A Note on Minimal Models
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A model $M$ is said to be minimal if there is no proper elementary submodel of $M$. We consider the size of an indiscernible set in a minimal model. In [2] Shelah showed that if a theory $T$ is $\omega$-stable then there is no infinite indiscernible set in a minimal model of $T$. On the other hand Marcus [1] constructed a theory having a minimal (and prime) model with an infinite indiscernible set. The theory is stable but non-superstable. In this note we show the following theorem:

**Theorem.** Let $T$ be superstable and let $A$ be any set. Then there is no minimal model over $A$ which has an infinite set of indiscernibles over $A$.

1. Notation

We fix a countable stable theory $T$. Our notations are fairly standard. $A, B, \ldots$ are used to denote small subsets of $\mathbb{C}$. $\overline{a}, \overline{b}, \ldots$ are used to denote finite sequences of elements in $\mathbb{C}$. $\varphi, \psi, \ldots$ are used to denote formulas (with parameter). $p, q, \ldots$ are used to denote types (with parameter). The nonforking extension of a stationary types $p$ to the domain $A$ is denoted by $p|A$. The type of $a$ over $A$ is denoted by $tp(a/A)$. $R^\infty(p)$ is the infinity rank of a type $p$. We simply write $R^\infty(a/A)$ instead of $R^\infty(tp(a/A))$. The set of realizations of a type $p$ (resp. a formula $\varphi$) in a model $M$ is denoted by $p^M$ (resp. $\varphi^M$).

2. Theorem and Proof

First we prove the following lemma:

**Lemma.** Let $T$ be superstable and let $A$ be any set. Let $I = \{a\} \cup J$ be an infinite Morley sequence of some stationary type $p \in S(A)$. Let $M$ be a model containing $I \cup A$. Suppose that $B$ is a maximal set satisfying $J \subset B \subset M$ and $B \downarrow_A a$. Then $B$ is an elementary submodel of $M$.

**Proof:** For the simplicity of the notation, we may assume that $A = \emptyset$. Take any consistent formula $\varphi(x, b_0)$ over $B$. By the Tarski criterion it is enough to see that $\varphi$ is satisfied by $B$. By the superstability of $T$ we can pick an element $b$ of $\varphi^M$ such that $R^\infty(b/B)$ is minimal.
Claim. $b$ is independent from $a$ over $B$.

Proof: Take a formula $\theta(x, b_1) \in tp(b/B)$ such that $R^\omega(b/B) = R^\omega(\theta)$. Without loss of generality, we can assume that $b_0 \subset b_1$. Suppose that $b$ and $a$ are not independent over $B$. By the superstability there is a finite sequences $\bar{b} \in B$ such that $ab \downarrow \bar{b}$ and $b_1 \subset \bar{b}$. Then we obtain that $b$ and $a$ are not independent over $\bar{b}$. So we can get a formula $\psi(x, \bar{b}, a)$ such that $\models \psi(b, \bar{b}, a)$, and if $\models \psi(b', \bar{b}, a)$ then $b' \not\in \bar{b}$. Let $I'(\bar{b}, a)$ denote $(\exists x)(\varphi(x, b_0) \land \psi(x, \bar{b}, a) \land \theta(x, b_1))$. On the other hand there is a finite subset $I'$ of $I$ such that $I - I'$ is the infinite Morley sequence of $p|\bar{b}$ since $\kappa(T)$ is finite. Moreover we can assume that $a \in I - I'$, since $\bar{b}$ and $a$ are independent. So we can pick some $a' \in J(\subset B)$ such that $I'(\bar{b}, a')$ holds. Therefore there is an element $b' \in \varphi^M$ such that $R^\omega(b'/\bar{b}) = R^\omega(b'/B)$ and $b' \not\in \bar{a'}$. But $R^\omega(\bar{b}/B) = R^\omega(b'/\bar{b}) < R^\omega(b'/\bar{b}) \leq R^\omega(b/B)$. This contradicts the minimality of $R^\omega(b/B)$. Hence $b$ and $a$ are independent over $B$.

So we have $b \in B$ by the maximality of $B$ and the above claim. Hence $\varphi$ is realised by the element $b$ of $B$. This completes the proof of the claim. 

Our theorem follows directly from the above lemma:

Theorem. Let $T$ be superstable and let $A$ be any set. Then there is no minimal model over $A$ which has an infinite set of indiscernibles over $A$.

Proof: Suppose that $M$ is a model containing a set $A$ and an infinite set $I$ of indiscernibles over $A$. We can assume that $I$ is an infinite Morley sequence over $A$ because $\kappa(T)$ is finite. By the lemma we get a proper elementary submodel of $M$. So $M$ is not minimal over $A$.

3. Example

The following example shows that our theorem cannot be extended to a stable theory. It is a slightly improvement of Marcus' one (see [1]).

Example: We construct a countable structure $M$ with the following conditions:

1. $M$ is minimal.
2. $M$ has an infinite indiscernible set and $\text{Th}(M)$ is stable
3. $M$ is not superstable.

Let $L_0$ be a language with an equality only. For $i < \omega$, let $L_{i+1} = \{P_{i+1}\} \cup \{R_{i+1}^n : n < \omega\} \cup L_i$, where $P_{i+1}$ is a unary predicate symbol and $R_{i+1}^n$'s are binary predicate symbols. For each $i < \omega$ we define inductively countable $L_i$-structures $M_i$ and countable subgroups $H_i$ of $\text{Aut}(M_i)$ satisfying the following properties:
(1) $P_{i+1}^{M_{i+1}} = M_{i+1} - M_i$.

(2) $R_{i+1}^n \subset P_{i+1}^{M_{i+1}} \times P_{i+1}^{M_{i+1}}$. For any $a \in P_{i+1}^{M_{i}}$ and $b \in P_{i+1}^{M_{i+1}}$ there is a predicate $R_{i+1}^n \in L_{i+1}$ such that $R_{i+1}^n(x, b)$ if and only if $x = a$.

(3) $M_0$ is a countable set. $H_0$ is a countable subgroup of permutation of $M_0$ which move only a finite number of elements.

(4) For all $f \in H_0$ and $i < \omega$ there is a unique extension of $f$ to an automorphism $f^* \in H_i$.

Now assume that $M_i$ and $H_i$ are defined as required. Let $M_{i+1} = \{b_f : f \in H_i\} \cup M_i$. Then $M_{i+1}$ is countable (because $H_i$ is so). Define a predicate $P_{i+1}^{M_{i+1}} = M_{i+1} - M_i$. Let $\{a_n : n < \omega\}$ be an enumeration of $P_{i}^{M_{i}}$. For every $n < \omega$ define a predicate $R_{i+1}^n M_{i+1} = \{(f(a_n), b_f) : f \in H_i\}$. Clearly $R_{i+1}^n$'s satisfy the condition (2). For $g \in H_i$ define a $g^*$ as follows:

\[
\begin{align*}
g^*(b_f) &= b_{g \cdot f} \quad \text{for each } b_f \in M_{i+1} - M_i, \\
g^*(a) &= g(a) \quad \text{for each } a \in M_i.
\end{align*}
\]

Then $g^*$ is an automorphism of $M_{i+1}$. In fact we can see that $(f(a), b_f) \in R_{i+1}^n$ iff $((g \cdot f)(a), b_{g \cdot f}) \in R_{i+1}^n$ iff $g^*(((f(a), b_f)) \in R_{i+1}^n$. Let $H_{i+1} = \{g^* : g \in H_i\}$. Then $H_{i+1}$ is a countable subgroup of $Aut(M_{i+1})$ since $H_i$ is so. Hence we can construct $M_i$'s and $H_i$'s.

Let $L = \bigcup L_i$. Let $M$ be an $L$-structure with $M = \bigcup M_i$.

(i) $M$ is a minimal model : Let $N$ be any submodel of $M$. Take any element $a$ of $M$. Since $M$ is the union of $P_i^{M_i}$'s there is minimum $i < \omega$ such that $a \in P_i^{M_i}$. Pick an arbitrary element $b$ of $P_{i+1}^{N}$. By the condition (2) there is some predicate $R \in L_{i+1}$ such that $R(x, b)$ holds if and only if $x = a$. Hence $a \in dcl(b) \subset N$, so $N = M$. Therefore $M$ is minimal.

(ii) $M_0$ is an indiscernible set : Let $\bar{a}, \bar{b}$ be any elements of $M_0$ with the same length. By the condition (3) there is an $f \in H_0$ such that $f(\bar{a}) = \bar{b}$. Moreover by (4) $f$ can be extended to an automorphism of $M$. So $tp(\bar{a}) = tp(\bar{b})$.

(iii) $Th(M)$ is not superstable : Let $\{a_n : n < \omega\}$ be an enumeration of $M_0$. For all $n < \omega$ let $\bar{a}_n = a_0 - a_1 - ... - a_n$. For all $n < \omega$ let $\varphi_n(x, \bar{a}_n)$ denote $R^n_1(a_0, x) \land ... \land R^n_1(a_n, x)$. Then $(\varphi_n)_{n<\omega}$ is a infinite chain of forking formulas. In fact, for each $n < \omega$, $\{\varphi_n(x, \bar{a}_{n-1}) : a \in M_0 - \{a_0, ..., a_{n-1}\}\}$ is a pairwise disjoint set. Hence $Th(M)$ is not superstable.

References
