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Analytic Zariski Decomposition

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1 Introduction

Let X be a projective variety and let D be a Cartier divisor on X. The following problem is fundamental in algebraic geometry.

Problem 1 *Study the linear system* $\mid \nu D \mid$ *for* $\nu \geq 1$.

To this problem, there is a rather well developped theory in the case of $\dim X = 1$. In the case of $\dim X = 2$, in early 60-th, O. Zariski reduced this problem to the case that D is nef(= numerically semipositive) by using his famous Zariski decomposition ([12]).

Recently Fujita, Kawamata etc. generalized the concept of Zariski decompositions to the case of $\dim X \geq 3$ ([2, 4]). The definition is as follows.

Definition 1 *Let X be a projective variety and let D be a* $\mathbf{R}$*-Cartier divisor on X. The expression*

$$D = P + N (P, N \in Div(X) \otimes \mathbf{R})$$

is called a Zariski decomposition of D, if the following conditions are satisfied.

1. *P is nef,*

2. *N is effective,*

3. $H^0(X, \mathcal{O}_X([\nu P])) \simeq H^0(X, \mathcal{O}_X([\nu D]))$ *holds for all* $\nu \in \mathbf{Z}_{\geq 0}$, *where* $[\ ]$*'s denote the integral parts of divisors.*

In the case of $\dim X = 2$, for any pseudoeffective divisor D on X, a Zariski decomposition of D exists ([12]). But in the case of $\dim X \geq 3$, although many useful applications of this decomposition have been known ([2, 4, 7]), as for the existence, very little has been known. There is the following (rather optimistic) conjecture.

Conjecture 1 *Let X be a normal projective variety and let D be a pseudoeffective $\mathbf{R}$-Cartier divisor on X. Then there exists a modification $f : Y \longrightarrow X$ such that f^*D admits a Zariski decomposition.*

The purpose of this paper show how to construct an analytic counterpart of Zariski decomposition. Please see [9, 10] for detail and further applications. In this paper, all algebraic varieties are defined over $\mathbf{C}$.

2 Statement of the results

The main idea in this paper is to use d-closed positive $(1,1)$-currents, instead of divisors. d-closed positive currents is far more general object than effective algebraic cycles. The advantage of using d-closed positive currents is in the flexibility and completeness of them.

Definition 2 *Let X be a normal projective variety and let D be a $\mathbf{R}$-Cartier divisor on X. D is called big if*

$$\kappa(D) := \limsup_{\nu \longrightarrow +\infty} \frac{\log \dim H^0(X, \mathcal{O}_X([\nu D]))}{\log \nu} = \dim X.$$

holds. D is called pseudoeffective ,if for any ample divisor H, $D + \varepsilon H$ is big for every $\varepsilon > 0$.

Definition 3 *Let M be a complex manifold of dimension n and let $A_c^{p,q}(M)$ denote the space of $C^\infty(p,q)$ forms of compact support on M with usual Fréchet space structure. The dual space $D^{p,q}(M) := A_c^{n-p,n-q}(M)^*$ is called* *the space of (p,q)-currents on M. The linear operators $\partial : D^{p,q}(M) \longrightarrow D^{p+1,q}(M)$ and $\bar{\partial} : D^{p,q}(M) \longrightarrow D^{p,q+1}(M)$ is defined by*

$$\partial T(\varphi) = (-1)^{p+q+1}T(\partial\varphi), T \in D^{p,q}(M), \varphi \in A_c^{n-p-1,n-q}(M)$$

and

$$\bar{\partial} T(\varphi) = (-1)^{p+q+1}T(\bar{\partial}\varphi), T \in D^{p,q}(M), \varphi \in A_c^{n-p,n-q-1}(M).$$

We set $d = \partial + \bar{\partial}$. $T \in D^{p,q}(M)$ is called closed if $dT = 0$. $T \in D^{p,p}(M)$ is called real if $T(\varphi) = T(\bar{\varphi})$ holds for all $\varphi \in A_c^{n-p,n-p}(M)$. A real current (p,p)-current T is called positive if $(\sqrt{-1})^{p(n-p)}T(\eta \wedge \bar{\eta}) \geq 0$ holds for all $\eta \in A_c^{p,0}(M)$.

Since codimension p subvarieties are considered to be closed positive (p,p)-currents, closed positive (p,p)-currents are considered as a completion of the space of codimension p subvarieties with respect to the topology of currents. For a $\mathbf{R}$ divisor D on a smooth projective variety X. We denote the class of D in $H^2(X,\mathbf{R})$ by $c_1(D)$.

Definition 4 *Let T be a closed positive (p,p)-current on the open unit ball $B(1)$ in $\mathbf{C}^n$ with centre O. The Lelong number $\Theta(T,O)$ of T at O is defined by*

$$\Theta(T,O) = \lim_{r \downarrow 0} \frac{1}{\pi^{n-p} r^{2(n-p)}} T(\chi(r)\omega^{n-p}),$$

where $\omega = \frac{\sqrt{-1}}{2}\sum_{i=1}^{n} dz_i \wedge d\bar{z}_i$ and $\chi(r)$ be the charcteristic function of the open ball of radius r with centre O in $\mathbf{C}^n$.

It is well known that the Lelong number is invariant under coordinate changes. Hence we can define the Lelong number for a closed positive (p,p)-current on a complex manifold. It is well known that if a closed positive current T is defined by a codimension p-subvariety the Lelong number $\Theta(T,x)$ coincides the multiplicities of the subvariety at x. In this sense the Lelong number is considered as the multiplicity of a closed positive current.

We note that thanks to Hironaka resolution of singularities, to solve the conjecture, we can restrict ourselves to the case that X is smooth. Our theorem is stated as follows.

Theorem 1 *Let X be a smooth projective variety and let L be a line bundle on X. Then there exists a closed positive $(1,1)$-current T such that*

1. *T represents $c_1(L)$ in $H^2(X,\mathbf{R})$,*
2. *For every modification $f : Y \longrightarrow X$ $\nu \in \mathbf{Z}_{\geq 0}$ and $y \in Y$,*

$$mult_y Bs \mid f^*(\nu L) \mid \geq \nu\Theta(f^*T, y)$$

holds.

We call T an Analytic Zariski decomposition(AZD) of L. Let

$$T = T_{abc} + T_{sing}$$

be the Lebesgue decomposition of T , where T_{abc}, T_{sing} denote the absolutely continuous part and the singular part of T respectively. As you see below, this decomposition corresponds to Zariski decomposition.

The relation between Zariski decomposition and AZD is described by the following corollary and proposition.

Corollary 1 *Let X be a smooth projective variety and let D be a nef and big $\mathbf{R}$ divisor on X. Then $c_1(D)$ can be represented by a closed positive $(1,1)$-current T with $\Theta(T) \equiv 0$.*

Proposition 1 *Let X be a smooth projective variety and let D be a $\mathbf{R}$ divisor on X such that $2\pi c_1(D)$ can be represented by a closed positive $(1,1)$ current T with $\Theta(T) \equiv 0$. Then D is nef.*

Let X, L be as in Theorem 1. Suppose that there exists a modification $f : Y \longrightarrow X$ such that there exists a Zariski decomposition $f^*L = P + N$ of f^*L on Y. Then by Cororally 1 there exists a closed positive $(1,1)$ current S such that $c_1(P) = [S]$ and $\Theta(S) \equiv 0$. Then the push-forward $T = f_*(S+N)$ is a AZD of L. The main advantage of AZD is that we can consider the existence without changing the space by modifications. One may ask whether AZD substitutes ZD(Zariski decomposition). In some case the answer is "Yes". In this paper, I would like to show some applications, too.

3 Outline of the proof of Theorem 1

Now I would like to show the outline of the proof of Theorem 1. Let X,L be as in Theorem 1. Let h be a C^∞-hermitian metric on L and let ω_∞ be the curvature form of h. Let ω_0 be a C^∞ Kähler form on X such that

$$\omega_0 - \omega_\infty > 0$$

holds on X. We set

$$\omega_t = (1 - e^{-t})\omega_\infty + e^{-t}\omega_0.$$

Let Ω be a C^∞ volume form on X. Now we consider the following initial value problem.

$$\begin{aligned} \frac{\partial u}{\partial t} &= \log \frac{(\omega_t + \sqrt{-1}\partial\bar{\partial}u)^n}{\Omega} - u \quad \text{on } X \times [0, t_0) && (1) \\ u &= 0 \qquad \text{on } X \times \{0\}, && (2) \end{aligned}$$

where $n = \dim X$ and t_0 is the maximal existence time for the C^∞ solution u. By the standard implicit function theorem T is positive. Since $\omega_0 - \omega_\infty > 0$, by direct calculation we have the partial differential inequality

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right) \leq \tilde{\Delta}\frac{\partial u}{\partial t} - \frac{\partial u}{\partial t},$$

where $\tilde{\Delta}$ dnotes the Laplacian with respect to the Kähler form $\omega_t + \sqrt{-1}\partial\bar{\partial}u$. Hence by maximum principle, there exists a positive constant C_0^+ such that

$$\frac{\partial u}{\partial t} \leq C_0^+ e^{-t}$$

holds on $X \times [0, t_0)$. But unfortunately, we do not have uniform lower bound for the solution u. Actually we cannot expect the uniform lower bound for u.

The above equation corresponds to the following Hamilton type equation:

$$\begin{aligned} \frac{\partial \omega}{\partial t} &= -\mathrm{Ric}_\omega - \omega + (\mathrm{Ric}\,\Omega + \mathit{curv}\, h) \quad \text{on } X \times [0, t_0) \\ \omega &= \omega_0 \qquad \text{om } X \times \{0\}. \end{aligned}$$

This equation preserves the Kählerity of ω. Hence it is meaningful to take the de Rham cohomology class $[\omega]$. By a calculation, we see that

$$[\omega] = (1 - e^{-t})2\pi c_1(L) + e^{-t}[\omega_0]$$

holds. Let $A(X)$ denote the Kähler cone of X. By the above equation, we see that $[\omega] \in A(X)$, if $t \in [0, t_0)$. Conversly we have:

Lemma 1 $T = \sup\{t \mid [\omega] \in A(X)\}$.

But this means that unless $2\pi c_1(L)$ sits on the closure of $A(X)$, we cannot expect $T = \infty$.

Hence we should consider a current solution

$$\omega_t + \sqrt{-1}\partial\bar{\partial}u$$

instead of a C^∞ solution, where $u : X \longrightarrow [-\infty, \infty)$ To construct a current solution we need to find the place where the estimate of the solution u breaks. We set

$$S = \cap_{\nu>0}\{x \in X \mid H^0(X, \mathcal{O}_X(\nu L)) \text{ does not separate } TX_x\}$$

and we expect that the solution u is C^∞ on $X - S$.

The natural way to construct such a singular solution is to construct the solution by as a limit of the solution of Dirichlet problems on relatively compact subdomains in $X - S$ which exhaust $X - S$. So we would like to apply the theory of Dirichlet problem for complex Monge-Ampère equations developped recently ([1]).

But in fact, we need to subtract a little bit larger set because $X - S$ is not strongly pseudoconvex. Otherwise the theory does not work (this phenomena is caused by the lack of good barriers for the estimates, if the domain is not pseudoconvex). Let $f_\nu : X_\nu \longrightarrow X$ be a resolution of Bs $\mid \nu L \mid$ and let

$$\mid f_\nu^*(\nu L) \mid = \mid P_\nu \mid + N_\nu$$

be the decomposition into the free part and the fixed part. The following lemma is well known and very useful.

Lemma 2 *(Kodaira's lemma) Let X be a smooth projective variety and let D be a big divisor on X. Then there exists an effective $\mathbf{Q}$-divisor E such that $D - E$ is an ample $\mathbf{Q}$-divisor.*

Then by Kodaira's lemma, we can find an effective divisor R_ν on X_ν such that for every sufficiently small positive rational number ε, $P_\nu - \varepsilon R_\nu$ is an ample $\mathbf{Q}$-divisor.

Let us take ν sufficiently large so that the free divisor P_ν is nef and big. Let $\Phi : X_\nu \longrightarrow \mathbf{P}^N$ be an embedding of X_ν into a projective space and let

$$\pi_\alpha : X_\nu \longrightarrow \mathbf{P}^n (\alpha = 1, \ldots, m)$$

be generic projections and we set

$$W_\alpha : \text{the ramification divisor of } \pi_\alpha$$

$$H_\alpha := \pi_\alpha^*(z_0 = 0),$$

where $[z_0 : \ldots : z_n]$ be the homogeneous coodinate of $\mathbf{P}^n$. For simplicity we shall denote the support of a divisor by the same notation as the one, if without fear of confusion. If we take m sufficiently large, we may assume the following conditions:

1. $\cap_{\alpha=1}^m (W_\alpha + H_\alpha) = \phi$,
2. $D := (F_\nu + \sum_{\alpha=1}^m (W_\alpha + H_\alpha))_{red}$ is an ample divisor with normal crossings,
3. D constains $S \cup R_\nu$,
4. $K_{X_\nu} + D$ is ample.

Then $U = X_\nu - D_\nu$ is strongly pseudoconvex and is identified with a Zariski open subset of X. Let K be a relatively compact strongly pseudoconvex subdomain of U with C^∞ boundary. Thanks to the condition 1 above, for K, we can apply the theory in [1] developped on strongly pseudoconvex domains with C^∞ boundary in a complex Euclidean space, although K is not inside $\mathbf{C}^n$.

Hence we can solve a Dirichlet problem for a complex Monge-Ampére equation on K. In our case the equation is parabolic, so we need to modify the theory. To get the C^0-estimate for the solution, we shall change the unknown. Let τ be a section of $\mathcal{O}_{X_\nu}(F_\nu)$ with divisor F_ν and let λ be a section of $\mathcal{O}_{X_\nu}(R_\nu)$ with divisor R_ν. Then there exists a hermitian metrics h_F,h_R on $\mathcal{O}_{X_\nu}(F_\nu)$, $\mathcal{O}_{X_\nu}(R_\nu)$ respectively such that

$$f_\nu^*\omega_\infty - \frac{1}{\nu} curv\, h_F - \varepsilon \cdot curv\, h_R$$

is a Kähler form on X_ν for every sufficiently small postive number ε. Let us change u by

$$v = u - (1 - e^{-t})(\frac{1}{\nu} \log h_N(\tau, \tau) - \varepsilon h_R(\lambda, \lambda)).$$

Then since

$$\omega_t' = (1 - e^{-t})(f_\nu^*\omega_\infty - \frac{1}{\nu} curv\, h_F - \varepsilon \cdot curv\, h_R) + e^{-t} f_\nu^*\omega_0$$

is uniformly positive on U, we can solve the 0-Dirichlet boundary value problem for v on $K \times [0, \infty)$ for any relatively compact strongly pseudoconvex subdomain K with C^∞ boundary.

Remark 1 *Here we need to worry about the Gibb's phenomena for the parabolic equation. But this is rather thechnical and not essential. Hence we shall omit it.*

Let us take an strongly pseudoconvex exhaustion $\{K_\mu\}$ of U and consider a family of Dirichlet problems of parabolic complex Monge-Ampére equation (1).

The next difficulty is the convergence of the solutions of this family of Dirichlet problems. Here we note that there exists a complete Kähler-Einstein form ω_D on U thanks to the conditions 3, 4 above and [6]. Then if we choose the boundary values properly, we can dominate the volume forms associated with the solutions from above by a constant times ω_D^n by maximum principle. This ensures the convergence.

Let $u \in C^\infty(U)$ be the solution of (1) on U. Then by the C^0-estimate of u, we see that u extends to a L^1-function on X for every t.

Now we set

$$T = \lim_{t \to \infty} (\omega_t + \sqrt{-1}\partial\bar{\partial}u),$$

where $\partial\bar{\partial}$ is taken in the sense of current.

Remark 2 *On the first look, T seems to depend on the choice of ν. But actually, T is indpendent of ν. This follows from the uniqueness property of the equation* (1).

Then we can verify that T is an AZD of D by using the C^0-estimate of u.

4 Basic properties of AZD

As a direct consequence of the construction, an AZD has following properties.

Proposition 2 *Let X be a smooth projective variety and let L be a big line bundle on X. Let T be an AZD of L, then T has the following properties.*

1. *Let $T = T_{abc} + T_{sing}$ denote the Lebesgue decomposition of T. Then there exists a reduced very ample divisor D on X such that T_{abc} is C^∞ on $X - D$.*

2. *T^n_{abc} is of Poincaré growth along D. In particular T^n_{abc} is integrable on X.*

3. *T is of finite order along D, i.e., only polynomial growth along D.*

Remark 3 *D need not be of normal crossings. Hence the word "Poincaré growth" means a little bit generalized sense, i.e. if we take any modification such that the total transform of D becomes of normal crossings, the pull-back of T^n_{abc} is of Poincaré growth along the total transform.*

Remark 4 *I think the third property of AZD should be the key to solve the conjecture in the introduction.*

By using Kodaira's lemma and Hörmander's L^2-estimate for $\bar{\partial}$-operator, we can easily get.

Proposition 3 *Let X,L,T be as in Proposition 1. Then for every modification $f : Y \longrightarrow X$ and any $y \in Y$,*

$$\Theta(f^*T, y) = \liminf_{\nu \longrightarrow +\infty} \nu^{-1} mult_y Bs \mid \nu L \mid$$

holds.

Proposition 2 means that although an AZD is not unique, but the singular part is in some sense unique and the AZD controlles the asymptotic behavior of the base shemes of the multilinear systems.

Instead of using AZD itself, sometimes it is more useful to use the "potential" of AZD.

Definition 5 *Let L be a line bundle on a complex manifold X. h is called a singular hermitian metric on L, if there exist a C^∞-hermitian metric h_0 on L and locally L^1-fuction φ such that*

$$h = e^{-\varphi} h_0$$

holds.

We note that for a singular hermitian meric it is meaningful to take curvature of it in the sense of current.

One of the most useful property of AZD is the following vanishing theorem.

Theorem 2 *Let X,L be as in Theorem 1 and let T be an AZD of D constructed as above. Let h be a singular hermitian metric on L such that $T = curv\, h$. For a positive integer m we set*

$$\mathcal{F}_m := \textit{sheaf of germs of local } L^2\textit{-holomorphic sections of } (\mathcal{O}_X(mD), h^{\otimes m}).$$

Then $\mathcal{F}_m$ is coherent sheaf on X and

$$H^p(X, K_X \otimes \mathcal{F}_m) = 0$$

holds for $p \geq 1$.

By Corollary 1, we get the following well known vanishing theroem.

Corollary 2 *([5]) Let X be a smooth projective manifold and let L be a nef and big line budnle. Then*

$$H^p(X, K_X \otimes L) = 0$$

holds for $p \geq 1$.

5 Some direct applications of AZD

In this section, we shall see that we can controle the asumptotic behavior of the multilinear systems associated with big line bundle in terms of its AZD.

Definition 6 *Let L be a line bundle over a projective n-fold X. We set*

$$vol(X, L) = \lim_{\nu \to +\infty} \nu^{-n} \dim H^0(X, \mathcal{O}_X(\nu L))$$

and call it the L-volume of X or the volume of X with respect to L.

We can express the volume in terms of AZD.

Theorem 3 *Let L be a big line bundle over a smooth projective n-fold X and let $T = T_{abc} + T_{sing}$ be an AZD of L constructed as in Section 3. Then we have*

$$vol(X, L) = \frac{1}{(2\pi)^n n!} \int_X T_{abc}^n$$

holds.

The following therem follows from the existence of AZD and Lebesgue-Fatou's lemma.

Theorem 4 *Let $\pi : X \longrightarrow S$ be a smooth projective family of projective varieties over a connected complex manifold S and let L be a relatively big line bundle on X. For $s \in S$, we set $X_s = \pi^{-1}(s)$ and $L_s = L \mid X_s$. Then $vol(X_s, L_s)$ is an uppersemicontinuous function on S.*

The following theorem follows from Theorem 2.

Theorem 5 *Let $\pi : X \longrightarrow S$ be a smooth projective family of projective varieties over a connected complex manifold S and let L be a line bundle on X. Suppose that $aL - K_X$ is relatively big for some $a > 0$. Then $vol(X_s, L_s)$ is a constant function on S.*

Proof of Theorem 3. Let X, L, T be as in Theorem 3. Let$D$ be as in Proposition 1. By taking a modification of X, we may assume that D is a divisor with simple normal crossings. By Kodaira's lemma there exists an effective $\mathbf{Q}$ divisor E such that $L - E$ is an ample $\mathbf{Q}$-line bundle. Let $\bar{\omega}$ be a Kähler form on X which represents $c_1(L - E)$. Let σ be a section of $\mathcal{O}_X(D)$ such that $(\sigma) = H$ and let h be a C^∞-hermitian metric on $\mathcal{O}_X(D)$. Then for a sufficiently small positive number c,

$$\omega = \bar{\omega} + c\sqrt{-1}\partial\bar{\partial}\log(-\log h(\sigma, \sigma))$$

is a complete Kähler form on $X - D$. We note that there exist positive constants C_1, C_2 such that

$$-C_1\omega < \mathrm{Ric}_\omega < C_2\omega$$

on $X - D$ by direct computation (actually ω has bounded geometry). Then by the L^2-Riemann-Roch inequality ([8]), we have that for every $\varepsilon > 0$ we have the inequality

$$\frac{1}{(2\pi)^n n!}\int_{X-D}(T_{abc} + \varepsilon\omega)^n \leq vol((1+\varepsilon)L - \varepsilon E) \leq (1+\varepsilon)^n vol(X, L).$$

Letting ε tend to 0, we have the inequality

$$\frac{1}{(2\pi)^n n!}\int_{X-D} T_{abc}^n \leq vol(X, L). \tag{3}$$

Let $f_\nu; X_\nu \longrightarrow X$ be a resolution of Bs $\mid \nu L \mid$ and let $\mid P_\nu \mid$ denote the free part of $\mid f_\nu^*(\nu L) \mid$. Assume that ν is sufficiently large so that P_ν is nef and big. Let ω_ν denote a semipositive first Chern form of $\mathcal{O}_{X_\nu}(f_\nu^*(\nu L))$. We set

$$T_\nu = \nu^{-1}(f_\nu)_*(\omega_\nu).$$

Then since $\mid P_\nu \mid$ is free, by Bertini's theorem, we get the sequence of inequalities:

$$T_{abc}^n \geq T_{abc}^{n-1}T_\nu \geq \cdots \geq T_\nu^n$$

on $X - D$. Hence we see that

$$\int_{X_\nu} (T_\nu)^n_{abc} = (2\pi)^n \nu^{-n} c_1^n(P_\nu) \leq \int_X T^n_{abc}$$

holds. We need the following proposition.

Proposition 4 *(Fujita) Let L be a big line bundle on a projective manifold X. Let $f_\nu : X_\nu \longrightarrow X$ be a resolutiona of $Bs \mid \nu L \mid$ and let*

$$\mid f_\nu^*(\nu L) \mid = \mid P_\nu \mid + F_\nu$$

be the decomposition into the free part and the fixed part. Then the equality

$$vol(X, L) = \frac{1}{n!} \lim_{\nu \to +\infty} \nu^{-n} P_\nu^n$$

holds.

By Proposition 4, we have that

$$vol(X, L) \leq \frac{1}{(2\pi)^n n!} \int_X T^n_{abc} \tag{4}$$

holds.

Combining (3) and (4), we complete the proof of Theorem 3.

Corollary 3 *Let X, L, T be as in Theorem 1 and let ω be a Kähler form on X. Then $\int_X T^k_{abc} \wedge \omega^{n-k}$ is finite on X.*

Proof of Theorem 4.

We may assume that $S = \{s \in \mathbf{C} \mid \mid s \mid < 1\}$.

Step 1. For the first we shall consider the case that L is relatively big. Let $T_s = (T_s)_{abc} + (T_s)_{sing} (s \in S)$ be the family of AZD's constructed by the flow for the positive current ω,

$$\begin{aligned} \frac{\partial \omega}{\partial t} &= -\mathrm{Ric}_\omega - \omega + (\mathrm{Ric}\, \Omega + curv\, h) \quad \text{on } X_s \times [0, \infty) \\ \omega &= \omega_0 \qquad \text{on } X_s \times \{0\}, \end{aligned}$$

where
$\mathrm{Ric}_\omega := -\sqrt{-1}\partial\bar{\partial}\log\omega^n$: the Ricci current of ω
h : a (relative) C^∞-hermitian metric on L,
ω_0 : a relative C^∞-Kähler form on X,
Ω a relative C^∞-volume form on X.
Since L is relatively big, as in Section 3, there exists a reduced divisor D on X such that

1. D is equidimensional over S,
2. $(X - D)$ admits a complete relative Kähler-Einstein metric ω_D of constant Ricci curvature -1,
3. ω is C^∞ on $X - D$.

Then by Lebesgue's bounded convergence theorem, we see that

$$vol(X_s, L_s) = \int_{X_s} (T_s)^n_{abc}$$

is an uppersemicontinuous function with respect to s. This completes the proof of Theorem 4.

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