Large indiscernible sets of a structure

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1 Introduction

An indiscernible set of a given structure is by definition a set $I$ such that every finite subset of the same cardinality has the same type. A singleton $I = \{a\}$ is trivially an indiscernible set, so it is called a trivial one. A transcienental basis of an algebraically closed field $K$ is a good example of a non-trivial indiscernible set. In this example, if $K$ is uncountable, then it has a large indiscernible set $I$, i.e. an indiscernible set $I$ with $|I| = |K|$. Generally speaking, if a theory $T$ is $\omega$-stable then every uncountable model of $T$ has such a large indiscernible set. However, in the structure $R = (R, 0, 1, +, \cdot)$, there is no non-trivial indiscernible set, i.e. $tp(a) = tp(b)$ implies $a = b$.

In this note we show that every $L$-structure $M$ can be embedded into a structure $M^*$ of an expanded language $L^*$ such that any $L^*$-structure $N \equiv M^*$ has a large indiscernible set. We also show that if $T$ is stable and non-$\omega$-stable then there is a model of power $\aleph_1$ which has no large indiscernible sets.

2 Preliminaries

In what follows, $T$ is a complete theory formulated in a countable language $L$. We give some necessary definitions and review some basic results.

Definition 1. (1) Let $I$ be a subset of a struture $M$. $I$ is said to be an indiscernible set if whenever $F \subseteq I$ and $G \subseteq I$ are finite sequences of the same length then $tp(F) = tp(G)$. 
We will say that an indiscernible set $I$ in a structure $M$ is large if $I$ has the same cardinality as $M$.

**Fact 1** (Theorem 2.8 of [S, CH.I, §2]). If $T$ is $\omega$-stable, then every uncountable model of $T$ includes a large indiscernible set.

If $T$ is not $\omega$-stable, then any $(a,\omega)$-model is uncountable. And any $(a,\kappa(T))$-prime model does not have indiscernible set of power greater than $\kappa(T)$. So we have:

**Fact 2.** If $T$ is a non-$\omega$-stable, superstable theory, then there is a model of power $\aleph_1$ without a large indiscernible set.

Let $T$ be the theory of refining equivalence relations. i.e., $T$ is the theory of the structure $(2^\omega, E_1, E_2, ...)$, where $E_i = \{(\eta_1, \eta_2) \in (2^\omega)^2 : \eta_1|_i = \eta_2|_i\}$. Then $T$ is a superstable theory with $|S(T)| = 2^{\aleph_0}$. Let $M$ be any uncountable elementary submodel of $(2^\omega, E_1, E_2, ...)$. $M$ has no large indiscernible sets.

**Definition 2.** A model $M \supset A$ is said to be $\ell$-atomic over $A$ if for every $\bar{a} \in M$, and every finite set $\Delta$ of formulas, $tp_\Delta(\bar{a}/A)$ is a principal type.

**Fact 3.** Let $T$ be stable.

1. For every set $A$, there is an $\ell$-atomic model over $A$.
2. Let $a_1$ and $a_2$ be independent over $M$. Let $M_i$ be an $\ell$-atomic model over $M \cup \{a_i\}$. Then $M_1$ and $M_2$ are independent over $M$.

## 3 Main Result

We want to extend fact 2 to a non-$\omega$-stable, stable theory $T$. The following lemma will play a crucial role.

**Lemma.** Let $T$ be a non-$\omega$-stable, stable theory and $\kappa \leq 2^{\aleph_0}$ an uncountable cardinal. Then there is a set $R$ of types over a set $A$, $|A| < \kappa$ such that whenever $B \supset A$ is a set with $|B| < \kappa$ and $S$ is a set of stationary types over $B$ with $|S| < \kappa$ then there is a non-algebraic type $r \in R$ which is almost orthogonal to any type in $S$.

**Proof.** This lemma remains true for a superstable theory, but we concentrate on an unsuperstable theory. (Superstable case is easier.) Since $T$ is not superstable, there are infinitely long continuous sequence $\{p_i : i \leq \alpha\}$ of types such that
(1) dom\( p_i \) is a countable set;

(2) \( p_i \) is a forking extension of \( p_j \), if \( i > j \);

(3) \( \alpha < \omega_1 \) is a countable limit ordinal;

(4) \( U(p_\alpha) < \infty \).

By choosing a subsequence of \( \{ p_i : i \leq \alpha \} \), we can assume that \( \alpha = \omega \). Now by the definition of forking, we can easily find a countable set \( A_0 \), and continuously many types \( \{ q_i : i < 2^{\aleph_0} \} \) over \( A_0 \) such that each \( q_i \) is \( U \)-ranked \( (U(q_i) < \infty) \). We can assume that each type \( q_i \) is stationary.

Suppose that our lemma does not hold. By induction on \( j < \omega \), we define a set \( A_j \) of cardinality \( \kappa \) and types \( q_{i,j} \in S(A_j) \ (i < 2^{\aleph_0}) \) such that for any \( i < 2^{\aleph_0}, k < j \),

\[
q_{i,k} \text{ is algebraic} \quad \text{or} \quad q_{i,j} \text{ is a forking extension of } q_{i,k}.
\]

For each \( i < 2^{\aleph_0} \), let \( q_{i,0} = q_i \). Suppose we have defined \( q_{i,k} \in S(A_k) \) for \( i < 2^{\aleph_0} \) and \( k < j \). Let \( \Lambda = \{ i < 2^{\aleph_0} : q_{i,j-1} \text{ is non-algebraic} \} \). Since we are assuming the negation of the statement in our lemma, there are a set \( B \supset A_{j-1}, |B| < \kappa \) and a set \( S \subset S(B), |S| < \kappa \) such that every \( q_{i,j-1} (i \in \Lambda) \) is not almost orthogonal to some \( s_i \in S \). For \( i \in \Lambda \), choose \( a_i \models q_{i,j-1}|B \) and \( b_i \models s_i \) such that \( a_i \) and \( b_i \) are dependent over \( B \). We can assume that if \( s_i = s_j \) then \( b_i = b_j \). Now let

\[
A_j = acl(A_{j-1} \cup \{ b_i : i \in \Lambda \});
\]

\[
q_{i,j} = \begin{cases} 
\text{tp}(a_i/A_j) & i \in \Lambda \\
\text{arbitrary extension of } q_{i,j-1} & i \notin \Lambda
\end{cases}
\]

Finally let \( A_\omega = \bigcup_{j<\omega} A_j \). Note that \( |A_\omega| < 2^{\aleph_0} \). (If \( \kappa = 2^{\aleph_0} \), then \( cf(\kappa) > \omega \), so \( |A_\omega| < \kappa = 2^\omega \). If \( \kappa < 2^{\aleph_0} \), then \( |A_\omega| \leq \kappa < 2^{\aleph_0} \).) Since \( q_i \) is \( U \)-ranked by (4), \( q^*_i = \bigcup_{j<\omega} q_{i,j} \in S(A_\omega) \) must be an algebraic type. (Otherwise there is an infinitely long forking sequence starting from \( q_i \).) So we have constructed continuously many distinct algebraic types over a fixed set \( A_\omega, |A_\omega| < 2^\omega \). However this is a contradiction, since we are assuming that \( L \) is countable.
Theorem A. Let \( T \) be a non-\( \omega \)-stable, stable theory. Then for any uncountable cardinal \( \kappa \leq 2^{\aleph_0} \), there is a model of power \( \kappa \) without a large indiscernible set.

Proof. Choose a set \( A \) and types \( R \subseteq S(A) \) which satisfy the condition in the above lemma. Let \( \lambda = |A| \). Clearly \( \lambda < \kappa \). We construct an elementary chain of models \( \{ M_i : i \leq \kappa \} \) such that each model \( M_i \) has cardinality \( \leq |i| + \lambda \). Without loss of generality, \( A \) is a model. Let \( M_0 = A \), and \( M_1 \) an arbitrary proper extension of \( M_0 \) with the same cardinality. Suppose that we have constructed \( \{ M_i : i < \alpha \} \). If \( \alpha \) is a limit ordinal, then let \( M_\alpha = \bigcup_{i<\alpha} M_i \). So we assume that \( \alpha = \beta + 1 \), and let

\[
S_\beta = \bigcup_{i<\beta} \{ q(x) \in S(M_\beta) : q \text{ is based on } M_i, q|M_i \text{ is realized in } M_\beta \}
\]

Clearly \( |S_\beta| \leq |\beta| + \lambda < \kappa \). By the property of \( R \), there is a type \( r \in R \) which is almost orthogonal to each type in \( S_\beta \). Let \( M_{\beta+1} \) be an \( \ell \)-atomic model over \( M_\beta \cup \{ e_\beta \} \), where \( e_\beta \) is a realization of \( r|M_\beta \). Of course we can assume \( |M_{\beta+1}| < |\beta+1| + \lambda \).

Claim. There is no large indiscernible set in \( M_\kappa \).

Suppose that there was a large indiscernible set \( I \subseteq M_\kappa \). By stability, there is a countable set \( I_0 \subseteq I \) such that \( J = I - I_0 \) is a Morley sequence over \( I_0 \). Choose \( M_i \) (\( i < \kappa \)) which includes \( I_0 \). Since \( M_i < \kappa \), we may assume that \( J \) is a Morley sequence over \( M_i \), by choosing a subset of \( J \) if necessary. Choose \( M_j \) (\( j < \kappa \)) which intersects with \( J \). Let \( a \in J \cap M_j \). Since \( |J| = \kappa \), there is \( b \in J \) which is independent from \( M_j \) over \( M_i \). Choose the least \( k \) such that \( b \) and \( M_k \) are dependent over \( M_i \). Then \( k \) is a successor ordinal greater than \( j \), and

1. \( b \) and \( M_k \) are dependent over \( M_{k-1} \);
2. \( b \) and \( M_{k-1} \) are independent over \( M_i \).

Remember that \( M_k \) is \( \ell \)-atomic over \( M_{k-1} \cup \{ e_{k-1} \} \). From (1), using fact 3, we know that \( b \) and \( e_{k-1} \) are dependent over \( M_{k-1} \). By our choice of \( e_{k-1} \), \( \text{tp}(e_{k-1}/M_{k-1}) \) is almost orthogonal to every type in \( S_{k-1} \), hence \( \text{tp}(b/M_{k-1}) \) does not belong to \( S_{k-1} \). Note that \( \text{tp}(b/M_i) \) is realized by \( a \in M_{k-1} \). Then we must have
(3) $\text{tp}(b/M_{k-1})$ is a forking extension of $\text{tp}(a/M_i)$.

(2) and (3) yield a contradiction.

Next theorem shows that theorem A cannot be extended to an unstable theory.

Theorem B. Let $M$ be an infinite $L$-structure. Then there is a structure $M^*$ for an expanded language $L^* \supset L$ with the following properties:

(i) $M$ is $0$-definable in $M^*$;

(ii) In any $L^*$-structure $N \equiv M^*$, there is a large indiscernible set in $N$.

Proof. For $i < \omega$, let $L_i = L \cup \{F_j(*) : j = 0, \ldots, i\} \cup \{U(*)\} \cup \{R_j(*, *, *) : j = 1, \ldots, i\}$, where $F_j$'s and $U$ are unary predicate symbols, and $R_j$'s are 3-ary predicate symbols. Let $L^* = \bigcup_{i < \omega} L_i$. We construct inductively countable $L_i$-structures $M_i$ and countable subgroups $S_i$ of $\text{Aut}(M_i)$ ($j < \omega$) with the following properties:

(1) $M_0 = F_0^{M_0} \cup U^{M_0}$, where $F_0^{M_0} = M$, and $U^{M_0}$ is an infinite set disjoint from $F_0^{M_0}$.

(2) $S_0$ is a countable subgroup of $\text{Aut}(M_0)$ such that for given finite sequences $\bar{a} \in U^{M_0}$ and $\bar{b} \in U^{M_0}$ of the same length, there is a $\sigma \in S_0$ with $\sigma(\bar{a}) = \bar{b}$. Any two automorphisms $f \in S_0$ and $g \in S_0$ differ at finitely many points.

(3) $M_{i+1} = M_i \cup F_{i+1}^{M_i}$,

(4) $S_i = \{\sigma[M_i : \sigma \in S_{i+1}]\}$.

Assume that we have already constructed $M_j$ and $S_j$ for $j < i$. Choose a bijective function $f_0 : F_i^{M_i-1} \rightarrow U^{M_i-1}$ arbitrarily and let

$$F_i^{M_i} = \{\sigma \circ f_0 \circ \sigma^{-1} : \sigma \in S_{i-1}\}.$$  

$F_i^{M_i}$ is a countable set of functions from $F_i^{M_i-1}$ to $U^{M_i-1}$. Define $R_i^{M_i} \subset F_i^{M_i} \times F_i^{M_i-1} \times U^{M_i-1}$ by

$$(f, a, b) \in R_i^{M_i} \Leftrightarrow f(a) = b.$$
Now let $M_i = M_{i-1} \cup F_i^{M_i}$. We can extend each $\tau \in S_{i-1}$ to an automorphism $\tau^*$ of $M_i$. Let $f = \sigma \circ f_0 \circ \sigma^{-1} \in S_{i-1}$. Then define

$$\tau^*(f) = \tau \circ f \circ \tau^{-1} = (\tau \sigma) \circ f_0 \circ (\tau \sigma)^{-1} \in S_{i-1}.$$ 

The following equivalence shows that $\tau^*$ is really an automorphism:

$$M_i \models R(f, a, b) \iff f(a) = b \iff \tau^*(f(\tau^*^{-1}(\tau^*(a)))) = \tau^*(b) \iff M_i \models R(\tau^*(f), \tau^*(a), \tau^*(b)).$$

Finally we set $M^* = \bigcup_{1<\omega} M_i$, $\tau^* = \text{Th}_{L^*}(M)$. Now it is sufficient to prove the following two claims.

Claim 1. In any model $N$ of $T^*$, $U^N$ is an indiscernible set.

It is sufficient to prove the statement for the case $N = M^*$. Let $\bar{a}, \bar{b} \in U^{M^*}$ be given. By the assumption on $S_0$, there is a $\sigma \in S_0$ such that $\sigma(\bar{a}) = \bar{b}$. $\sigma$ can be extended to an automorphism of $M^*$. So $\bar{a} \equiv \bar{b}$.

Claim 2. If $N \models T^*$, then there is a large indiscernible set.

Clearly $U^N \cup \bigcup_i F_i^N$ has the same cardinality as $N$, or the complement $N - (U^N \cup \bigcup_i F_i^N)$ has the same cardinality as $N$. The second case clearly implies that $N - (U^N \cup \bigcup_i F_i^N)$ is a large indiscernible set. Let the second case hold. Note that an element in $F_{i+1}$ gives a bijection between $F_i^N$ and $U^N$. Then we see that $U^N$ has the same cardinality as $N$. By claim 1, $U^N$ is a large indiscernible set in this case.

Remark. (i) Any model of $T = \text{Th}(\mathbb{Z}, <)$ has a large indiscernible sequence. (ii) The construction of $M^*$ was inspired by [F], in which Fuhrken showed the existence of an uncountable complete theory without the omitting types property. Note that our $T^*$ is not stable: By our choice of $S_0$ and $F_1$, there is a sequence $\{(f_i, g_i) : i < \omega\} \subset F_1^{M^*} \times F_1^{M^*}$ such that the formulas $\forall y \in F_0 (R(f_i, x, y) \leftrightarrow R(g_i, x, y)) (i < \omega)$ define a strictly decreasing subsets of $F_0$.

Question. Does theorem A remain true, if we we replace ‘large indiscernible set’ by ‘uncountable indiscernible set’?
4 References

