<table>
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<th>Title</th>
<th>Large indiscernible sets of a structure (Mathematical Logic and Applications'92)</th>
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<tr>
<td>Author(s)</td>
<td>Tsuboi, Akito</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 1993(1993), 818: 134-140</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1993-01</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/83128">http://hdl.handle.net/2433/83128</a></td>
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<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
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Kyoto University
Large indiscernible sets of a structure

Akito Tsuboi

1 Introduction

An indiscernible set of a given structure is by definition a set $I$ such that every finite subset of the same cardinality has the same type. A singleton $I = \{a\}$ is trivially an indiscernible set, so it is called a trivial one. A transcendental basis of an algebraically closed field $K$ is a good example of a non-trivial indiscernible set. In this example, if $K$ is an uncountable, then it has a large indiscernible set $I$, i.e. an indiscernible set $I$ with $|I| = |K|$. Generally speaking, if a theory $T$ is $\omega$-stable then every uncountable model of $T$ has such a large indiscernible set. However, in the structure $R = (\mathbb{R}, 0, 1, +, \cdot)$, there is no non-trivial indiscernible set, i.e. $tp(a) = tp(b)$ implies $a = b$.

In this note we show that every $L$-structure $M$ can be embedded into a structure $M^*$ of an expanded language $L^*$ such that any $L^*$-structure $N \equiv M^*$ has a large indiscernible set. We also show that if $T$ is stable and non-$\omega$-stable then there is a model of power $\aleph_1$ which has no large indiscernible sets.

2 Preliminaries

In what follows, $T$ is a complete theory formulated in a countable language $L$. We give some necessary definitions and review some basic results.

Definition 1. (1) Let $I$ be a subset of a structure $M$. $I$ is said to be an indiscernible set if whenever $F \subseteq I$ and $G \subseteq I$ are finite sequences of the same length then $tp(F) = tp(G)$.
(2) We will say that an indiscernible set \( I \) in a structure \( M \) is large if \( I \) has the same cardinality as \( M \).

**Fact 1** (Theorem 2.8 of [S, CH.I, §2]). *If \( T \) is \( \omega \)-stable, then every uncountable model of \( T \) includes a large indiscernible set.*

If \( T \) is not \( \omega \)-stable, then any \((a, \omega)\)-model is uncountable. And any \((a, \kappa(T))\)-prime model does not have indiscernible set of power greater than \( \kappa(T) \). So we have:

**Fact 2.** *If \( T \) is a non-\( \omega \)-stable, superstable theory, then there is a model of power \( \aleph_1 \) without a large indiscernible set.*

Let \( T \) be the theory of refining equivalence relations. i.e., \( T \) is the theory of the structure \( (2^\omega, E_1, E_2, \ldots) \), where \( E_i = \{ (\eta_1, \eta_2) \in (2^\omega)^2 : \eta_1 \upharpoonright i = \eta_2 \upharpoonright i \} \). Then \( T \) is a superstable theory with \( |S(T)| = 2^{\aleph_0} \). Let \( M \) be any uncountable elementary submodel of \( (2^\omega, E_1, E_2, \ldots) \). \( M \) has no large indiscernible sets.

**Definition 2.** A model \( M \supseteq A \) is said to be \( \ell \)-atomic over \( A \) if for every \( \bar{a} \in M \), and every finite set \( \Delta \) of formulas, \( tp_\Delta(\bar{a}/A) \) is a principal type.

**Fact 3.** *Let \( T \) be stable.*

(1) For every set \( A \), there is an \( \ell \)-atomic model over \( A \).

(2) Let \( a_1 \) and \( a_2 \) be independent over \( M \). Let \( M_i \) be an \( \ell \)-atomic model over \( M \cup \{ a_i \} \). Then \( M_1 \) and \( M_2 \) are independent over \( M \).

### 3 Main Result

We want to extend fact 2 to a non-\( \omega \)-stable, stable theory \( T \). The following lemma will play a crucial role.

**Lemma.** *Let \( T \) be a non-\( \omega \)-stable, stable theory and \( \kappa \leq 2^{\aleph_0} \) an uncountable cardinal. Then there is a set \( R \) of types over a set \( A \), \( |A| < \kappa \) such that whenever \( B \supseteq A \) is a set with \( |B| < \kappa \) and \( S \) is a set of stationary types over \( B \) with \( |S| < \kappa \) then there is a non-algebraic type \( r \in R \) which is almost orthogonal to any type in \( S \).*

**Proof.** This lemma remains true for a superstable theory, but we concentrate on an unsuperstable theory. (Superstable case is easier.) Since \( T \) is not superstable, there are infinitely long continuous sequence \( \{ p_i : i \leq \alpha \} \) of types such that
(1) dom$p_i$ is a countable set;

(2) $p_i$ is a forking extension of $p_j$, if $i > j$;

(3) $\alpha < \omega_1$ is a countable limit ordinal;

(4) $U(p_\alpha) < \infty$.

By choosing a subsequence of $\{p_i : i \leq \alpha\}$, we can assume that $\alpha = \omega$. Now by the definition of forking, we can easily find a countable set $A_0$, and continuously many types $\{q_i : i < 2^{\aleph_0}\}$ over $A_0$ such that each $q_i$ is $U$-ranked ($U(q_i) < \infty$). We can assume that each type $q_i$ is stationary.

Suppose that our lemma does not hold. By induction on $j < \omega$, we define a set $A_j$ of cardinality $< \kappa$ and types $q_{i,j} \in S(A_j)$ ($i < 2^{\aleph_0}$) such that for any $i < 2^{\aleph_0}$, $k < j$,

\[ q_{i,k} \text{ is algebraic or } q_{i,j} \text{ is a forking extension of } q_{i,k}. \]

For each $i < 2^{\aleph_0}$, let $q_{i,0} = q_i$. Suppose we have defined $q_{i,k} \in S(A_k)$ for $i < 2^{\aleph_0}$ and $k < j$. Let $\Lambda = \{i < 2^{\aleph_0} : q_{i,j-1} \text{ is non-algebraic}\}$. Since we are assuming the negation of the statement in our lemma, there are a set $B \supset A_{j-1}$, $|B| < \kappa$ and a set $S \subset S(B)$, $|S| < \kappa$ such that every $q_{i,j-1}$ ($i \in \Lambda$) is not almost orthogonal to some $s_i \in S$. For $i \in \Lambda$, choose $a_i \models q_{i,j-1}|B$ and $b_i \models s_i$ such that $a_i$ and $b_i$ are dependent over $B$. We can assume that if $s_i = s_j$ then $b_i = b_j$. Now let

\[ A_j = \text{acl}(A_{j-1} \cup \{b_i : i \in \Lambda\}); \]

\[ q_{i,j} = \begin{cases} \text{tp}(a_i/A_j) & i \in \Lambda \\ \text{arbitrary extension of } q_{i,j-1} & i \notin \Lambda \end{cases} \]

Finally let $A_\omega = \bigcup_{j<\omega} A_j$. Note that $|A_\omega| < 2^{\aleph_0}$. (If $\kappa = 2^{\aleph_0}$, then $\text{cf}(\kappa) > \omega$, so $|A_\omega| < \kappa = 2^\omega$. If $\kappa < 2^{\aleph_0}$, then $|A_\omega| \leq \kappa < 2^{\aleph_0}$.) Since $q_i$ is $U$-ranked by (4), $q_i^* = \bigcup_{j<\omega} q_{i,j} \in S(A_\omega)$ must be an algebraic type. (Otherwise there is an infinitely long forking sequence starting from $q_i$.) So we have constructed continuously many distinct algebraic types over a fixed set $A_\omega$, $|A_\omega| < 2^\omega$. However this is a contradiction, since we are assuming that $L$ is countable.
Theorem A. Let $T$ be a non-$\omega$-stable, stable theory. Then for any uncountable cardinal $\kappa \leq 2^{\aleph_0}$, there is a model of power $\kappa$ without a large indiscernible set.

Proof. Choose a set $A$ and types $R \subset S(A)$ which satisfy the condition in the above lemma. Let $\lambda = |A|$. Clearly $\lambda < \kappa$. We construct an elementary chain of models $\{M_i : i \leq \kappa\}$ such that each model $M_i$ has cardinality $\leq |i| + \lambda$. Without loss of generality, $A$ is a model. Let $M_0 = A$, and $M_1$ an arbitrary proper extension of $M_0$ with the same cardinality. Suppose that we have constructed $\{M_i : i < \alpha\}$. If $\alpha$ is a limit ordinal, then let $M_\alpha = \bigcup_{i<\alpha} M_i$. So we assume that $\alpha = \beta + 1$, and let

$$S_\beta = \bigcup_{i<\beta} \{q(x) \in S(M_\beta) : q \text{ is based on } M_i, q|M_i \text{ is realized in } M_\beta\}$$

Clearly $|S_\beta| \leq |\beta| + \lambda < \kappa$. By the property of $R$, there is a type $r \in R$ which is almost orthogonal to each type in $S_\beta$. Let $M_{\beta+1}$ be an $\ell$-atomic model over $M_\beta \cup \{e_\beta\}$, where $e_\beta$ is a realization of $r|M_\beta$. Of course we can assume $|M_{\beta+1}| < |\beta + 1| + \lambda$.

Claim. There is no large indiscernible set in $M_\kappa$.

Suppose that there was a large indiscernible set $I \subset M_\kappa$. By stability, there is a countable set $I_0 \subset I$ such that $J = I - I_0$ is a Morley sequence over $I_0$. Choose $M_i$ ($i < \kappa$) which includes $I_0$. Since $M_i < \kappa$, we may assume that $J$ is a Morley sequence over $M_i$, by choosing a subset of $J$ if necessary. Choose $M_j$ ($j < \kappa$) which intersects with $J$. Let $a \in J \cap M_j$. Since $|J| = \kappa$, there is $b \in J$ which is indepent from $M_j$ over $M_i$. Choose the least $k$ such that $b$ and $M_k$ are dependent over $M_i$. Then $k$ is a successor ordinal greater than $j$, and

1. $b$ and $M_k$ are dependent over $M_{k-1}$;

2. $b$ and $M_{k-1}$ are independent over $M_i$.

Remember that $M_k$ is $\ell$-atomic over $M_{k-1} \cup \{e_{k-1}\}$. From (1), using fact 3, we know that $b$ and $e_{k-1}$ are dependent over $M_{k-1}$. By our choice of $e_{k-1}$, $\text{tp}(e_{k-1}/M_{k-1})$ is almost orthogonal to every type in $S_{k-1}$, hence $\text{tp}(b/M_{k-1})$ does not belong to $S_{k-1}$. Note that $\text{tp}(b/M_i)$ is realized by $a \in M_{k-1}$. Then we must have
(3) $\text{tp}(b/M_{k-1})$ is a forking extension of $\text{tp}(a/M_i)$.

(2) and (3) yield a contradiction.

Next theorem shows that theorem A cannot be extended to an unstable theory.

**Theorem B.** Let $M$ be an infinite $L$-structure. Then there is a structure $M^*$ for an expanded language $L^* \supset L$ with the following properties:

1. $M$ is $\emptyset$-definable in $M^*$;
2. In any $L^*$-structure $N \equiv M^*$, there is a large indiscernible set in $N$.

**Proof.** For $i < \omega$, let $L_i = L \cup \{F_j(*) : j = 0, \ldots, i\} \cup \{U(*)\} \cup \{R_j(*, *, *) : j = 1, \ldots, i\}$, where $F_j$'s and $U$ are unary predicate symbols, and $R_j$'s are 3-ary predicate symbols. Let $L^* = \bigcup_{i<\omega} L_i$. We construct inductively countable $L_i$-structures $M_i$ and countable subgroups $S_i$ of $\text{Aut}(M_i)$ ($j < \omega$) with the following properties:

1. $M_0 = F_0^{M_0} \cup U^{M_0}$, where $F_0^{M_0} = M$, and $U^{M_0}$ is an infinite set disjoint from $F_0^{M_0}$.

2. $S_0$ is a countable subgroup of $\text{Aut}(M_0)$ such that for given finite sequences $\bar{a} \in U^{M_0}$ and $\bar{b} \in U^{M_0}$ of the same length, there is a $\sigma \in S_0$ with $\sigma(\bar{a}) = \bar{b}$. Any two automorphisms $f \in S_0$ and $g \in S_0$ differ at finitely many points.

3. $M_{j+1} = M_j \cup F_{j+1}^{M_{j+1}}$,

4. $S_j = \{\sigma[M_j] : \sigma \in S_{j+1}\}$.

Assume that we have already constructed $M_j$ and $S_j$ for $j < i$. Choose a bijective function $f_0 : F_{i-1_i}^{M_{i-1}} \rightarrow U^{M_{i-1}}$ arbitrarily and let

$$F_i^{M_i} = \{\sigma \circ f_0 \circ \sigma^{-1} : \sigma \in S_{i-1}\}.$$ 

$F_i^{M_i}$ is a countable set of functions from $F_{i-1}^{M_{i-1}}$ to $U^{M_{i-1}}$. Define $R_i^{M_i} \subset F_i^{M_i} \times F_{i-1}^{M_{i-1}} \times U^{M_{i-1}}$ by

$$(f, a, b) \in R_i^{M_i} \iff f(a) = b.$$
Now let $M_i = M_{i-1} \cup F_i$. We can extend each $\tau \in S_{i-1}$ to an automorphism $\tau^*$ of $M_i$. Let $f = \sigma \circ f_0 \circ \sigma^{-1} \in S_{i-1}$. Then define

$$\tau^*(f) = \tau \circ f \circ \tau^{-1} = (\tau \sigma) \circ f_0 \circ (\tau \sigma)^{-1} \in S_{i-1}.$$ 

The following equivalence shows that $\tau^*$ is really an automorphism:

$$M_i \models R(f, a, b) \iff f(a) = b \iff \tau^*(f(\tau^*(a))) = \tau^*(b) \iff M_i \models R(\tau^*(f), \tau^*(a), \tau^*(b)).$$

Finally we set $M^* = \bigcup_{i<\omega} M_i$, $\tau^* = \text{Th}_{L^*}(M)$. Now it is sufficient to prove the following two claims.

**Claim 1.** In any model $N$ of $T^*$, $U^N$ is an indiscernible set.

It is sufficient to prove the statement for the case $N = M^*$. Let $\bar{a}, \bar{b} \in U^{M^*}$ be given. By the assumption on $S_0$, there is a $\sigma \in S_0$ such that $\sigma(\bar{a}) = \bar{b}$. $\sigma$ can be extended to an automorphism of $M^*$. So $\bar{a} \equiv \bar{b}$.

**Claim 2.** If $N \models T^*$, then there is a large indiscernible set.

Clearly $U^N \cup \bigcup_i F_i^N$ has the same cardinality as $N$, or the complement $N - (U^N \cup \bigcup_i F_i^N)$ has the same cardinality as $N$. The second case clearly implies that $N - (U^N \cup \bigcup_i F_i^N)$ is a large indiscernible set. Let the second case hold. Note that an element in $F_{i+1}$ gives a bijection between $F_i^N$ and $U^N$. Then we see that $U^N$ has the same cardinality as $N$. By claim 1, $U^N$ is a large indiscernible set in this case.

**Remark.** (i) Any model of $T = \text{Th}(\mathbb{Z}, <)$ has a large indiscernible sequence. (ii) The construction of $M^*$ was inspired by [F], in which Fuhrken showed the existence of an uncountable complete theory without the omitting types property. Note that our $T^*$ is not stable: By our choice of $S_0$ and $F_1$, there is a sequence $\{(f_i, g_i) : i < \omega\} \subset F_1^{M^*} \times F_1^{M^*}$ such that the formulas $\forall y \in F_0(R(f_i, x, y) \leftrightarrow R(g_i, x, y)) (i < \omega)$ define a strictly decreasing subsets of $F_0$.

**Question.** Does theorem A remain true, if we we replace 'large indiscernible set' by 'uncountable indiscernible set'?
4 References

