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Relative Intrinsic Distance and Hyperbolic Imbedding

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Let Y be a complex space and X a complex subspace with compact closure $\overline{X}$. Let d_X and d_Y denote the intrinsic pseudo-distances of X and Y, respectively, (see [3]). We say that X is *hyperbolically imbedded* in Y if, for every pair of distinct points p, q in the closure $\overline{X} \subset Y$, there exist neighborhoods U_p and U_q of p and q in Y such that $d_X(U_p \cap X, U_q \cap X) > 0$. (In applications, X is usually a relatively compact open domain in Y.) It is clear that a hyperbolically imbedded complex space X is hyperbolic. The condition of hyperbolic imbedding says that the distance $d_X(p_n, q_n)$ remains positive when two sequences $\{p_n\}$ and $\{q_n\}$ in X approach two distinct points p and q of the boundary $\partial X = \overline{X} - X$. The concept of hyperbolic imbedding was first introduced in Kobayashi [3] to obtain a generalization of the big Picard theorem. The term "hyperbolic imbedding" was first used by Kiernan [2].

We shall now introduce a pseudo-distance $d_{X,Y}$ on $\overline{X}$ so that X is hyperbolically imbedded in Y if and only if $d_{X,Y}$ is a distance.

Let $\mathcal{F}_{X,Y}$ be the family of holomorphic maps $f: D \to Y$ such that $f^{-1}(X)$ is either empty or a singleton. Thus, $f \in \mathcal{F}_{X,Y}$ maps all of D, with the exception of possibly one point, into X. The exceptional point is of course mapped into $\overline{X}$.

We define a pseudo-distance $d_{X,Y}$ on $\overline{X}$ in the same way as d_Y, but using only chains of holomorphic disks belonging to $\mathcal{F}_{X,Y}$:

$$(1) \qquad d_{X,Y}(p,q) = \inf_{\alpha} l(\alpha), \qquad p, q \in \overline{X},$$

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where the infimum is taken over all chains α of holomorphic disks from p to q which belong to $\mathcal{F}_{X,Y}$. If p or q is in the boundary of X, such a chain may not exist. In such a case, $d_{X,Y}(p,q)$ is defined to be ∞. For example, if X is a convex bounded domain in $\mathbf{C}^n$, any holomorphic disk passing through a boundary point of X goes outside the closure $\overline{X}$, so that $d_{X,\mathbf{C}^n}(p,q) = \infty$ if p is a boundary point of X. On the other hand, if X is Zariski-open in Y, any pair of points p, q in $\overline{X} = Y$ can be joined by a chain of holomorphic disks beloning to $\mathcal{F}_{X,Y}$, so that $d_{X,Y}(p,q) < \infty$.

Since

$$\mathrm{Hol}(D,X) \subset \mathcal{F}_{X,Y} \subset \mathrm{Hol}(D,Y),$$

we have

$$(2) \qquad d_Y \leq d_{X,Y} \leq d_X,$$

where the second inequality holds on X while the first is valid on $\overline{X}$.

For the punctured disk $D^* = D - \{0\}$, we have

$$(3) \qquad d_{D^*,D} = d_D.$$

The inequality $d_{D^*,D} \geq d_D$ is a special case of (2). Using the identity map $\mathrm{id}_D \in \mathcal{F}_{D^*,D}$ as a holomorphic disk joining two points of D yeilds the opposite inequality.

Let $X' \subset Y'$ be another pair of complex spaces with $\overline{X}'$ compact. If $f: Y \to Y'$ is a holomorphic map such that $f(X) \subset X'$, then

$$(4) \qquad d_{X',Y'}(f(p), f(q)) \leq d_{X,Y}(p,q) \qquad p, q \in \overline{X}.$$

We can also define the infinitesimal form $F_{X,Y}$ of $d_{X,Y}$ in the same way as the infinitesimal form F_Y of d_Y, again using $\mathcal{F}_{X,Y}$ instead of $\mathrm{Hol}(D,Y)$.

Theorem. *A complex space X is hyperbolically imbedded in Y if and only if $d_{X,Y}(p,q) > 0$ for all pairs $p, q \in \overline{X}$, $p \neq q$.*

Proof. From $d_{X,Y} \leq d_X$ it follows that if $d_{X,Y}$ is a distance, then X is hyperbolically imbedded in Y.

Let E be any length function on Y. In order to prove the converse, it suffices to show that there is a positive constant c such that $cE \leq F_{X,Y}$ on $\overline{X}$. Suppose that there is no such constant. Then there exist a sequence of tangent vectors v_n of $\overline{X}$, a sequence of holomorphic maps $f_n \in \mathcal{F}_{X,Y}$ and a sequence of tangent vectors e_n of D with Poincaré length $\|e_n\| \searrow 0$ such that $f_n(e_n) = v_n$. Since D is homogeneous, we may assume that e_n is a vector at the origin of D.

In constructing $\{f_n\}$, instead of using the fixed disk D and varying vectors e_n, we can use varying disks D_{R_n} and a fixed tangent vector e at the origin with $R_n \nearrow \infty$. (We take e to be the vector d/dz at the origin of D, which has the Euclidean length 1. Let $|e_n|$ be the Euclidean length of e_n, and $R_n = 1/|e_n|$. Instead of $f_n(z)$ we use $f_n(|e_n|z)$.) Let $\mathcal{F}^{R_n}_{X,Y}$ be the family of holomorphic maps $f: D_{R_n} \to Y$ such that $f^{-1}(X)$ is either empty or a singleton. Having replaced D, e_n by D_{R_n}, e, we may assume that $f_n \in \mathcal{F}^{R_n}_{X,Y}$ and $f_n(e) = v_n$. We want to show that a suitable subsequence of $\{f_n\}$ converges to a nonconstant holomorphic map $f: \mathbf{C} \to \overline{X}$.

By applying Brody's lemma [1] to each f_n and a constant $0 < c < \frac{1}{4}$ we obtain holomorphic maps $g_n \in \mathrm{Hol}(D_{R_n}, Y)$ such that

(a) $g_n^* E^2 \leq cR_n^2 ds^2_{R_n}$ on D_{r_n} and the equality holds at the origin 0;

(b) $\mathrm{Image}(g_n) \subset \mathrm{Image}(f_n)$.

Since g_n is of the form $g = f_n \circ \mu_{r_n} \circ h_n$, where h_n is an automorphism of D_{R_n} and μ_{r_n}, $(0 < \mu_{r_n} < 1$, is the multiplication by r_n, each g_n is also in $\mathcal{F}_{X,Y}$.

Now, as in the proof of Brody's theorem [1] we shall construct a nonconstant holomorphic map $h: \mathbf{C} \to Y$ to which a suitable subsequence of $\{g_n\}$ converges. In fact, since

$$g_n^* E^2 \leq cR_n^2 ds^2_{R_n} \leq cR_m^2 ds^2_{R_m} \qquad \text{for} \quad n \geq m,$$

the family $\mathcal{F}_m = \{g_n | D_{R_m}, n \geq m\}$ is equicontinuous for each fixed m. Since the family $\mathcal{F}_1 = \{g_n | D_{R_1}\}$ is equicontinuous, the Arzela-Ascoli theorem implies that we can extract a subsequence which converges to a map $h_1 \in \mathrm{Hol}(D_{R_1}, Y)$. (We note that this is where we use the compactness of $\overline{X}$.) Applying the same theorem to the corresponding sequence in $\mathcal{F}_2$, we extract a subsequence which converges to a map $h_2 \in \mathrm{Hol}(D_{R_2}, Y)$. In this way we obtain maps $h_k \in \mathrm{Hol}(D_{R_k}, Y), k = 1, 2, \cdots$ such that each h_k is an extension of h_{k-1}. Hence, we have a map $h \in \mathrm{Hol}(\mathbf{C}, Y)$ which extends all h_k.

Since $g_n^* E^2$ at the origin 0 is equal to $(cR_n^2 ds^2_{R_n})_{z=0} = 4c dz d\bar{z}$, it follows that

$$(h^* E^2)_{z=0} = \lim_{n \to \infty} (g_n^* E^2)_{z=0} = 4c dz d\bar{z} \neq 0,$$

which shows that h is nonconstant.

Since $g_n^* E^2 \leq cR_n^2 ds^2_{R_n}$, in the limit we have

$$h^* E^2 \leq 4c dz d\bar{z}.$$

By suitably normalizing h we obtain

$$h^* E^2 \leq dz d\bar{z} \quad \text{with the equality holding at} \quad z = 0.$$

We may assume that $\{g_n\}$ itself converges to h. Since h is the limit of of $\{g_n\}$, clearly $h(\mathbf{C}) \subset \overline{X}$. Let p, q be two points of $h(\mathbf{C})$, say $p = h(a)$ and $q = h(b)$. Taking a subsequence and suitable points a, b we may assume that $g_n(a), g_n(b) \in X$. Then $\lim g_n(0) = p$ and $\lim g_n(a) = q$ and

$$d_X(g_n(a), g_n(b)) \leq d_{D_{R_n}}(a,b) \to 0 \quad \text{as} \quad n \to \infty,$$

contradicting the assumption that X is hyperbolically imbedded in Y. Q.E.D.

This relative distance $d_{X,Y}$ simplifies the proof of the big Picard theorem as formulated in [3].

Bibliography

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