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Large indiscernible sets of a structure

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1 Introduction

An indiscernible set of a given structure is by definition a set I such that every finite subset of the same cardinality has the same type. A sigleton $I = \{a\}$ is trivially an indiscernible set, so it is called a trivial one. A transciendental basis of an algeraically closed field K is a good example of a non-trivial indiscernible set. In this example, if K is an regular uncountable, then it has a large indiscernible set I, i.e. an indiscernible set I with $|I| = |K|$. Generally speaking, if a theory T is ω-stable then every uncountable model of T has such a large indiscernible set. However, in the structure $\mathbf{R} = (\mathbf{R}, 0, 1, +, \cdot)$, there is no non-trivial indiscernible set, i.e. $\mathrm{tp}(a) = \mathrm{tp}(b)$ implies $a = b$.

In this note we show that every L-structure M can be embedded into a structure M^* of an expanded language L^* such that any L^*-structure $N \equiv M^*$ has a large indiscernible set. We also show that if T is stable and non-ω-stable then there is a model of power $\aleph_1$ which has no large indiscernible sets.

2 Preliminaries

In what follows, T is a complete theory formulated in a countable language L. We give some necessary definitions and review some basic results.

Definition 1. (1) Let I be a subset of a struture M. I is said to be an indiscernible set if whenever $F \subset I$ and $G \subset I$ are finite sequences of the same length then $\mathrm{tp}(F) = \mathrm{tp}(G)$.

(2) We will say that an indiscernible set I in a structure M is large if I has the same cardinality as M.

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Fact 1 (Theorem 2.8 of [S, CH.I, §2]). *If T is ω-stable, then every uncountable model of T includes a large indiscernible set.*

If T is not ω-stable, then any (a,ω)-model is uncountable. And any (a,$\kappa(T)$)-prime model does not have indiscernible set of power greater than $\kappa(T)$. So we have:

Fact 2. *If T is a non-ω-stable, superstable theory, then there is a model of power $\aleph_1$ without a large indiscernible set.*

Let T be the theory of refining equivalence relations. i.e., T is the theory of the structure $(2^\omega, E_1, E_2, \ldots)$, where $E_i = \{(\eta_1, \eta_2) \in (2^\omega)^2 : \eta_1|i = \eta_2|i\}$. Then T is a superstable theory with $|\mathrm{S}(T)| = 2^{\aleph_0}$. Let M be any uncountable elementary submodel of $(2^\omega, E_1, E_2, \ldots)$. M has no large indiscernible sets.

Definition 2. A model $M \supset A$ is said to be ℓ-atomic over A if for every $\bar{a} \in M$, and every finite set Δ of formulas, $\mathrm{tp}_\Delta(\bar{a}/A)$ is a principal type.

Fact 3. *Let T be stable.*
(1) For every set A, there is an ℓ-atomic model over A.
(2) Let a_1 and a_2 be independent over M. Let M_i be an ℓ-atomic model over $M \cup \{a_i\}$. Then M_1 and M_2 are independent over M.

3 Main Result

We want to extend fact 2 to a non-ω-stable, stable theory T. The following lemma will play a crucial role.

Lemma. *Let T be a non-ω-stable, stable theory and $\kappa \leq 2^{\aleph_0}$ an uncountable cardinal. Then there is a set R of types over a set A, $|A| < \kappa$ such that whenever $B \supset A$ is a set with $|B| < \kappa$ and S is a set of stationary types over B with $|S| < \kappa$ then there is a non-algebraic type $r \in R$ which is almost orthogonal to any type in S.*

Proof. This lemma remains true for a superstable theory, but we concentrate on an unsuperstable theory. (Superstable case is easier.) Since T is not superstable, there are infinitely long continuous sequence $\{p_i : i \leq \alpha\}$ of types such that

(1) $\mathrm{dom} p_i$ is a countable set;

(2) p_i is a forking extension of p_j, if $i > j$;

(3) $\alpha < \omega_1$ is a countable limit ordinal;

(4) $U(p_\alpha) < \infty$.

By choosing a subsequence of $\{p_i : i \leq \alpha\}$, we can assume that $\alpha = \omega$. Now by the definition of forking, we can easily find a countable set A_0, and continuously many types $\{q_i : i < 2^{\aleph_0}\}$ over A_0 such that each q_i is U-ranked ($U(q_i) < \infty$). We can assume that each type q_i is stationary.

Suppose that our lemma does not hold. By induction on $j < \omega$, we define a set A_j of cardinality $< \kappa$ and types $q_{i,j} \in \mathrm{S}(A_j)$ ($i < 2^{\aleph_0}$) such that for any $i < 2^{\aleph_0}$, $k < j$,

$$q_{i,k} \text{ is algebraic } \text{ or } q_{i,j} \text{ is a forking extension of } q_{i,k}.$$

For each $i < 2^{\aleph_0}$, let $q_{i,0} = q_i$. Suppose we have defined $q_{i,k} \in \mathrm{S}(A_k)$ for $i < 2^{\aleph_0}$ and $k < j$. Let $\Lambda = \{i < 2^{\aleph_0} : q_{i,j-1} \text{ is non-algebraic}\}$. Since we are assuming the negation of the statement in our lemma, there are a set $B \supset A_{j-1}, |B| < \kappa$ and a set $S \subset \mathrm{S}(B)$, $|S| < \kappa$ such that every $q_{i,j-1}$ ($i \in \Lambda$) is not almost orthogonal to some $s_i \in S$. For $i \in \Lambda$, choose $a_i \models q_{i,j-1}|B$ and $b_i \models s_i$ such that a_i and b_i are dependent over B. We can assume that if $s_i = s_j$ then $b_i = b_j$. Now let

$$A_j = \mathrm{acl}(A_{j-1} \cup \{b_i : i \in \Lambda\});$$

$$q_{i,j} = \begin{cases} \mathrm{tp}(a_i/A_j) & i \in \Lambda \\ \text{arbitrary extension of } q_{i,j-1} & i \notin \Lambda \end{cases}$$

Finally let $A_\omega = \bigcup_{j<\omega} A_j$. Note that $|A_\omega| < 2^{\aleph_0}$. (If $\kappa = 2^{\aleph_0}$, then $\mathrm{cf}(\kappa) > \omega$, so $|A_\omega| < \kappa = 2^\omega$. If $\kappa < 2^{\aleph_0}$, then $|A_\omega| \leq \kappa < 2^{\aleph_0}$.) Since q_i is U-ranked by (4), $q_i^* = \bigcup_{j<\omega} q_{i,j} \in \mathrm{S}(A_\omega)$ must be an algebraic type. (Otherwise there is an infinitely long forking sequence starting from q_i.) So we have constructed continuously many distinct algebraic types over a fixed set A_ω, $|A_\omega| < 2^\omega$. However this is a contradiction, since we are assuming that L is countable.

Theorem A. *Let T be a non-ω-stable, stable theory. Then for any uncountable cardinal $\kappa \le 2^{\aleph_0}$, there is a model of power κ without a large indiscernible set.*

Proof. Choose a set A and types $R \subset S(A)$ which satisfy the condition in the above lemma. Let $\lambda = |A|$. Clearly $\lambda < \kappa$. We construct an elementary chain of models $\{M_i : i \le \kappa\}$ such that each model M_i has cardinality $\le |i| + \lambda$. Without loss of generality, A is a model. Let $M_0 = A$, and M_1 an arbitrary proper extension of M_0 with the same cardinality. Suppose that we have constructed $\{M_i : i < \alpha\}$. If α is a limit ordinal, then let $M_\alpha = \bigcup_{i<\alpha} M_i$. So we assume that $\alpha = \beta + 1$, and let

$$S_\beta = \bigcup_{i<\beta} \{q(x) \in S(M_\beta) : q \text{ is based on } M_i, q|M_i \text{ is realized in } M_\beta\}$$

Clearly $|S_\beta| \le |\beta| + \lambda < \kappa$. By the property of R, there is a type $r \in R$ which is almost orthogonal to each type in S_β. Let $M_{\beta+1}$ be an ℓ-atomic model over $M_\beta \cup \{e_\beta\}$, where e_β is a realization of $r|M_\beta$. Of course we can assume $|M_{\beta+1}| < |\beta + 1| + \lambda$.

Claim. *There is no large indiscernible set in M_κ.*

Suppose that there was a large indiscernible set $I \subset M_\kappa$. By stability, there is a countable set $I_0 \subset I$ such that $J = I - I_0$ is a Morley sequence over I_0. Choose M_i $(i < \kappa)$ which includes I_0. Since $M_i < \kappa$, we may assume that J is a Morley sequence over M_i, by choosing a subset of J if necessary. Choose M_j $(j < \kappa)$ which intersects with J. Let $a \in J \cap M_j$. Since $|J| = \kappa$, there is $b \in J$ which is indepent from M_j over M_i. Choose the least k such that b and M_k are dependent over M_i. Then k is a successor ordinal greater than j, and

(1) b and M_k are dependent over M_{k-1};

(2) b and M_{k-1} are independent over M_i.

Remember that M_k is ℓ-atomic over $M_{k-1} \cup \{e_{k-1}\}$. From (1), using fact 3, we know that b and e_{k-1} are dependent over M_{k-1}. By our choice of e_{k-1}, $\mathrm{tp}(e_{k-1}/M_{k-1})$ is almost orthogonal to every type in S_{k-1}, hence $\mathrm{tp}(b/M_{k-1})$ does not belong to S_{k-1}. Note that $\mathrm{tp}(b/M_i)$ is realized by $a \in M_{k-1}$. Then we must have

(3) $\text{tp}(b/M_{k-1})$ is a forking extension of $\text{tp}(a/M_i)$.

(2) and (3) yield a contradiction.

Next theorem shows that theorem A cannot be extended to an unstable theory.

Theorem B. *Let M be an infinite L-structure. Then there is a structure M^* for an expanded language $L^* \supset L$ with the following properties:*

(i) M is $\emptyset$-definable in M^*;

(ii) In any L^*-structure $N \equiv M^*$, there is a large indiscernible set in N.

Proof. For $i < \omega$, let $L_i = L \cup \{F_j(*) : j = 0, ..., i\} \cup \{U(*)\} \cup \{R_j(*, *, *) : j = 1, ..., i\}$, where F_i's and U are unary predicate symbols, and R_j's are 3-ary predicate symbols. Let $L^* = \bigcup_{i<\omega} L_i$. We construct inductively countable L_j-structures M_j and countable subgroups S_j of $\text{Aut}(M_j)$ ($j < \omega$) with the following properties:

(1) $M_0 = F_0^{M_0} \cup U^{M_0}$, where $F_0^{M_0} = M$, and U^{M_0} is an infinite set disjoint from $F_0^{M_0}$.

(2) S_0 is a countable subgroup of $\text{Aut}(M_0)$ such that for given finite sequences $\bar{a} \in U^{M_0}$ and $\bar{b} \in U^{M_0}$ of the same length, there is a $\sigma \in S_0$ with $\sigma(\bar{a}) = \bar{b}$. Any two automorphisms $f \in S_0$ and $g \in S_0$ differ at finitely many points.

(3) $M_{j+1} = M_j \cup F_{j+1}^{M_{j+1}}$,

(4) $S_j = \{\sigma \lceil M_j : \sigma \in S_{j+1}\}$.

Assume that we have already constructed M_j and S_j for $j < i$. Choose a bijective function $f_0 : F_{i-1}^{M_{i-1}} \to U^{M_{i-1}}$ arbitrarily and let

$$F_i^{M_i} = \{\sigma \circ f_0 \circ \sigma^{-1} : \sigma \in S_{i-1}\}.$$

$F_i^{M_i}$ is a countable set of functions from $F_{i-1}^{M_{i-1}}$ to $U^{M_{i-1}}$. Define $R_i^{M_i} \subset F_i^{M_i} \times F_{i-1}^{M_{i-1}} \times U^{M_{i-1}}$ by

$$(f, a, b) \in R_i^{M_i} \Leftrightarrow f(a) = b.$$

Now let $M_i = M_{i-1} \cup F_i^{M_i}$. We can extend each $\tau \in S_{i-1}$ to an automorphism τ^* of M_i. Let $f = \sigma \circ f_0 \circ \sigma^{-1} \in S_{i-1}$. Then define

$$\tau^*(f) = \tau \circ f \circ \tau^{-1} = (\tau\sigma) \circ f_0 \circ (\tau\sigma)^{-1} \in S_{i-1}.$$

The following equivalence shows that τ^* is really an automorphism:

$$\begin{aligned} M_i \models R(f, a, b) &\Leftrightarrow f(a) = b \\ &\Leftrightarrow \tau^*(f(\tau^{*-1}(\tau^*(a))) = \tau^*(b) \\ &\Leftrightarrow M_i \models R(\tau^*(f), \tau^*(a), \tau^*(b)). \end{aligned}$$

Finally we set $M^* = \bigcup_{i<\omega} M_i$, $T^* = \mathrm{Th}_{L^*}(M)$. Now it is sufficient to prove the following two claims.

Claim 1. *In any model N of T^*, U^N is an indiscernible set.*

It is suffinient to prove the statement for the case $N = M^*$. Let $\bar{a}, \bar{b} \in U^{M^*}$ be given. By the assumption on S_0, there is a $\sigma \in S_0$ such that $\sigma(\bar{a}) = \bar{b}$. σ can be extended to an automorphism of M^*. So $\bar{a} \equiv \bar{b}$.

Claim 2. *If $N \models T^*$, then there is a large indiscernible set.*

Clearly $U^N \cup \bigcup_i F_i^N$ has the same cardinality as N, or the complement $N - (U^N \cup \bigcup_i F_i^N)$ has the same cardinality as N. The second case clearly implies that $N - (U^N \cup \bigcup_i F_i^N)$ is a large indiscernible set. Let the second case hold. Note that an element in F_{i+1} gives a bijection between F_i^N and U^N. Then we see that U^N has the same cardinality as N. By claim 1, U^N is a large indiscernible set in this case.

Remark. (i) Any model of $T = \mathrm{Th}(\mathbf{Z}, <)$ has a large indiscernible *sequence*. (ii) The construction of M^* was inspired by [F], in which Fuhrken showed the existence of an uncountable complete theory without the omitting types property. Note that our T^* is not stable: By our choice of S_0 and F_1, there is a sequence $\{(f_i, g_i) : i < \omega\} \subset F_1^{M^*} \times F_1^{M^*}$ such that the formulas $\forall y \in F_0(R(f_i, x, y) \leftrightarrow R(g_i, x, y))$ $(i < \omega)$ define a strictly decreasing subsets of F_0.

Question. Does theorem A remain true, if we we replace 'large indiscernible set' by 'uncountable indiscernible set'?

4 References

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