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MARKOV'S PRINCIPLE, CHURCH'S THESIS AND LINDELÖF'S THEOREM

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Abstract. The principal results of this paper are: in constructive mathematics (1) the theorem "Mappings from a complete metric space into a metric space are sequentially continuous" can be proved using a disjunctive form of Church's thesis only, and (2) the theorem "Every open cover of a complete separable metric space has an enumerable subcover" can be proved using the Extended Church's Thesis only; Markov's principle is not needed.

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In constructive mathmatics which is formalized by, for example $\mathbf{EL} + \mathrm{AC}_0$ [1], the extended Church's thesis, where A is almost negative:

ECT_0. $\forall x(Ax \to \exists y Bxy) \to \exists z \forall x(Ax \to \exists u(Tzxu \wedge B(x, Uu)))$

yields, in combination with Markov's principle:

MP. $\forall \alpha[\neg\neg\exists n(\alpha n \neq 0) \to \exists n(\alpha n \neq 0)]$,

interesting mathematical consequences, in particular the KLST-theorem ([3, Chapter 9], [6, 7.2.11])

> *Assuming* $\mathrm{ECT}_0 + \mathrm{MP}$, *all mappings from complete separable metric spaces into metric spaces are continuous*

and a version of Lindelöf's theorem ([3, Theorem 1, 9.3], [6, 7.2.15])

> *Assuming* $\mathrm{ECT}_0 + \mathrm{MP}$, *every open cover of a complete separable metric space has an enumerable subcover.*

Recently, the author investigated the interrelation between a weak version of Markov's principle:

WMP. $\forall \alpha[\forall \beta(\neg\neg\exists n(\beta n \neq 0) \vee \neg\neg\exists n(\alpha n \neq 0 \wedge \beta n = 0)) \to \exists n(\alpha n \neq 0)]$

and certain continuity principles [1, 2].

In this paper, we shall show that MP is equivalent to WMP and a disjunctive version of Markov's principle:

$\mathrm{MP}^{\vee}$. $\forall \alpha\beta[\neg\neg\exists n(\alpha n \neq 0 \vee \beta n \neq 0) \to \neg\neg\exists n(\alpha n \neq 0) \vee \neg\neg\exists n(\beta n \neq 0)]$,

and that the lesser limited principle of omniscience, here we call it SEP according to [5, 6]:

SEP. $\forall \alpha\beta[\neg\exists n(\alpha n \neq 0 \vee \beta n \neq 0) \to \neg\exists n(\alpha n \neq 0) \vee \neg\exists n(\beta n \neq 0)]$

[1] For detailed exposition of **EL** and AC, see [5, 6]; also we follow notations and convensions in [5, 6]; for example α, β, γ, δ range over $\mathbf{N} \to \mathbf{N}$.

implies $\mathrm{MP}^{\vee}$. (WMP and $\mathrm{MP}^{\vee}$ is equivalent to WLPE and LLPE in [4], respectively; see [2] and compare $\mathrm{MP}^{\vee}$ and [4, Theorem 4.1 (j)]. So these results correspond to the results in [4].) Then we shall prove that WMP is derivable from Church's thesis for disjunctions:

$\mathrm{CT}_0^{\vee}$. $\forall x(Ax \vee Bx) \to \exists \alpha \in \mathrm{TREC}\forall x((\alpha x = 0 \to Ax) \wedge (\alpha x \neq 0 \to Bx))$,

So assuming $\mathrm{CT}_0^{\vee}$, MP is equivalent to $\mathrm{MP}^{\vee}$. Using these results we shall prove

> *Assuming* $\mathrm{CT}_0^{\vee}$, *all mappings from complete metric spaces into metric spaces are sequentially continuous*

and

> *Assuming* ECT_0, *any open cover of a complete separable metric space has an enumerable subcover.*

Now we turn to our first result [4].

Proposition 1

(1) $\mathrm{MP} \Leftrightarrow \mathrm{WMP} + \mathrm{MP}^{\vee}$;

(2) $\mathrm{SEP} \Rightarrow \mathrm{MP}^{\vee}$.

Proof. (1). It is easy to see that $\mathrm{MP} \Rightarrow \mathrm{WMP} + \mathrm{MP}^{\vee}$. To see the converse, let α be such that $\neg\neg\exists n(\alpha n \neq 0)$, and for arbitrary β define γ by

$$\gamma n := \begin{cases} 1 & \text{if } \alpha n \neq 0 \wedge \beta n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\neg\neg\exists n(\beta n \neq 0 \vee \gamma n \neq 0)$. Appling $\mathrm{MP}^{\vee}$, we have

$$\neg\neg\exists n(\beta n \neq 0) \vee \neg\neg\exists n(\gamma n \neq 0)$$

or

$$\neg\neg\exists n(\beta n \neq 0) \vee \neg\neg\exists n(\alpha n \neq 0 \wedge \beta n = 0).$$

Since β is arbitrary, we have $\exists n(\alpha n \neq 0)$ by WMP.

(2). Let α and β be such that $\neg\neg\exists n(\alpha n \neq 0 \vee \beta n \neq 0)$, and define γ and δ by

$$\gamma n := \begin{cases} 1 & \text{if } \forall i < n(\alpha i = 0 \wedge \beta i = 0) \wedge \alpha n \neq 0 \wedge \beta n = 0, \\ 0 & \text{otherwise;} \end{cases}$$

$$\delta n := \begin{cases} 1 & \text{if } \forall i < n(\alpha i = 0 \wedge \beta i = 0) \wedge \alpha n = 0 \wedge \beta n \neq 0, \\ 0 & \text{otherwise,} \end{cases}$$

Then $\neg(\exists n(\gamma n \neq 0) \wedge \exists n(\delta n \neq 0))$. Hence either $\neg\exists n(\gamma n \neq 0)$ or $\neg\exists n(\delta n \neq 0)$ by SEP. In the former case, if $\neg\exists n(\beta n \neq 0)$, then $\neg\exists n(\alpha n \neq 0)$, a contradiction; hence $\neg\neg\exists n(\beta n \neq 0)$. Similarly, in the latter case, we have $\neg\neg\exists n(\alpha n \neq 0)$. □

Lemma 1 $\mathrm{CT}_0^\vee$ *proves*

$$\forall x(\neg\neg\exists n Txxn \vee Ax) \rightarrow \exists y(\exists n Tyyn \wedge Ay).$$

Proof. Assume $\forall x(\neg\neg\exists n Txxn \vee Ax)$ and apply $\mathrm{CT}_0^\vee$. Then we find a total recursive γ such that

$$\forall x((\gamma x = 0 \rightarrow \neg\neg\exists n Txxn) \wedge (\gamma x \neq 0 \rightarrow Ax)).$$

Let $\{y\}$ be a partial recursive function such that

$$\gamma x \neq 0 \quad \leftrightarrow \quad \exists n Tyxn,$$

and suppose that $\gamma y = 0$. Then $\neg\neg\exists n Tyyn$ and $\neg\exists n Tyyn$, a contradiction. Hence $\gamma y \neq 0$ and therefore $\exists n Tyyn$ and Ay. □

Proposition 2 *Assuming* $\mathrm{CT}_0^\vee$, WMP *holds.*

Proof. Let the β in $\forall\beta(\neg\neg\exists n(\beta n \neq 0) \vee \neg\neg\exists n(\alpha n \neq 0 \wedge \beta n = 0))$ range over the characteristic function of $Txxn$ as predicate in n. Then

$$\forall x(\neg\neg\exists n Txxn \vee \neg\neg\exists n(\alpha n \neq 0 \wedge \neg Txxn)).$$

Apply Lemma 1. Then we find y such that

$$\exists n Tyyn \wedge \neg\neg\exists n(\alpha n \neq 0 \wedge \neg Tyyn).$$

Choose n so that $Tyyn$ and suppose that $\forall k < n(\alpha k = 0)$. Then $\neg\exists n(\alpha n \neq 0 \wedge \neg Tyyn)$, a contradiction. Hence $\exists k < n(\alpha k \neq 0)$. □

Theorem 1 *Assuming* $\mathrm{CT}_0^\vee$, MP *is equivalent to* $\mathrm{MP}^\vee$.

Proof. By Proposition 1 (1) and Proposition 2. □

Theorem 2 *Assuming* $\mathrm{CT}_0^\vee$, *all mappings from complete metric spaces into metric spaces are sequentially continuous.*

Proof. By $\mathrm{CT}_0^\vee \Rightarrow \neg\forall$-PEM ([5, Corollary 1, 4.3.4]), Proposition 2, and [2, Corollary 1]. □

Let $M \equiv (X, \rho)$ be a complete separable metric space with basis $\langle p_n \rangle_n$. Let $\{x\}$ be a total recursive function such that

$$A_M(x) := \forall n(\rho(p_{\{x\}(n)}, p_{\{x\}(n+1)}) < 2^{-n-1}); \qquad (*)$$

we write $[x]_M$ for $\lim \langle p_{\{x\}(n)} \rangle_n$.

If $M_0 := \{y \in \mathbf{R} : y > 0\}$ with metric inherited from $\mathbf{R}$, we let

$$S_n := U([j_1 n]_{M_0}, [j_2 n]_M);$$

here $U(r, c)$ denotes the open ball with radius r and ceter c, and j_1, j_2 are the projections for the pairing $j(x, y)$.

Here and in the sequel, the use of the notation $[x]_M$ is tacitly taken to imply that x satisfies (*).

A partial recursive α is said to be *effective covering of open sheres of* M if

$$\forall [x]_M(\boldsymbol{E}(\alpha x) \wedge [x]_M \in S_{\alpha x}).$$

Proposition 3 *Assuming* $\mathrm{CT}_0^\vee$, *every effective covering of open spheres of a complete separable metric space has an enumerable subcover.*

Proof. Let $\langle p_n \rangle_n$ be a basis for a complete separable metric space M. We put

$$\gamma(x,y,n) \simeq \gamma_1(x,y)(n) \simeq \begin{cases} \{x\}(n) & \text{if } \neg\exists k \leq nTyyk, \\ \{x\}(\min_{k\leq n} Tyyk) & \text{if } \exists k \leq nTyyk. \end{cases}$$

Then

$$A_M(x) \to \forall y A_M(\gamma_1(x,y))$$

Let R be the r.e. predicate defined by

$$R(x,y) := \exists k[Tyyk \wedge \forall i \leq k \boldsymbol{E}\gamma(x,y,i) \wedge \forall i < k(\rho(p_{\gamma(x,y,i)}, p_{\gamma(x,y,i+1)}) < 2^{-i-1})].$$

So if $R(x,y)$, then $A_M(\gamma_1(x,y))$. Let δ_1 enumerate $\{j(x,y) : R(x,y)\}$, and put

$$\delta m := \gamma_1(j_1(\delta_1 m), j_2(\delta_1 m)).$$

Let α be an effective covering. Then

$$A_M(x) \to \forall y[\boldsymbol{E}(\alpha(\gamma_1(x,y))) \wedge [\gamma_1(x,y)]_M \in S_{\alpha(\gamma_1(x,y))}].$$

We shall prove that $\{S_{\alpha(\delta m)} : m \in \mathbf{N}\}$ is a covering. To see this, let $rx := j_1(\alpha x)$ and $cx := j_2(\alpha x)$, and assume that $A_M(x)$. Then

$$\forall y[\rho([x]_M, [\gamma_1(x,y)]_M) > 0 \to \exists nTyyn]$$

and

$$\forall y[\rho([x]_M, [\gamma_1(x,y)]_M) < r(\gamma_1(x,y)) - \rho([\gamma_1(x,y)]_M, [c(\gamma_1(x,y))]_M) \to [x]_M \in S_{\alpha(\gamma_1(x,y))}];$$

hence

$$\forall y(\exists nTyyn \vee [x]_M \in S_{\alpha(\gamma_1(x,y))}).$$

Apply Lemma 1. Then we find z such that

$$\exists nTzzn \wedge [x]_M \in S_{\alpha(\gamma_1(x,z))}.$$

and therefore $R(x,z)$. Hence $\alpha\delta$ is an enumerable subcovering of α. □

Theorem 3 *Assuming* ECT_0, *any open cover of a complete separable metric space has an enumerable subcover.*

Proof. Similar to the proof of [6, Theorem 7.2.15] using Proposition 3. □

References

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