A Note on Minimal Models

KOICHIRO IKEDA (池田光利)
Institute of Mathematics
University of Tsukuba

A model $M$ is said to be minimal if there is no proper elementary submodel of $M$. We consider the size of an indiscernible set in a minimal model. In [2] Shelah showed that if a theory $T$ is $\omega$-stable then there is no infinite indiscernible set in a minimal model of $T$. On the other hand Marcus [1] constructed a theory having a minimal (and prime) model with an infinite indiscernible set. The theory is stable but non-superstable. In this note we show the following theorem:

**Theorem.** Let $T$ be superstable and let $A$ be any set. Then there is no minimal model over $A$ which has an infinite set of indiscernibles over $A$.

1. Notation

We fix a countable stable theory $T$. Our notations are fairly standard. $A, B, \ldots$ are used to denote small subsets of $C$. $\bar{a}, \bar{b}, \ldots$ are used to denote finite sequences of elements in $C$. $\varphi, \psi, \ldots$ are used to denote formulas (with parameter). $p, q, \ldots$ are used to denote types (with parameter). The nonforking extension of a stationary types $p$ to the domain $A$ is denoted by $p|A$. The type of $a$ over $A$ is denoted by $tp(a/A)$. $R^\infty(p)$ is the infinity rank of a type $p$. We simply write $R^\infty(a/A)$ instead of $R^\infty(tp(a/A))$. The set of realizations of a type $p$ (resp. a formula $\varphi$) in a model $M$ is denoted by $p^M$ (resp. $\varphi^M$).

2. Theorem and Proof

First we prove the following lemma:

**Lemma.** Let $T$ be superstable and let $A$ be any set. Let $I = \{a\} \cup J$ be an infinite Morley sequence of some stationary type $p \in S(A)$. Let $M$ be a model containing $I \cup A$. Suppose that $B$ is a maximal set satisfying $J \subset B \subset M$ and $B \downarrow_A a$. Then $B$ is an elementary submodel of $M$.

**Proof:** For the simplicity of the notation, we may assume that $A = \emptyset$. Take any consistent formula $\varphi(x, b_0)$ over $B$. By the Tarski criterion it is enough to see that $\varphi$ is satisfied by $B$. By the superstability of $T$ we can pick an element $b$ of $\varphi^M$ such that $R^\infty(b/B)$ is minimal.
CLAIM. \(b\) is independent from \(a\) over \(B\).

PROOF: Take a formula \(\theta(x, \overline{b}_1) \in tp(b/B)\) such that \(R^\infty(b/B) = R^\infty(\theta)\). Without loss of generality, we can assume that \(b_0 \subseteq b_1\). Suppose that \(b\) and \(a\) are not independent over \(B\). By the superstability there is a finite sequences \(b \in B\) such that \(ab \upharpoonright b\) and \(b_1 \subset b\). Then we obtain that \(b\) and \(a\) are not independent over \(b\). So we can get a formula \(\psi(x, \overline{b}, a)\) such that \(\models \psi(b, \overline{b}, a)\), and if \(\models \psi(b', \overline{b}, a)\) then \(b' \not\models a\). Let \(I'(\overline{b}, a)\) denote \((\exists x)(\varphi(x, b_0) \land \psi(x, \overline{b}, a) \land \theta(x, b_1))\). On the other hand there is a finite subset \(I'\) of \(I\) such that \(I - I'\) is the infinite Morley sequence of \(p|b\) since \(\kappa(T)\) is finite. Moreover we can assume that \(a \in I - I'\), since \(b\) and \(a\) are independent. So we can pick some \(a' \in J(\subset B)\) such that \(I'(\overline{b}, a')\) holds. Therefore there is an element \(b' \in \varphi^M\) such that \(R^\infty(b'/b) = R^\infty(b'/B)\) and \(\not\models a'\). But \(R^\infty(b'/B) = R^\infty(b'/\overline{b}) < R^\infty(b'/\overline{b})\) \(\leq R^\infty(b/B)\). This contradicts the minimality of \(R^\infty(b/B)\). Hence \(b\) and \(a\) are independent over \(B\).

So we have \(b \in B\) by the maximality of \(B\) and the above claim. Hence \(\varphi\) is realised by the element \(b\) of \(B\). This completes the proof of the claim. \(\blacksquare\)

Our theorem follows directly from the above lemma:

**Theorem.** Let \(T\) be superstable and let \(A\) be any set. Then there is no minimal model over \(A\) which has an infinite set of indiscernibles over \(A\).

**Proof:** Suppose that \(M\) is a model containing a set \(A\) and an infinite set \(I\) of indiscernibles over \(A\). We can assume that \(I\) is an infinite Morley sequence over \(A\) because \(\kappa(T)\) is finite. By the lemma we get a proper elementary submodel of \(M\). So \(M\) is not minimal over \(A\). \(\blacksquare\)

3. Example

The following example shows that our theorem can not be extended to a stable theory. It is a slightly improvement of Marcus’ one (see [1]).

**EXAMPLE:** We construct a countable structure \(M\) with the following conditions: i) \(M\) is minimal, ii) \(M\) has an infinite indiscernible set and iii) \(Th(M)\) is stable but non-superstable. Let \(L_0\) be a language with an equality only. For \(i < \omega\), let \(L_{i+1} = \{P_{i+1}\} \cup \{R_{i+1}^n : n < \omega\} \cup L_i\), where \(P_{i+1}\) is a unary predicate symbol and \(R_{i+1}^n\)'s are binary predicate symbols. For each \(i < \omega\) we define inductively countable \(L_i\)-structures \(M_i\) and countable subgroups \(H_i\) of \(Aut(M_i)\) satisfying the following properties:
(1) $P_{i+1}^{M_{i+1}} = M_{i+1} - M_i$.

(2) $R_{i+1}^N \subset P_{i+1}^{M_{i+1}} \times P_{i+1}^{M_{i+1}}$. For any $a \in P_i^{M_i}$ and $b \in P_{i+1}^{M_{i+1}}$ there is a predicate $P_{i+1}^N \in L_{i+1}$ such that $\models P_{i+1}^N(x, b)$ if and only if $x = a$.

(3) $M_0$ is a countable set. $H_0$ is a countable subgroup of permutation of $M_0$ which move only a finite number of elements.

(4) For all $f \in H_0$ and $i < \omega$ there is a unique extension of $f$ to an automorphism $f^* \in H_i$.

Now assume that $M_i$ and $H_i$ are defined as required. Let $M_{i+1} = \{ b_f : f \in H_i \} \cup M_i$. Then $M_{i+1}$ is countable (because $H_i$ is so). Define a predicate $P_{i+1}^{M_{i+1}} = M_{i+1} - M_i$. Let $\{ a_n : n < \omega \}$ be an enumeration of $P_i^{M_i}$. For every $n < \omega$ define a predicate $P_{i+1}^{N_{i+1}} = \{ (f(a_n), b_f) : f \in H_i \}$. Clearly $P_{i+1}^N$'s satisfy the condition (2). For $g \in H_i$ define a $g^*$ as follows:

\[
\begin{cases}
  g^*(b_f) = b_{g \cdot f} & \text{for each } b_f \in M_{i+1} - M_i, \\
  g^*(a) = g(a) & \text{for each } a \in M_i.
\end{cases}
\]

Then $g^*$ is an automorphism of $M_{i+1}$. In fact we can see that $(f(a), b_f) \in R_{i+1}^N$ iff $((g \cdot f)(a), b_{g \cdot f}) \in R_{i+1}^N$ iff $g^*((f(a), b_f)) \in R_{i+1}^N$. Let $H_{i+1} = \{ g^* : g \in H_i \}$. Then $H_{i+1}$ is a countable subgroup of $Aut(M_{i+1})$ since $H_i$ is so. Hence we can construct $M_i$'s and $H_i$'s.

Let $L = \bigcup L_i$. Let $M$ be an $L$-structure with $M = \bigcup M_i$.

(i) $M$ is a minimal model: Let $N$ be any submodel of $M$. Take any element $a$ of $M$. Since $M$ is the union of $P_i^{M_i}$'s there is minimum $i < \omega$ such that $a \in P_i^M$. Pick an arbitrary element $b$ of $P_{i+1}^N$. By the condition (2) there is some predicate $R \in L_{i+1}$ such that $R(x, b)$ holds if and only if $x = a$. Hence $a \in dcl(b) \subset N$, so $N = M$. Therefore $M$ is minimal.

(ii) $M_0$ is an indiscernible set: Let $\bar{a}, \bar{b}$ be any elements of $M_0$ with the same length. By the condition (3) there is an $f \in H_0$ such that $f(\bar{a}) = \bar{b}$. Moreover by (4) $f$ can be extended to an automorphism of $M$. So $tp(\bar{a}) = tp(\bar{b})$.

(iii) $Th(M)$ is not superstable: Let $\{ a_n : n < \omega \}$ be an enumeration of $M_0$. For all $n < \omega$ let $\bar{a}_n = a_0 \ldots a_{n-1}$. For all $n < \omega$ let $\varphi_n(x, \bar{a}_n)$ denote $\bigwedge_{i=1}^n (a_{0}, x) \wedge \ldots \wedge (a_{n}, x)$. Then $(\varphi_n)_{n<\omega}$ is a infinite chain of forking formulas. In fact, for each $n < \omega$, $\{ \varphi_n(x, \bar{a}_{n-1})^{C} : a \in M_0 - \{ a_0, \ldots, a_{n-1} \} \}$ is a pairwise disjoint set. Hence $Th(M)$ is not superstable.

References
