Title	A degenerate metric induced on the dual of nef line bundles(HOLOMORPHIC MAPPINGS, DIOPHANTINE GEOMETRY and RELATED TOPICS : in Honor of Professor Shoshichi Kobayashi on his 60th Birthday)
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Citation	数理解析研究所講究録 (1993), 819: 218-223
Issue Date	1993-01
URL	http://hdl.handle.net/2433/83146
Right	
Type	Departmental Bulletin Paper
Textversion	publisher

数理解析研究所講究録
819 巻 1993 年 218-223

A degenerate metric induced on the dual of nef line bundles

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In this note we explain the notion of nef line bundles and the relation of it and positive currents. Next we discuss the convergence of metrics induced on the dual of nef line bundles to a certain degenerate metric and its regularization. The proofs of results stated in this note and their applications will appear in elsewhere.

1 Nef line bundles and positive currents

Let (X, ω_X) be an n dimensional compact complex manifold with the smooth hermitian metric ω_X and let (F, h) be a holomorphic line bundle on X with the smooth hermitian metric h. The curvature form of F relative to h defined by the equation $\Theta(h) = -dd_c \log h$ ($d_c = i(\bar\partial - \partial)/2$). F is said to be semi-positive if F has a smooth metric h whose curvature form $\Theta(h)$ is a semi-positive form on X.

This geometric semi-positivity of line bundles is however not so flexible in view of the smoothness of metrics. In algebraic geometry a more flexible notion of semi-positivity is numerical effectivity. A holomorphic line bundle (E, e) on a projective algebraic manifold M is said to be ***nef*** (numerically effective) if $E \cdot C = \int_C c_{R,1}(E) \geqq 0$ for any irreducible reduced curve C in M where $c_{R,1}(E) = [\Theta(e)/2\pi]$ in $H^2(X, R)$ is the first Chern class of E. It is clear that any semi-positive line bundle on a projective algebraic manifold is nef. However the converse is not true in general. In fact we can describe the nefness in terms of a certain positive current with singularity which is a limit of smooth positive forms. Precisely we can formulate the nefness in terms of positive current as follows.

Let (X, ω_X) and (F, h) be as above. We say F is nef if for any positive number ε there is a smooth hermitian metric h_ε so that $\Theta(h_\varepsilon) + \varepsilon\,\omega_X$ is positive on X. Since h and h_ε are conformally equivalent each other, this definition is equivalent to the following one : for any positive number ε there is a smooth function f_ε so that $\Theta(h) + \varepsilon\,\omega_X + dd_C f_\varepsilon$ is positive on X. By setting $T_\varepsilon = \Theta(h) + \varepsilon\,\omega_X + dd_C f_\varepsilon$, it can be verified that T_ε converges weakly to a positive current $T = \Theta(h) + dd_C f$ with an L^1-function f (in this case it is not necessary that ω_X is Kähler). Since f has singularity generally, $h^{-1}e^f$ is a degenerate metric of the dual F^* of F. If ω_X is Kähler, then F is nef if and only if the first Chern class $c_{R,1}(F)$ of F is contained in the closure of the Kähler cone of X in view of $\partial\bar{\partial}$-lemma on compact Kähler manifolds. Moreover if X is projective algebraic ,then the above nefness is equivalent to the original one in algebraic geometry in view of Kleiman's criterion (cf. [N] , § 2 and [D] ,§6) .

2 A sequence of smooth metrics on the dual of nef line bundles

Let (X, ω_X) be an n dimensional compact complex manifold and let (F, h) be a nef line bundle on X. From the above observation ,for any $\varepsilon > 0$ there is a smooth function f_ε so that $T_\varepsilon = \Theta(h) + \varepsilon\omega_X + dd_c f_\varepsilon > 0$. We have already seen the convergence of T_ε as currents . However not much is known about the convergence of the metrics $h_\varepsilon = h e^{-f_\varepsilon}$ of F or the metrics $h_\varepsilon^* = h^{-1} e^{f_\varepsilon}$ of F^*. Here we discuss the convergence of h_ε^*. A reason why the convergence of h_ε^* is easier than that of h_ε is explained as follows. We say a smooth function f almost plurisubharmonic relative to a fixed metric ω_* on X if $\omega_* + dd_c f > 0$ on X. Since X is compact,we can assume $\int_X e^{f_\varepsilon}\omega_X^n = 1$. Since the functions f_ε are almost plurisubharmonic relative to $\omega_* = K\omega_X$ with $K > 0$,by the above normalization we can conclude that the fuctions $\eta_\varepsilon := e^{f_\varepsilon}$ are uniformly bounded from above. This implies that the functions η_ε and their radical roots are almost plurisubharmonic relative to a certain metric ω_{**}. Hence we can apply the discussion of the Dirichlet form acting on locally uniformly bounded plurisubharmonic functions by Fukushima and Okada [F-O] to those functions. In fact using the compactness of X we can show the following theorem which is not true in a local nature.

Theorem *Let (F, h) be a nef line bundle on an n dimensional compact complex manifold (X, ω_X). Then there exist a positive constant K, a decreasing sequence of positive numbers $\{\delta_k\}$ with $\delta_k \to 0$ as $k \to \infty$, a sequence of smooth positive functions $\{\eta_k\}$ on X and a non-negative function η on X so that*

(1) $0 < \eta_k \leqq K$ *on* X *and* $0 \leqq \eta \leqq K$ *a.e. on* X

(2) $\eta_k^{\frac{1}{2}}$ *converges to* $\eta^{\frac{1}{2}}$ *in* $W^{1,2}(X)$ *(* L^2 *- Sobolev space of order 1)*

(3) $\Theta(h) + \delta_k \omega_X + dd_c \log \eta_k > 0$

(4) $dd_c \log(h^{-1}\eta)$ *is a positive current on* X.

In particular the set $E_\eta = \{\eta = 0\}$ *is measure zero.*

In the above theorem the set E_η is not empty in general (cf. [DPS]). This means that any nef line bundle is not always semi-positive in the above sense. Moreover it is not so clear whether E_η is an analytic subset of X.

3 A regularization of the degenerate metric on nef line bundles

Let (F, h) be a holomorphic line bundle on an n dimensional compact complex manifold (X, ω_X). We say F quasi semi-positive if F admits a hermitian metric a (not necessarily smooth) so that locally $-\log a$ is plurisubharmonic. For instance any effective Cartier divisor of X is quasi semi-positive and any nef line bundle on X is also from the above observation. Hence any quasi semi-positive line bundle is not always semi-positive. However in a special case we can regularize the metric of any quasi semi-positive line bundle as follows.

Theorem *Let (X, ω_X) be an n dimensional compact complex manifold which is a compactification of a complete Kähler manifold (Y, ω_Y) whose universal covering is an n dimensional complex vector space (C^n, ω_e) with the Euclidean metric ω_e. Then for any fixed non-negative integer k, any quasi semi-positive line bundle on X admits a hermitian metric $a_{(k)}$ on Y so that (i) $a_{(k)}$ is of class C^k and bounded along $X \setminus Y$ (ii) locally $-\log a_{(k)}$ is plurisubharmonic.*

The case $k = \infty$ is still remained . However it is known that we can take $k = \infty$ if Y is a complex torus. This problem is related to the regularization of positive currents or the approximation of positive currents by smooth positive currents on compact Kähler manifolds. The reader should be refered to [D] and [DPS] with respect to those problems.

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