} \oplus \mathfrak{a} \oplus \overline{\mathfrak{n}} \), where \( \overline{\mathfrak{n}} = \theta \mathfrak{n} \). Correspondingly, we put \( S_- = MAN \).

Then the Lie algebra homomorphism

\[
-\sqrt{-1}\lambda : \mathfrak{s}_- \longrightarrow \sqrt{-1} \mathbb{R}, \quad X_0 + H + X_- \longrightarrow -\sqrt{-1}\lambda(H)
\]

lifts to the unitary character of \( S_- \):

\[
S_- \longrightarrow U(1), \quad m \exp H \overline{\mathfrak{n}} \longrightarrow e^{-\sqrt{-1}\lambda(H)}.
\]

We define a one-dimensional representation \( \xi_\lambda \) of \( S_- \) by

\[
\xi_\lambda : S_- \longrightarrow \mathbb{C}^\times, \quad m \exp H \overline{\mathfrak{n}} \longrightarrow e^{-(\sqrt{-1}\lambda + \rho)(H)}.
\]

Let \( L_\lambda \) be the \( C^\infty \)-line bundle over \( G/S_- \) associated to the one-dimensional representation \( \xi_\lambda \) of \( S_- \). Then we can identify the space of all \( C^\infty \)-sections of \( L_\lambda \) with

\[
C^\infty(L_\lambda) = \{ f \in C^\infty(G); f(xb) = \xi_\lambda(b)^{-1}f(x), x \in G, b \in S_- \}.
\]

For any \( f \in C^\infty(L_\lambda) \) we put

\[
\|f\|^2 = \int_K |f(k)|^2 dk,
\]

where \( dk \) is a Haar measure on \( K \). Let \( V_\lambda \) be the completion of \( C^\infty(L_\lambda) \) with respect to the norm. For \( g \in G, f \in C^\infty(L_\lambda) \) and \( x \in G \), we define

\[
\pi_\lambda(g)f(x) = f(g^{-1}x).
\]

Then \( \pi_\lambda \) can be uniquely extended to a unitary operator on \( V_\lambda \), which we also denote by \( \pi_\lambda \).

For each \( \alpha \in \Sigma^+ \) we can find nonzero root vectors \( E_{\alpha,i} \in \mathfrak{g}_{\alpha} \) \( (i = 1, \ldots, m_\alpha) \) such that

\[
B(E_{\alpha,i}, \theta E_{\alpha,j}) = -\delta_{ij},
\]

where \( \delta_{ij} \) is Kronecker's delta. Put \( E_{-\alpha,i} = -\theta E_{\alpha,i} \) and introduce differentiable coordinates on \( \mathfrak{n} \) and \( \overline{\mathfrak{n}} \) as follows:

\[
\mathbb{R}^m \longrightarrow \mathfrak{n}, \quad x = (x_{\alpha,i})_{\alpha \in \Sigma^+, i=1,\ldots,m_\alpha} \longrightarrow \sum_{\alpha \in \Sigma^+} \sum_{i=1}^{m_\alpha} x_{\alpha,i} E_{\alpha,i}
\]
\[ \mathbb{R}^m \rightarrow \bar{n}, \quad y = (y_{\alpha,i})_{\alpha \in \Sigma^+, i=1, \ldots, m_\alpha} \mapsto \sum_{\alpha \in \Sigma^+} \sum_{i=1}^{m_\alpha} y_{\alpha,i} E_{\alpha,i}, \]

where \( m = \dim \mathfrak{n} \). Put

\[ n_x = \exp \sum_{\alpha \in \Sigma^+} \sum_{i=1}^{m_\alpha} x_{\alpha,i} E_{\alpha,i} \in N \]  
\[ \bar{n}_y = \exp \sum_{\alpha \in \Sigma^+} \sum_{i=1}^{m_\alpha} y_{\alpha,i} E_{\alpha,i} \in \bar{N}. \]

We define a map \( L \) of \( C^\infty(L_\lambda) \) into \( C^\infty(N) \) by

\[ Lf(n) = f(n) \quad \text{for} \quad f \in C^\infty(L_\lambda). \]

Then, defining a norm on \( C^\infty(N) \) with respect to a Haar measure on \( N \), one can show that

\[ \|f\|^2 = \|Lf\|^2, \]

when the Haar measures are suitably normalized.

Let \( \mathcal{H}_\lambda \) be the completion of the image of \( C^\infty(L_\lambda) \) by \( L \). Then one can show that \( L \) is extended to an isometry of \( V_\lambda \) onto \( \mathcal{H}_\lambda \). Define a representation \((U_\lambda, \mathcal{H}_\lambda)\) of \( G \) such that the following diagram commutes for any \( g \in G \):

\[ \begin{array}{ccc}
V_\lambda & \xrightarrow{L} & \mathcal{H}_\lambda \\
\pi_\lambda(g) \downarrow & & \downarrow U_\lambda(g) \\
V_\lambda & \xrightarrow{L} & \mathcal{H}_\lambda.
\end{array} \]

For \( g \in NMA\bar{N} \), we write as

\[ g = n(g)m(g)a(g)\bar{n}(g). \]

Then

\[ U_\lambda(g)F(x) = e^{(\sqrt{-1}\lambda + \rho)\log a(g^{-1}n_x)} F(n(g^{-1}n_x)) \]

for \( F \in L(C^\infty(L_\lambda)) \).
2 Quantization

We retain the notation of §1. Moreover, for \( z = (z_{\alpha,i})_{\alpha \in \Sigma^+, i = 1 \ldots m_\alpha} \), we put

\[
X = \sum_{\alpha \in \Sigma^+} \sum_{i=1}^{m_\alpha} x_{\alpha,i} E_{\alpha,i}.
\]

(2.1)

In this section we compute the differential representation \( dU_\lambda \) of \( U_\lambda \) and quantize the Hamiltonian functions for \( Y \in m \oplus \alpha \) or \( \mathfrak{n} \).

We decompose \( \text{Ad}(e^{-X})Y \) as

\[
\text{Ad}(e^{-X})Y = X_+ + X_0 + H + X_-
\]

(2.2)

with \( X_+ \in \mathfrak{n}, X_0 \in m, H \in \alpha \) and \( X_- \in \overline{\mathfrak{n}} \).

Then, for \( Y \in \mathfrak{g} \) and \( F \in C_c^\infty(N) \), \( dU_\lambda(Y) \) is given by

\[
dU_\lambda(Y)F(x) = -\left( \sqrt{-1} \left\{ \lambda, \text{Ad}(n_x)^{-1}Y \right\} + \left\{ p, \text{Ad}(n_x)^{-1}Y \right\} \right) F(x)
- \sum_{\alpha \in \Sigma^+} \sum_{i=1}^{m_\alpha} c_{\alpha,i} \partial_{\alpha,i} F(x),
\]

(2.3)

where \( x = (x_{\alpha,i}), n_x = \exp X, \partial_{\alpha,i} = \partial/\partial x_{\alpha,i} \) and

\[
c_{\alpha,i} = B \left( \frac{\text{ad}X}{1 - e^{-\text{ad}X}} X_+, E_{-\alpha,i} \right).
\]

Define a 1-form \( \varphi \) by

\[
\varphi = \left\{ \lambda, g^{-1}dg \right\}
- \left\{ \text{Ad}^*(\overline{n})\lambda, n(g)^{-1}dn(g) \right\} + \left\{ \lambda, a(g)^{-1}da(g) \right\},
\]

where \( d \) is the exterior derivative on \( G \) and \( \overline{n} = m(g)a(g)\overline{n}(g)(m(g)a(g))^{-1} \).

Since the second term is an exact 1-form, we choose

\[
\alpha_{s_-} = \left\{ \text{Ad}^*(\overline{n})\lambda, n(g)^{-1}dn(g) \right\}.
\]

and parametrize \( n(g) \) as \( n(g) = \exp X \), where \( X \) is of the form (2.1). Let

\[
p_{\alpha,i} = \alpha_{s_-}(\partial_{\alpha,i})
\]

i.e. \( p_{\alpha,i} \) is the coefficient of \( dx_{\alpha,i} \) in \( \alpha_{s_-} : \alpha_{s_-} = \sum_{\alpha,i} p_{\alpha,i} dx_{\alpha,i}. \) Then \( p_{\alpha,i} \) is given by

\[
p_{\alpha,i} = B \left( \frac{e^{\text{ad}X} - 1}{\text{ad}X} \text{Ad}(\overline{n}) H_\lambda, E_{\alpha,i} \right).
\]
Using $c_{\alpha,i}$ and $p_{\alpha,i}$, we have, for $Y \in \mathfrak{g}$,

\[ H(g : Y) = \langle \lambda, \text{Ad}(n_x)^{-1}Y \rangle + \sum_{\alpha \in \Sigma^+} \sum_{i=1}^{m_{\alpha}} c_{\alpha,i}p_{\alpha,i}, \]  

(2.4)

where $g \in NMA\overline{N}$ and $n(g) = n_x = \exp X$.

Now, using (2.4), we quantize the Hamiltonian function for $Y \in m \oplus \alpha$ or $\mathfrak{n}$, replacing $x_{\alpha,i}$ and $\sqrt{-1}p_{\alpha,i}$ in $H(g : Y)$ by $x_{\alpha,i} \times$ (multiplication operator) and $\partial_{\alpha,i}$, respectively, (canonical quantization!) and choosing an operator ordering between $x_{\alpha,i}'s$ and $\partial_{\alpha,i}'s$.

**Proposition 2.1.** For $Y \in m \oplus \alpha$ or $\mathfrak{n}$, we define quantized Hamiltonians $H(Y)$ as follows:

(i) For $Y \in m \oplus \alpha$,

\[ H(Y) = \langle \lambda, Y \rangle - \frac{\sqrt{-1}}{2} \sum_{\alpha \in \Sigma^+} \sum_{i=1}^{m_{\alpha}} \{ c_{\alpha,i} \partial_{\alpha,i} + \partial_{\alpha,i} \circ c_{\alpha,i} \}; \]

(ii) For $Y \in \mathfrak{n}$,

\[ H(Y) = -\sqrt{-1} \sum_{\alpha \in \Sigma^+} \sum_{i=1}^{m_{\alpha}} \partial_{\alpha,i} \circ c_{\alpha,i}, \]

where $\circ$ denotes the composition of operators. Then the quantized Hamiltonian coincides with $\sqrt{-1}dU_{\lambda}(Y)$.

**Remark.** If $Y \in \mathfrak{n}$, since $\partial_{\alpha,i}c_{\alpha,i} = 0$, we obtain

\[ H(Y) = -\sqrt{-1} \sum_{\alpha \in \Sigma^+} \sum_{i=1}^{m_{\alpha}} c_{\alpha,i} \partial_{\alpha,i} \]

\[ = -\frac{\sqrt{-1}}{2} \sum_{\alpha \in \Sigma^+} \sum_{i=1}^{m_{\alpha}} \{ c_{\alpha,i} \partial_{\alpha,i} + \partial_{\alpha,i} \circ c_{\alpha,i} \}. \]

But we do not adopt these quantizations in the present paper.
3 Path integrals

In this section we show that the path integrals with Hamiltonian functions with $Y \in m \oplus a$ or $n$ give the kernel function of the unitary operator constructed in §1. For detail, we refer the reader to [6].

The path integral is, symbolically, given by

$$\int D(x, p) \exp \left( \sqrt{-1} \int_0^T \gamma^* \alpha_{\epsilon} - H(g : Y) dt \right)$$

for $Y \in g$, where $\gamma$ denotes certain paths in the phase space [3].

Here we divide the time interval $[0, T]$ into $N$ small intervals

$$\left[ \frac{k-1}{N} T, \frac{k}{N} T \right] \quad (k = 1, \ldots, N).$$

On each small interval $[\frac{k-1}{N} T, \frac{k}{N} T]$, Proposition 2.1 indicates that we should take the following ordering of Hamiltonian functions $H_k(g : Y)$ with $Y \in m \oplus a$ or $n$.

(i) For $Y \in m \oplus a$,

$$H_k(g : Y) = \langle \lambda, Y \rangle + \frac{1}{2} \sum_{\alpha \in \Sigma^+} \sum_{i=1}^{m_\alpha} (c_{\alpha,i}^k p_{\alpha,i}^{k-1} + p_{\alpha,i}^{k-1} c_{\alpha,i}^{k-1}),$$

where $c_{\alpha,i}^k = \alpha(Y) x_{\alpha,i}^k$.

(ii) For $Y \in n$,

$$H_k(g : Y) = \sum_{\alpha \in \Sigma^+} \sum_{i=1}^{m_\alpha} p_{\alpha,i}^{k-1} c_{\alpha,i}^{k-1},$$

where

$$c_{\alpha,i}^k = B \left( \frac{\text{ad} X^k}{e^{\text{ad} X^k} - 1} Y, E_{-\alpha,i} \right)$$

and $X^k = \sum_{\alpha, i} x_{\alpha,i}^k E_{\alpha,i}$.

Now the computation of the path integral.

For $x = (x_{\alpha,i}), x' = (x'_{\alpha,i})$ given, let $x_0^{\alpha,i} = x_{\alpha,i}, x_N^{\alpha,i} = x'_{\alpha,i}$. We put

$$dx^j = \prod_{\alpha \in \Sigma^+} \prod_{i=1}^{m_\alpha} dx_{\alpha,i}^j$$

and

$$dp^j = \frac{1}{(2\pi)^m} \prod_{\alpha \in \Sigma^+} \prod_{i=1}^{m_\alpha} dp_{\alpha,i}^j$$

for brevity, where $m = \text{dim } n$. Remark that the Haar measure $dx$ on $N$ equals the Haar measure $dn$ given in §1, up to constant multiple.
A. Path integral for $Y \in m \oplus a$

Recall that if $Y \in m \oplus a$, then $H_k(g : Y)$ is given by

$$H_k(g : Y) = \langle \lambda, Y \rangle + \frac{1}{2} \sum_{\alpha \in \Sigma^+} \sum_{i=1}^{m_{\alpha}} p_{\alpha,i}^{k-1} (c_{\alpha,i}^k + c_{\alpha,i}^{k-1}),$$

where $c_{\alpha,i}^k = \alpha(Y)x_{\alpha,i}^k$.

B. Path integral for $Y \in n$

Recall that if $Y \in n$, then $H_k(g : Y)$ is given by

$$H_k(g : Y) = \sum_{\alpha \in \Sigma^+} \sum_{i=1}^{m_{\alpha}} p_{\alpha,i}^{k-1} c_{\alpha,i}^{k-1},$$

where

$$c_{\alpha,i}^k = B\left(\frac{\mathrm{ad}X^k}{\mathrm{ad}X^k - 1} Y, E_{-\alpha,i}\right)$$

and $X^k = \sum_{\alpha,i} x_{\alpha,i}^k E_{\alpha,i}$. Now we assume that

$$C^0n \supset C^1n \supset C^2n \supset C^3n = \{0\},$$

where $C^0n = n$ and $C^{i+1}n = [n, C^i n]$.

Then, computing the path integrals as in §0, we obtain

**Theorem 3.1.** (i) For $Y \in m \oplus a$, taking the ordering of the Hamiltonian function $H(g : Y)$ ($g \in NMAN$) described in this section, the path integral with the Hamiltonian gives the kernel function of the operator $U_\lambda(\exp TY)$.

(ii) Assume that the length of the central descending series of $n$ is $\leq 3$ (see (3.1)). Then for $Y \in n$, taking the ordering of the Hamiltonian function $H(g : Y)$ ($g \in NMAN$) described in this section, the path integral with the Hamiltonian gives the kernel function of the operator $U_\lambda(\exp TY)$. 
4 Intertwining Operator

In this section we take another real polarization and show that the formal intertwining operator between the two representations can be obtained from the path integral.

Let $\lambda$ be the same element of $a^*$ as in §1. We take another real polarization $s_+ = m \oplus a \oplus n$. Correspondingly, we put $S_+ = MAN$. Then the Lie algebra homomorphism

$$-\sqrt{-1}\lambda : s_+ \longrightarrow -\sqrt{-1}R,$$

$$X_0 + H + X_+ \longmapsto -\sqrt{-1}\lambda(H)$$

lifts to the unitary character of $S_+$:

$$S_+ \longrightarrow U(1), \quad m \exp H n \longmapsto e^{-\sqrt{-1}\lambda(H)}.$$

We define a one-dimensional representation $	ilde{\xi}_{\lambda}$ of $S_+$ by

$$\tilde{\xi}_{\lambda} : S_+ \longrightarrow C^x, \quad m \exp H n \longmapsto e^{(-\sqrt{-1}\lambda + \rho)(H)}.$$

Let $(H_{\tilde{\lambda}}, U_{\tilde{\lambda}})$ be the unitary representation of $G$ which is constructed from $\tilde{\xi}_{\lambda}$ as in §1, instead of $\xi_{\lambda}$. Note that $\tilde{F} \in H_{\tilde{\lambda}}$ is a function on $\overline{N}$, on which we introduced coordinates by (1.6).

For $g \in \overline{N}MAN$, we write as

$$g = \overline{n}(g)m'(g)a'(g)n'(g) \quad (4.1)$$

and parametrize $\overline{n}(g)$ as $\overline{n}(g) = \overline{n}_y = \exp Y$, where $Y$ is of the form

$$Y = \sum_{\alpha \in \Sigma^+} \sum_{i=1}^{m_\alpha} y_{\alpha,i} E_{-\alpha,i}. \quad (4.2)$$

Then for $g \in G$ and $\tilde{F} \in H_{\tilde{\lambda}}$ the action is

$$U_{\tilde{\lambda}}(g)\tilde{F}(y) = e^{(\sqrt{-1}\lambda - \rho)\log a'(g^{-1}\overline{n}_y)}\tilde{F}(m'(g^{-1}\overline{n}_y)), \quad (4.3)$$

where $y = (y_{\alpha,i})$ and $\overline{n}_y = \exp \sum_{\alpha \in \Sigma^+} y_{\alpha,i} E_{-\alpha,i}$. If we use the parametrization (4.1), then $\varphi$ is given by

$$\varphi = \langle \lambda, g^{-1}dg \rangle = \langle \text{Ad}^*(n')\lambda, \overline{n}(g)^{-1}d\overline{n}(g) \rangle + \langle \lambda, a'(g)^{-1}da'(g) \rangle,$$

where $n' = m'(g)a'(g)n'(g)(m'(g)a'(g))^{-1}$. Since the second term is an exact 1-form, we choose

$$\alpha_{s_+} = \langle \text{Ad}^*(n')\lambda, \overline{n}(g)^{-1}d\overline{n}(g) \rangle.$$
Fixing $y' = (y'_{\alpha,i})$ and $y = (y_{\alpha,i})$, we can explicitly compute the path integral with Hamiltonian function for $Y' \in \mathfrak{m} \oplus \mathfrak{a}$ or $\overline{\mathfrak{n}}$, in the same way as in §3.

For $g \in NMAN \cap NMAN$, write $g$ in two ways:

$$g = n(g)\overline{n}m(g)a(g) = \overline{n}'(g)n'm'(g)a'(g).$$

Then we have

$$\alpha_{s_{-}} - \alpha_{s_{+}} = \{\lambda, a^{-1}da\},$$

where $a = a(\overline{n}'(g)^{-1}n(g))$.  

We parametrize $n(g) = n_{x} = \exp X$ and $\overline{n}'(g) = \overline{n}_{y} = \exp Y$, where $X$ (or $Y$) is of the form (2.1) (or (4.2), respectively), and fix $x' = (x_{\alpha,i})$, $x = (x_{\alpha,i})$, $y' = (y_{\alpha,i})$ and $y = (y_{\alpha,i})$.

Then using (4.4) and proceeding analogously to the argument in §0, we can show that an integral operator with kernel function

$$\exp((-\sqrt{-1} + \rho)\log a(\overline{n}_{\overline{\nu}^{1}}n_{x}))$$

coincides with the formal intertwining operator $A(S_{+} : S_{-} : 1 : \sqrt{-1}\lambda)$ given in [9][10]. The integral operator with kernel function (4.5) is not well-defined in the sense that the integral

$$\int_{N} e^{-\sqrt{-1}\lambda\rho} \log a(n'_{y}^{-1}n_{x}) F(x) dx$$

need not converge for $F \in \mathcal{H}_{\lambda}$. Knapp and Stein showed in [9][10] that if one regularizes the integral suitably, then the regularized operator, $\mathcal{A}(S_{+} : S_{-} : 1 : \sqrt{-1}\lambda)$ in their notation, is a well-defined intertwining operator and is invertible, i.e., the following diagram commutes for all $g \in G$.

$$\mathcal{H}_{\lambda} \xrightarrow{A(S_{+} : S_{-} : 1 : \sqrt{-1}\lambda)} \mathcal{H}_{\tilde{\lambda}}$$

$$U_{\lambda}(g) \downarrow \quad \downarrow U_{\tilde{\lambda}}(g)$$

$$\mathcal{H}_{\lambda} \xrightarrow{A(S_{+} : S_{-} : 1 : \sqrt{-1}\lambda)} \mathcal{H}_{\tilde{\lambda}}$$

**Theorem 4.1.** The path integral with the action defined by (4.5) provides the formal intertwining operator $A(S_{+} : S_{-} : 1 : \sqrt{-1}\lambda)$, where $A(S_{+} : S_{-} : 1 : \sqrt{-1}\lambda)$ is given by

$$A(S_{+} : S_{-} : 1 : \sqrt{-1}\lambda)f(\tilde{n}_{y}) = \int_{N} f(\tilde{n}_{y}n_{x}) dx \quad \text{for } f \in \mathcal{V}_{\lambda}.$$
when the indicated integrals are convergent.

We can compute the path integral for $Y \in \mathfrak{g}$ using the polarization given in this section in the same way as in §3.

Thus, considering the composition

$$A(S_+: S_- : 1 : \sqrt{-1}\lambda)^{-1} \circ U_{\lambda}(\exp TY) \circ A(S_+: S_- : 1 : \sqrt{-1}\lambda),$$

we can obtain the unitary operators $U_{\lambda}(\exp TY)$ for $Y \in \mathfrak{g}$ by the path integrals.

REFERENCES


