数理解析研究所講究録
第 819 巻 1993 年 120-135

Recent Results in Hyperbolic Geometry and Diophantine Geometry

Pit-Mann Wong*

Introduction In his monograph "Hyperbolic Manifolds and Holomorphic Mappings", Kobayashi [K] raised the question of whether the complement of a generic curve of degree $d \geq 5$ in $\mathbf{P}_2(\mathbf{C})$ is Kobayashi hyperbolic. The problem is still open at this time but some progress have been made towards this problem. The purpose of this note is to describe some of these developments. In recent years there also emerged evidence that the theories of hyperbolic geometry and diophantine geometry are closely related. Indeed the underlying complex manifolds of all known Mordellic varieties (following Lang [L], a *projective variety* V defined over an algebraic number field **K** is said to be *Mordellic* if the **K**-rational points V(**K**) is finite; an *affine variety* defined over **K** is said to be *Mordellic* if the number of **K**-integral points is finite) are hyperbolic. We shall indicate also in this note how one may "translate" a proof of "hyperbolicity" into a proof of "finiteness". The main principle is this:

> "*if a proof that a variety is hyperbolic is based entirely on the standard Second Main Theorem of Value Distribution Theory then the proof can be translated into a proof of finiteness of the corresponding variety defined over an algebraic number field*".

The basic correspondence is Vojta's observation that the Second Main Theorem of Value Distribution Theory correspondes to the Thue-Siegel-Roth-Schmidt Theorem in the Theory of Diophantine Approximations. For further details of this correspondence we refer the reader to Vojta [V1] and Ru-Wong [RW].

§ 1 The case of 4 or more components

Let $\mathcal{S}(d)$ be the space of curves of degree $d \geq 5$ in $\mathbf{P}_2(\mathbf{C})$, then $\mathcal{S}(d)$ is a projective variety of dimension $\{(d+1)(d+2)/2\} - 1 = d(d+3)/2$. Kobayashi's problem is to show that:

"*there exists a Zariski closed subset $\mathcal{F}$, of strictly lower dimension, of $\mathcal{S}(d)$ such that* $\mathbf{P}_2(\mathbf{C})$ *- C is Kobayashi-hyperbolic and hyperbolically embedded in* $\mathbf{P}_2(\mathbf{C})$ *for all* $C \in \mathcal{S}(d) - \mathcal{F}$".

---

* Research supported in part by NSF grant # DMS 91-00872

More generally, let $\mathcal{S}(d_1, ..., d_k)$ be the space of configurations of curves $(C_1, ..., C_k)$ with degree $C_i = d_i$ and $d_1 + ... + d_k \geq 5$, the problem is to show that

"*there exists a Zariski closed set $\mathcal{F}_k$, of strictly lower dimension, in $\mathcal{S}(d_1, ..., d_k)$ such that* $\mathbf{P}_2(\mathbf{C})$ - *C is Kobayashi-hyperbolic and hyperbolically embedded in* $\mathbf{P}_2(\mathbf{C})$ *for all $C \in \mathcal{S}(d_1, ..., d_k) - \mathcal{F}_k$*".

An indication that the conjecture might be true is the following result of Zaidenberg [Z]:

Theorem (Zaidenberg) *The set of curves of degree d (≥ 5) in* $\mathbf{P}_2(\mathbf{C})$ *with Kobayashi hyperbolic complement (in fact hyperbolically embeddedness) is (non-empty) open, in the classical topology, in the space $\mathcal{S}(d)$.*

Classically it is known that the complement of d (≥ 5) lines in general position in $\mathbf{P}_2(\mathbf{C})$ is Kobayashi-hyperbolic and hyperbolically embedded. Zaidenberg obtained his result by deformation, indeed he showed that small deformation of the complement of d (≥ 5) lines in general position preserves hyperbolicity. For compact manifolds it is a result of Brody [B] that hyperbolicity is preserved under small deformation. Zaidenberg's result can be interpreted as a non-compact (but with compactification) version of Brody's Theorem.

Definition 1 Let C be a curve in $\mathbf{P}_2(\mathbf{C})$ with (reduced) irreducible components $C_1, ..., C_q$. Then C is said to be *set theoretically in general position* if no point is contained in more that 2 irreducible components of C.

Definition 2 Let C be a curve in $\mathbf{P}_2(\mathbf{C})$ with (reduced) irreducible components $C_1, ..., C_q$. Then C is said to be *geometrically in general position* if it is set theoretically in general position and if the components intersect transversally, i.e. the components have no common tangents at the points of intersection.

The following result is to some extent well-known (cf. [DSW]):

Theorem 3 *Let C be a curve in* $\mathbf{P}_2(\mathbf{C})$ *with (reduced) irreducible components $C_1, ..., C_q$. Then*

(*i*) *if $q \geq 5$ and if C is set theoretically in general position, then* $\mathbf{P}_2(\mathbf{C})$ - *C is Kobayashi-hyperbolic and hyperbolically embedded in* $\mathbf{P}_2(\mathbf{C})$;

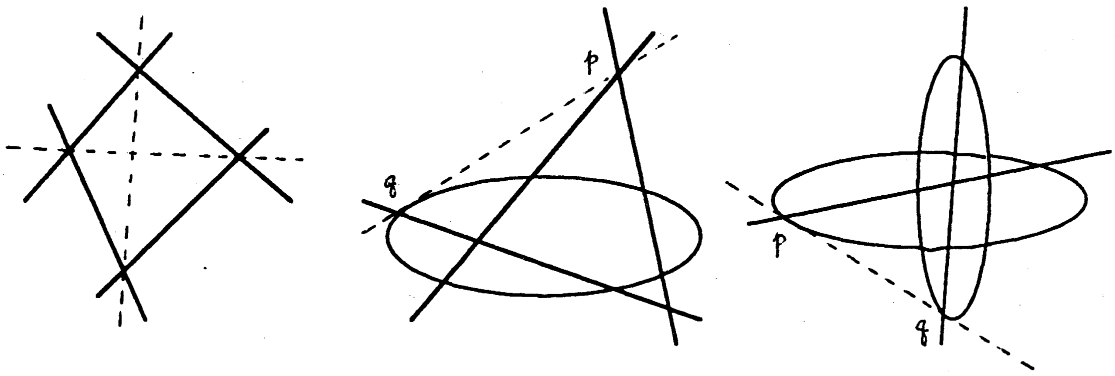
(*ii*) *if* $q = 4$ *and if every irreducible component of C is smooth and geometrically in general position,then* $\mathbf{P}_2(\mathbf{C})$ - *C is Kobayashi-hyperbolic and hyperbolically embedded with* 3 *exceptional cases*:

(*a*) *C is a union of* 4 *lines*:

(*b*) *C consists of* 3 *lines* (L_1, L_2 *and* L_3) *and* 1 *smooth quadric* (Q) *such that the line joining the intersection point p of* L_1, L_2 *and one of the intersection points q of* L_3 *and* Q *is tangent to* Q;

(*c*) *C consists of* 2 *lines* (L_1, L_2) *and* 2 *smooth quadrics* (Q_1 *and* Q_2) *such that the two lines pass through a point p on* Q_1 *and a point q of* Q_2 *where the line joining p and q is a bitangent of C*:

The figures below is helpful in visualizing the 3 exceptional cases:



The dotted lines are isomorphic to $\mathbf{P}_1(\mathbf{C})$ minus two distinct points, hence the complements of the configurations are clearly not hyperbolic.

The proof of Theorem 1 is based on the works of M. Green ([Gm1], [Gm2]). First we give the following definition:

Definition 4 Let C be a divisor in a projective manifold with (reduced) irreducible components $C_1, \ldots, C_q$. Then C is said to be *hyperbolically stratified* if for any partition I and J of $\{1, \ldots, q\}$ (i.e. $I \cap J = \emptyset$, $I \cup J = \{1, \ldots, q\}$) the following condition is satisfied:

$$\cup_{i \in I} C_i - \cup_{j \in J} C_j$$

is Kobayashi-hyperbolic.

It is well-known that Kobayashi-hyperbolicity implies Brody-hyperbolicity and the two concepts are equivalent for compact manifolds; the following lemma of Green [Gm1] gives a sufficient condition for the reverse implication in the case of complements of divisors.

Lemma *Let C be a divisor which is hyperbolically stratified in a projective manifold M. Then V = M - C is Kobayashi hyperbolic and is hyperbolically embedded in M if V is Brody hyperbolic.*

The assumptions in Theorem 1 guarantee that C is hyperbolically stratified in $\mathbf{P}_2(\mathbf{C})$. Thus it is enough to show that $\mathbf{P}_2(\mathbf{C})$ - C is Brody hyperbolic. Let $C_i = \{\mathbf{z} \in \mathbf{P}_2(\mathbf{C}) \mid P_i(\mathbf{z}) = 0\}$ where the P_i's are homogeneous polynomials of the same degree. Using an argument of Green [Gm2] one can show that every entire holomorphic curve in the complement of C is algebraically degenerate. Namely, using the fact that the transcendence degree of the rational function field of $\mathbf{P}_2(\mathbf{C})$ is 2 implies that the rational functions P_1/P_0, P_2/P_0, P_3/P_0 are algebraically dependent. If f is an entire holomorphic curve in the complement of C then $g_i = P_i(f)/P_0(f)$ are non-vanishing entire functions satisfying a polynomial relation. Borel's lemma then implies that the g_i's are algebraically dependent and hence f is also algebraically dependent, i.e. the image f(C) is contained in an algebraic curve of $\mathbf{P}_2(\mathbf{C})$. By a direct argument (cf. § 3 below) one sees that every algebraic curve intersects the components of C in at least 3 distinct points (with 3 exceptional cases listed in the Theorem), this shows that the entire curve f must be a constant.

The case where C has 5 or more components, set theoretically in general position, is easier as every algebraic curve in $\mathbf{P}_2(\mathbf{C})$ must intersects C in at least 3 distinct points and so there is no exceptional cases. In this case the Theorem also follows immediately from a Second Main Theorem of Eremenko and Sodin [ES]:

Theorem (Eremenko-Sodin) *Let* $f : \mathbf{C} \to \mathbf{P}_n(\mathbf{C})$ *be a holomorphic map and let C be a divisor with irreducible components* $C_1, \ldots, C_q$ *which is set theoretically in general position. Let* Q_i *be a defining polynomial (of degree* d_i*) of* C_i. *If* $Q_i(f) \not\equiv 0$ *for all i then*

$$(q - 2n)\, \mathrm{T}(f, r) \leq \sum_{i=1}^{q} d_k{}^{-1} \mathrm{N}(f, C_i, r) + o(\mathrm{T}(f, r)).$$

Indeed, Eremenko-Sodin's Theorem implies that the complement of a divisor D with at least 2n + 1 components, set theoretically in general position, is Brody-hyperbolic. The condition that the components are set theoretically in general position implies that D is hyperbolically stratified hence the complement is Kobayashi-hyperbolic by Green's lemma. However, the analogue in diophantine approximation of the SMT of Eremenko-Sodin is still open:

Conjecture: *Let C be a divisor in* $\mathbf{P}_n(\mathbf{K})$ *where* $\mathbf{K}$ *is an algebraic number field such that the components* $C_1, ..., C_q$ *are set theoretically in general position. Then the estimate*

$$(q - 2n)\, h(\mathbf{x}) \leq \sum_{i=1}^{q} d_k^{-1} N(\mathbf{x}, C_i) + O(1)$$

holds for all but finitely many points $\mathbf{x} \in \mathbf{P}_n(\mathbf{K}) - C$.

The conjecture is open even in the case where n = 2 and C is a curve. On the other hand the analogue of Borel's lemma in diophantine approximations is known (cf. § 2), hence we prefer the proof sketched above.

§ 2 The case of 3 generic quadrics

The complement of 3 quadrics was first studied by Grauert; the hyperbolicity of the complement of 3 generic quadrics is established recently in [DSW].

Theorem 4 *Let* $C_i = \{z \in \mathbf{P}^2(\mathbf{C}) \mid P_i(x) = 0, P_i$ *is a homogeneous polynomial of degree* $2\}$, $(i = 0, 1, 2)$ *be 3 quadrics in generic position. Then* $\mathbf{P}_2(\mathbf{C}) - \cup_{0 \leq i \leq 2} C_i$ *is Kobayashi-hyperbolic and hyperbolically embedded in* $\mathbf{P}_2(\mathbf{C})$.

The generic conditions can be explicitly described as follows. Two quadrics $Q_i = \{z \in \mathbf{P}_2(\mathbf{C}) \mid P_i(x) = 0$, P_i is a homogeneous polynomial of degree $2\}$, i = 0, 1, are said to be in *general position* if they are smooth and the intersection $Q_0 \cap Q_1$ consists of 4 distinct points $\{A_{01}^1, ..., A_{01}^4\}$. (This condition is equivalent to set theoretically in general position and, since the quadrics are smooth, also equivalent to geometrically in general position as defined in the previous section). By joining any two distinct points of these 4 points we get 6 distinct lines. Two distinct lines of these 6 lines is said to be *a pair* if all 4 points of intersection $Q_0 \cap Q_1$ are on these two lines. In these way, these 6 lines are grouped into 3 distinct pairs of lines:

$$\{L_{01}^i \mid 1 \leq i \leq 2\},\ \{J_{01}^i \mid 1 \leq i \leq 2\} \text{ and } \{K_{01}^i \mid 1 \leq i \leq 2\}.$$

Note that the condition of being a pair is equivalent to (say the pair $\{L_{01}^i \mid 1 \leq i \leq 2\}$) the existence of constants a and b such that

$$L_{01}^1 \cup L_{01}^2 = \{\mathbf{x} \in \mathbf{P}_2(\mathbf{C}) \mid aP_0(\mathbf{x}) + bP_1(\mathbf{x}) = 0\}.$$

Simply put, *the pair of lines considered as a quadric is in the linear system of quadrics generated by* Q_0 *and* Q_1.

Three smooth quadrics $Q_i = \{z \in \mathbf{P}_2(\mathbf{C}) \mid P_i(\mathbf{x}) = 0$, P_i is a homogeneous polynomial of degree $2\}$, $(i = 0, 1, 2)$, are said to be *in general position* if any distinct pair is in general position and if the 12 points $Q_0 \cap Q_1 = \{A_{01}^1, \ldots, A_{01}^4\}$, $Q_1 \cap Q_2 = \{A_{12}^1, \ldots, A_{12}^4\}$ and $Q_2 \cap Q_0 = \{A_{20}^1, \ldots, A_{20}^4\}$ are distinct. For 3 quadrics in general position we have 18 distinct lines grouped into 9 pairs :

$$\{L_{01}^i \mid 1 \le i \le 2\},\ \{J_{01}^i \mid 1 \le i \le 2\} \text{ and } \{K_{01}^i \mid 1 \le i \le 2\},$$
$$\{L_{12}^i \mid 1 \le i \le 2\},\ \{J_{12}^i \mid 1 \le i \le 2\} \text{ and } \{K_{12}^i \mid 1 \le i \le 2\},$$
$$\{L_{20}^i \mid 1 \le i \le 2\},\ \{J_{20}^i \mid 1 \le i \le 2\} \text{ and } \{K_{20}^i \mid 1 \le i \le 2\}.$$

Notice that we have 3 linear system of quadrics: $\mathcal{L}_{01} = \{a_{01}P_0 + b_{01}P_1\}$, $\mathcal{L}_{12} = \{a_{12}P_1 + b_{12}P_2\}$ and $\mathcal{L}_{20} = \{a_{20}P_0 + b_{20}P_1\}$ and, the general position assumption implies that if we take two quadrics from different linear systems then the intersection consists of 4 distinct points but cannot contain any of the 12 points $\{A_{01}^1, \ldots, A_{01}^4, A_{12}^1, \ldots, A_{12}^4, A_{20}^1, \ldots, A_{20}^4\}$. This implies, in particular, that only 3 of the 18 lines can pass through each of the 12 points. Each pair of lines determines a point and we have 9 points

$$A_{01} = L_{01}^1 \cap L_{01}^2,\ B_{01} = J_{01}^1 \cap J_{01}^2,\ C_{01} = K_{01}^1 \cap K_{01}^2$$
$$A_{12} = L_{12}^1 \cap L_{12}^2,\ B_{12} = J_{12}^1 \cap J_{12}^2,\ C_{12} = K_{12}^1 \cap K_{12}^2$$
$$A_{20} = L_{20}^1 \cap L_{20}^2,\ B_{20} = J_{20}^1 \cap J_{20}^2,\ C_{20} = K_{20}^1 \cap K_{20}^2.$$

The set of 3 smooth quadrics in general position is clearly Zariski open in the space of 3 quadrics.

Definition 5 Three smooth quadrics are said to be in *generic position* if

(i) they are in general position,

(ii) none of the 18 lines is tangent to any of the 3 quadrics,

(iii) a line through a point of intersection of two of the quadrics is not a tangent of the third quadric and

(iv) the following conditions are satisfied:

$\{A_{01}, B_{01}, C_{01}\}$ is not contained in the 6 lines in the linear system $\mathcal{L}_{12}$ and $\mathcal{L}_{20}$,
$\{A_{12}, B_{12}, C_{12}\}$ is not contained in the 6 lines in the linear system $\mathcal{L}_{20}$ and $\mathcal{L}_{01}$,
$\{A_{20}, B_{20}, C_{20}\}$ is not contained in the 6 lines in the linear system $\mathcal{L}_{01}$ and $\mathcal{L}_{12}$.

The set of 3 smooth quadrics in generic position is Zariski open in the space of 3 quadrics because each of the conditions above is a close condition. We refer the reader to [DSW] for the proof.

We sketch the proof Theorem 4 below and refer the readers to [DSW] for more details. First we make a very important reduction which, in the case of compact manifolds is due to Brody [B]:

Lemma (Brody) *Let $(M\ ds^2)$ be a compact complex hermitian manifold which is not Kobayshi-hyperbolic. Then there exists a holomorphic map $f: \mathbf{C} \to M$ such that*

$$\int_{\Delta_r} f^*(ds^2) \leq O(r^2)$$

where Δ_r is the disk of radius r in **C** *centered at the origin.*

In our situation, even though $M = \mathbf{P}_2(\mathbf{C}) - \cup_{0 \leq i \leq 2} Q_i$ is not compact, it does have a smooth completion $\mathbf{P}_2(\mathbf{C})$. Brody's proof actually applies (because a sequence of holomorphic curves in M can of course be considered as a sequence of holomorphic curves in $\mathbf{P}_2(\mathbf{C})$, hence existence of convergent subsequences is not a problem). First note that the generic condition implies that $\cup_{0 \leq i \leq 2} Q_i$ is hyperbolically stratified (definition 3 in § 1). Thus M is Kobayashi-hyperbolic if and only if it is Brody-hyperbolic. If M were not hyperbolic then there is a non-constant holomorphic curve $f : \mathbf{C} \to M$. We may assume that $f'(0) \neq 0$. Let $f_r(\zeta) = f(r\zeta)$ for all $\zeta \in \Delta$ = unit disk (centered at the origin) in **C**, then $|f_r'(0)| \to \infty$. By Brody's reparametrization, there exists a sequence of holomorphic maps $g_r : \Delta_r \to M$, with $|g_r'(0)| = 1$. Here for simplicity we denote by $|\ \ |$ the norm of the complete metric on $M = \mathbf{P}_2(\mathbf{C}) - \cup_{0 \leq i \leq 2} Q_i$ defined by

$$dt^2 = \frac{1}{|P_0P_1P_2|^{2+\varepsilon}}\, ds^2$$

where ds^2 is the Fubini-Study metric. Since $\mathbf{P}_2(\mathbf{C})$ is compact, a subsequence of $\{g_r\}$ does converge to a holomorphic map $g : \mathbf{C} \to \mathbf{P}_2(\mathbf{C})$. The maps $\{g_r\}$ actually are obtained from $\{f_r\}$ by repara-metrization with $f_r(0) = g_r(0)$, hence f and g actually have the same image (not pointwise but as a set). In particular, g is an entire curve in M. It is clear that the condition $|g_r'(0)| = 1$ implies that

$$\int_{\Delta_r} g^*(dt^2) \leq O(r^2).$$

Since $ds^2 \leq c\,dt^2$ for some constant c, we have

$$T(g, r) = \int_0^r \frac{dt}{t} \int_{\Delta_t} g^*(ds^2) \leq c \int_0^r \frac{dt}{t} \int_{\Delta_r} g^*(dt^2) \leq c\, O(r^2).$$

In the terminology of Nevanlinna Theory the map g is said to be an exponential map of finite order ≤ 2 (finite order 2 for short). In other words, in order to prove Theorem 4 it is sufficient to show that

"*every entire holomorphic curve f*: $\mathbf{C} \to M$ *of finite order is constant*".

Remark (i) The above proof works whenever the manifold has a smooth completion and the "infinity" is hyperbolically stratified. (ii) Note that in the proof above, f and g have the same image, hence f is algebraically non-degenerate if and only if g is algebraically non-degenerate.

As in the case of Theorem 1 in § 1, to show that an entire curve f (of finite order in this case) in M is constant we first use Nevanlinna Theory to show that it is algebraically degenerate and then use the generic condition to show that the entire curve f has to be constant.

Lemma *Let* $\{Q_i \mid 1 \leq i \leq 3\}$ *be* 3 *quadrics in generic position and let* f: $\mathbf{C} \to \mathbf{P}_2(\mathrm{C}) - \cup_{0\leq i\leq 2} Q_i$ *be a holomorphic map. Then f is quadratically degenerate, in fact the image of f must be contained in a quadric in the linear system* $\{a_0Q_0 + a_1Q_1 + a_2Q_2\}$.

Let $Q_i = \{\mathbf{z} \in \mathbf{P}^2(\mathrm{C}) \mid P_i(\mathbf{z}) = 0$ where P_i is a homogeneous polynomial of degree $2\}$. The branching (or ramification) divisor is defined to be:

$$B = \{\mathbf{z} \in \mathbf{P}^2(\mathrm{C}) \mid \det(\partial P_i/\partial z_j\,(\mathbf{z}) = 0\}.$$

The degree of B is 3. If B consists of 3 lines then by the generic condition, each of the line intersects the 3 given quadrics at at least 3 distinct points. If B consists of 1 irreducible (hence smooth) quadrics and 1 line then as before the line intersects the 3 given quadrics at at least 3 distinct points; if the quadric Q intersects the 3 given quadrics at only 2 distinct points then one of them is a point of intersection of 2 of the 3 given quadrics. But any two of the given quadrics intersects transversally and so Q cannot be non-singular at that point. If B is an irreducible cubic intersecting the 3 given quadrics at only 2 points then both points must be points of intersections of the given quadrics; otherwise it intersects one of the given quadric at only one point which is impossible unless B is reducible. Thus, if there is a non-constant holomorphic map from $\mathbf{C}$ into $\mathbf{P}_2(\mathbf{C}) - \cup_{0\leq i\leq 2} Q_i$ the image cannot be contained in the branching divisor B.

We may assume that the map f is of finite order. We shall need the following special cases of a well-known technical lemma of Ahlfors:

Lemma *(i) Let* $f = [\exp p_0, \exp p_1] : \mathbf{C} \to \mathbf{P}_1(\mathbf{C})$ *be a holomorphic of finite order where*

$$p_i(\zeta) = \alpha_i\zeta^n + \text{lower order terms},\ 1 \le i \le 2$$

are polynomials such that at least one of the $\alpha_i \neq 0$. *Then the characteristic function of f satisfies*

$$\lim_{r\to\infty} \frac{T(\phi,r)}{r^n} = \frac{|\alpha_0-\alpha_1|}{\pi}.$$

(ii) Let $\phi = [\exp p_0, \exp p_1, \exp p_2] : \mathbf{C} \to \mathbf{P}_2(\mathbf{C})$ *be a holomorphic of finite order where*

$$p_i(\zeta) = \alpha_i\zeta^n + \text{lower order terms},\ 1 \le i \le 3$$

are polynomials such that at least one of the $\alpha_i \neq 0$. *Then the characteristic function of f satisfies*

$$\lim_{r\to\infty} \frac{T(\phi,r)}{r^n} = \frac{|\alpha_0-\alpha_1| + |\alpha_1-\alpha_2| + |\alpha_2-\alpha_0|}{2\pi}.$$

The main tool of the proof of the theorem is the Second Main Theorem (SMT) of Nevanlinna Theory:

Second Main Theorem *Let* $f : \mathbf{C} \to \mathbf{P}_n(\mathbf{C})$ *be a linearly non-degenerate (i.e. the image* $f(\mathbf{C})$ *is not contained in a hyperplane) holomorphic map. Let* $\{L_i \mid i = 1, ..., q\}$ *be q hyperplanes in general position. Then*

$$(q - n - 1)\, T(f, r) \le \sum_{i=1}^{q} N(f, L_i, r) + o(T(f, r))$$

for all r > 0 and outside an exceptional set E of finite Lebesgue measure. If f is of finite order then the exceptional set E is empty.

Proof of Theorem 4. Suppose that the image of f is not contained in the linear system $\{aQ_0 + bQ_1 + cQ_2\}$. Consider the map $P = [P_0, P_1, P_2] : \mathbf{P}_2(\mathbf{C}) \to \mathbf{P}_2(\mathbf{C})$ where $Q_i = \{P_i = 0\}$. Then P is a morphism because the P_i's have no common zeros. Hence the composite $\phi = P\circ f : \mathbf{C} \to \mathbf{P}_2(\mathbf{C})$ is linearly non-degenerate. Since the P_i's are of degree 2 and P is a morphism, it is well-known and easily verified that

$$(*) \qquad T(\phi, r) = 2T(f, r).$$

Since $P_i\circ f$ is non-vanishing, the map $\phi = P\circ f$ is an entire curve in $\mathbf{P}_2(\mathbf{C}) - \cup_{0\le i\le 2} H_i$ where $H_i = \{[w_0, w_1, w_2] \mid w_i = 0\}$ are the coordinate hyperplanes. Thus ϕ is of the form [exp

p_0, exp p_1, exp p_2] where $p_i(\zeta) = \alpha_i\zeta^n$ + lower order terms, $1 \le i \le 3$, are polynomials such that at least one of the $\alpha_i \neq 0$ (this is so because ϕ is of finite order and all its components are non-vanishing, hence it must be of integral order*). The there maps

$$\phi_{01} = [P_0\circ f, P_1\circ f], \phi_{12} = [P_1\circ f, P_2\circ f] \text{ and } \phi_{20} = [P_2\circ f, P_0\circ f]$$

are holomorphic maps from **C** into $\mathbf{P}_1(\mathbf{C})$. The lemma of Ahlfors implies that

$$(**) \quad 3 \lim_{r\to\infty} T(\phi,r)/r^n = 2\{\lim_{r\to\infty} T(\phi_{01},r)/r^n + \lim_{r\to\infty} T(\phi_{12},r)/r^n + \lim_{r\to\infty} T(\phi_{20},r)/r^n\}.$$

Now we apply the SMT to the 12 lines consisting of (any) two pairs of lines from each of the linear system $\mathcal{L}_1 = \{aQ_0 + bQ_1\}$, $\mathcal{L}_2 = \{aQ_1 + bQ_2\}$ and $\mathcal{L}_3 = \{aQ_2 + bQ_0\}$. These 12 lines, denoted by L_i ($1 \le i \le 12$), are in general position. Hence we have

$$(***) \quad 9\, T(f, r) \le \sum_{i=1}^{12} N(f, L_i, r) + o(T(f, r)).$$

Suppose that $\{L_1, L_2\}$ and $\{L_3, L_4\}$ (resp. $\{L_5, L_6\}$ and $(L_7, L_8\}$; resp. $\{L_9, L_{10}\}$ and $(L_{11}, L_{12}\})$ are the two pairs in $\mathcal{L}_1$ (resp. $\mathcal{L}_2$; resp. $\mathcal{L}_3$). Then there exists constants a and b such that $L_1L_2 = aP_0 + bP_1$. Thus $N(f, L_1, r) + N(f, L_2, r) = N(f, L_1L_2, r) = N(f, aP_0 + bP_1, r)$. On the other hand, $N(f, aP_0 + bP_1, r) = N(\phi_{01}, [a, b], r)$. Now apply the SMT to ϕ_{01} and the 3 points [0, 1], [1, 0] and [a, b], we have

$$\begin{aligned} T(\phi_{01}, r) &\le N(\phi_{01}, [0, 1], r) + N(\phi_{01}, [1, 0], r) + N(\phi_{01}, [a, b], r) + o(T(\phi_{01}, r)) \\ &= N(\phi_{01}, [a, b], r) + o(T(\phi_{01}, r)). \end{aligned}$$

But the First Main Theorem of Nevanlinna Theory gives the reverse inequality

$$N(\phi_{01}, [a, b], r) \le T(\phi_{01}, r) + O(1).$$

Thus we must have

$$\begin{aligned} \lim_{r\to\infty} T(\phi_{01},r)/r^n &= \lim_{r\to\infty} N(\phi_{01},[a,b],r)/r^n = \lim_{r\to\infty} N(f,aP_0+bP_1,r)/r^n \\ &= \lim_{r\to\infty} N(f,L_1,r)/r^n + \lim_{r\to\infty} N(f,L_2,r)/r^n \\ &= \lim_{r\to\infty} N(f,L_1,r)/r^n + \lim_{r\to\infty} N(f,L_2,r)/r^n \end{aligned}$$

Analogously we get the estimate for $T(\phi_{12},r)$ in terms of $N(f,L_5,r)$, $N(f,L_6,r)$ (also $N(f,L_7,r)$ and $N(f,L_8,r)$) resp. $T(\phi_{20},r)$ in terms of $N(f,L_9,r)$, $N(f,L_{10},r)$ (also $N(f,L_{11},r)$ and $N(f,L_{12},r)$). From (*), (**) and (**) we arrive at the following contradiction:

$$\begin{aligned} 9 \lim_{r\to\infty} T(f,r)/r^n &\le \sum_{i=1}^{12} \lim_{r\to\infty} N(f,L_i,r)/r^n \\ &= 2 \lim_{r\to\infty} \{T(\phi_{01},r) + T(\phi_{12},r) + T(\phi_{20},r)\}/r^n \\ &= 4 \lim_{r\to\infty} T(\phi,r)/r^n = 8 \lim_{r\to\infty} T(f,r)/r^n \end{aligned}$$

* This fact an be proved directly in this special case or one can use the general result of S. Mori that an entire curve of finite order and of maximal defect must be of integral order.

Thus the supposition that f is quadratically non-degenerate is wrong and the lemma is verified. QED

The previous lemma implies that the image of f is contained in a quadric of the form $Q = aQ_0 + bQ_1 + cQ_2$. We can show that f must be constant by a direct argument. If the quadric Q is irreducible (hence smooth), we claim that Q intersects the union of the 3 given quadrics in at least 3 distinct points. Suppose the contrary, then Q intersects the 3 given quadrics at only two points p and q and we may assume without loss of generality that $p \in Q_0 \cap Q_1$ and $q \in Q_1 \cap Q_2$ (because it must intersects all 3). If two quadrics intersects transversally then there are 4 points of intersections, thus Q must be tangent to Q_0 (resp. Q_2) at p (resp. q) and it must be tangent to Q_1 at either p or q, say at p for definiteness). But Q_0 and Q_1 intersects transversally, hence Q cannot be tangent to both at p. This contradiction shows that Q must intersects the 3 given quadrics in at least 3 points. If Q is reducible then it consists of a pair of lines (or one double line). But any line must intersects the 3 quadrics in at least 3 distinct points by the generic conditions. This shows that every entire holomorphic curve $f : \mathbf{C} \to M = \mathbf{P}_2(\mathbf{C}) - \cup_{0 \le i \le 2} Q_i$ is constant, i.e. M is Brody-hyperbolic. Theorem 4 now follows from Green's lemma and the fact that $Q = \cup_{0 \le i \le 2} Q_i$ is hyperbolically stratified.

§ 3 Diophantine Geometry

Let **K** be an algebraic number field. Let $\mathcal{S}(d)$ be the space of curves of degree $d \ge 5$ in $\mathbf{P}_2$ defined over **K**. The conjecture corresponding to the conjecture of Kobayashi is the following:

"*There exists a Zariski closed subset $\mathcal{F}$, of strictly lower dimension, of $\mathcal{S}(d)$ such that for all $C \in \mathcal{S}(d) - \mathcal{F}$, $\mathbf{P}_2(\mathbf{K}) - C$ contains at most finitely many* **K**-*integral points*".

More generally, let $\mathcal{S}(d_1, ..., d_k)$ be the space of configurations of curves $(C_1, ..., C_k)$ with degree $C_i = d_i$ and $d_1 + ... + d_k \ge 5$, then

"*There exists a Zariski closed set $\mathcal{F}_k$, of strictly lower dimension, in $\mathcal{S}(d_1, ..., d_k)$ such that for all $C \in \mathcal{S}(d_1, ..., d_k) - \mathcal{F}_k$, $\mathbf{P}_2(\mathbf{K}) - C$ contains at most finitely many* **K**-*integral points*".

First we recall the definition of finiteness of integral points for affine varieties (cf. [V1] and [Si]).

Definition 6 Let X be a non-singular projective variety defined over **K**. Let C be a divisor on X with at worst simple normal crossing singualrities and let V = X - C. Choose an embedding

$$\phi : X \to \mathbf{P}_N(\mathbf{K})$$

such that $\phi(C) = \phi(X) \cap \{[w_0, ..., w_N] \in \mathbf{P}_N(\mathbf{K}) \mid w_0 = 0\}$. Then $\phi(V)$ is embedded as a closed sub-variety of the affine space $\mathbf{K}^N$. The affine variety V = X - C is said to *contain finitely many* **K***-integral points* (or *Mordellic*) if

$$\phi(V) \cap \mathcal{O}_{\mathbf{K}}{}^N$$

is a finite set where $\mathcal{O}_{\mathbf{K}}{}^N$ is the N-fold Cartesian product of the ring of **K**-integers. More generally, let S be a finite set of valuations on **K** containing all the archimedean valuations on **K**. Then the set of S-integral points, denoted $\mathcal{O}_S = \mathcal{O}_{S,\mathbf{K}}$, is defined to be the set of elements x in **K** such that $\upsilon(x) \leq 1$ for all $\upsilon \notin \mathcal{O}_S$. The affine variety V = X - C is said to *contain finitely many S-integral points* if

$$\phi(V) \cap \mathcal{O}_{S,\mathbf{K}}{}^N$$

is a finite set.

We refer the reader to the papers of Silverman [Si] and Vojta [V1] for the proof that the definition of finiteness given above is well-defined (independent of the choice of the embedding ϕ).

Remark 7 For an affine open subset U of V, a set of integral points of V (remember the embedding ϕ) may not be a set of integral points of U (because ϕ is not an embedding of U as a closed subvariety of an affine space). Thus it is possible that U has only finitely many integral points (in some embedding of U as a closed subvariety in an affine space) yet it contains infinitely many integral points of V. For instance U = **K** - {0, 1} is an open subset of V = **K** and obviously contains infinitely many integral points of **K** but **K** - {0, 1} when embedded in $\mathbf{K}^2$ (e.g. by the map $x \to (x, 1/x(x-1))$) has only finitely many integral points (Thue-Siegel). On the other hand, for Zariski closed subset C of V, an embedding of V in $\mathbf{K}^N$ as a closed subvariety also restricted to an embedding of C as a closed subvariety and indeed the set of integral points of C are contained in the set of integral points of V. In particular, V contains infinitely many integral points if we can find a closed subvariety containing infinitely many integral points; conversely, if V contains only finitely many integral points then the same is true for any closed subvariety of V.

We shall give a proof of the Theorem in diophantine geometry corresponding to Theorem 1 in § 1. The proof is based on the lemma of solutions of the unit equation (we include Borel's lemma for comparison), we refer the reader to Vojta [V1], van der Poorten [vdP] and Schlickewei [Schl] for the proof (see also Ru [R] lemma 3.5).

Lemma (*i*) (*Unit-Equation*) *Let* $\{a_i\}$ *be non-zero elements of* $\mathbf{K}$. *Then all but finitely many S-integral solutions* $\{(u_1, ..., u_n) \mid u_i \in \mathcal{O}_{S,\mathbf{K}}\}$ (*more generally,* $u_i \in \Gamma$ *where* Γ *is a finitely generated subgroup of* $\mathbf{K} - \{0\}$) *of the equation*

$$\sum_{i=1}^{n} a_i u_i = 1$$

is contained in a diagonal hyperplane

$$H_I = \{\mathbf{x} \mid \sum_{i \in I} x_i = 0\}$$

where I is a subset of $\{1, ..., n\}$ *consisting of at least* 2 *elements.*

(ii) (*Borel's Lemma*) *Let* $\{a_i\}$ *be non-zero complex numbers. Let* $\{u_i\}$ *be entire non-vanishing functions satisfying the equation*

$$\sum_{i=1}^{n} a_i u_i = 1$$

then the image of the entire curve $f = (u_1, ..., u_n)$ (*where* $\{u_i\}$ *are the entire non-vanishing solutions of the unit equation*) *is contained in a diagonal hyperplane.*

It is well-known that Borel's lemma follows from the standard SMT as stated in § 2. On the other hand, the lemma on the unit equation follows from Roth-Schmidt's Theorem. As mentioned in the introduction, the SMT correspondes to Roth-Schmidt's Theorem in Vojta's dictionary. Indeed in Ru-Wong [RW], Roth-Schmidt's Theorem was reformulated in the form of SMT and, using this reformulation one can easily translated the proof of Borel's lemma (using the SMT) to a proof of the lemma of the unit equation.

We shall use the unit-equation to give a proof of the counterpart of Theorem 1 in § 1.

Theorem 6 *Let C be a curve in* $\mathbf{P}_2$ *defined over an algebraic number field* $\mathbf{K}$. *Let* C_1, ..., C_q *be the* (*reduced*) *irreducible components of C. Then*

(*i*) *if* $q \geq 5$ *and C is set theoretically in general postion then* $\mathbf{P}_2(\mathbf{K}) - C$ *is Mordellic*;

(*ii*) *if* $q = 4$ *and if the components of C are smooth and geometrically in general position then* $\mathbf{P}_2(\mathbf{K}) - C$ *is Mordellic with 3 exceptions*:

(*a*) *C is a union of* 4 *lines*:

(b) C consists of 3 lines (L_1, L_2 and L_3) and 1 smooth quadric (Q) such that the line joiningthe intersection point p of L_1, L_2 and one of the intersection points q of L_3 and Q is tangent to Q;

(c) C consists of 2 lines (L_1, L_2) and 2 smooth quadrics (Q_1 and Q_2) such that the two lines pass through a point p on Q_1 and a point q of Q_2 where the line joining p and q is a bitangent of C:

Proof. We treat case (ii) first as case (i) follow easily from the proof of (ii). If we can show that $\mathbf{P}_2(\mathbf{L})$ - C contains at most finitely many S'-integral points for some finite algebraic extension **L** of **K** and S' extension to **L** of the set of valutions S, then (a priori) $\mathbf{P}_2(\mathbf{K})$ - C contains at most finitely many S-integral points (cf. [V1] lemma 1.4.5). By adjoining the coordinates of the points of intersection if necessary we may assume without loss of generality (and for the convenience of exposition) that **K** already contains these coordinates.

Let $\{C_i \mid 0 \leq i \leq 3\}$ be the components of C. For i = 0, 1, 2, 3 let P_i be a homogeneous polynomial with coefficients in $\mathcal{O}_{S,\mathbf{K}}$ and deg P_i = d (for all i) such that $C_i = \{\mathbf{z} \in \mathbf{P}_2(\mathbf{K}) \mid P_i(\mathbf{z}) = 0\}$. Since transcendence degree of $\mathbf{P}_2$ is 2, the rational functions P_1/P_0, P_2/P_0, P_3/P_0 are algebraically dependent. Hence there exists a polynomial R such that

$$R(P_1/P_0, P_2/P_0, P_3/P_0) \equiv 0$$

where we may assume that the coefficients of R are in **K**. Thus we have

$$\sum_{i=1}^{n} a_i R_i/R_0 = 1$$

where $a_i \neq 0$ and each R_i is a monomial in $\{P_0, P_1, P_2, P_3\}$. Let $\mathcal{J}$ be the set of S-integral points of $\mathbf{P}_2(\mathbf{K})$ - C. Since a_iR_i/R_0 is a regular funtion on outside the curve $\mathbf{P}_2(\mathbf{K})$ - C, there exists a ∈ **K** such that $aa_iR_i/R_0(x) \in \mathcal{O}_S$ for all $\mathbf{x} \in \mathcal{J}$ and for all $1 \leq i \leq n$ (cf. [V1] lemma 1.4.6, see also [R] § 3). The lemma of the unit-equation implies that the solutions $\{(R_1/R_0(x), \ldots, R_n/R_0(x)) \mid \mathbf{x} \in \mathcal{J}\}$ of the equation

$$\sum_{i=1}^{n} a_i R_i/R_0(x) = 1$$

is contained in a diagonal hyperplane. This is equivalent to the condition that the set of S-integral points $\mathcal{J}$ of $\mathbf{P}_2(\mathbf{K})$ - C is contained in an algebraic curve D in $\mathbf{P}_2(\mathbf{K})$.

Let D' be any irreducible component of D. Then $D' \cap (\cup C_i)$ contains at least 2 distinct points because C is in general position and D' must intersect every component. If C onsists of 4 lines (exceptional case (a)) then it is possible that D' intersects C in exactly 2 points (for instance D' is the line joining the point of intersection p of C_1,C_2 and the point of intersection q of C_3, C_4). In this case we cannot conclude that $\mathbf{P}_2(\mathbf{K})$ - C has only finitely many S-integral points (because a rational curve minus 2 points contains infinitely many integral points).

We now assume that at least one of the component of C (say C_4) is a smooth quadric. Suppose that D' intersects C in 2 distinct points then these points must be intersection points of the components of C, say $p \in C_1 \cap C_2$ and $q \in C_3 \cap C_4$ (this is so because D' must intersect each component of C). If D' have distinct tangents at the point p then $\pi^{-1}(p)$ consists of two distinct points where $\pi : D'' \to D'$ is the normalization (desingularization) of D'. Thus $\pi^{-1}(p) \cup \pi^{-1}(q)$ consists of at least 3 points so that $D'' - \pi^{-1}(p) \cup \pi^{-1}(q)$ contains at most finitely many S-integral points by the Theorem of Thue and Siegel. It follows that D' contains at most finitely many integral points of $\mathbf{P}_2(\mathbf{K}) - C$ (cf. [V1] theorem 1.4.11) and we are done in this case. Thus we may assume that D' have no distinct tangents at the point p. Since C is geometrically in general position and all of its components are smooth, D' cannot be tangent to both C_1 and C_2 at p. Say D' is not tangent to C_1 at p. Then D' must intersect C_1 at a point r other than p (in which case we are done because p, q, r are 3 distinct points and any curve with 3 points deleted contains at most finitely integral points) unless both C_1 and D' are lines. If C_2 is not a line (hence a smooth quadric) then D' must be tangent to C_2 at p otherwise D' would intersect C_2 at a point r other than p and we are done. Thus we have two cases to consider: (b) C_2 is a line or (c) C_2 is a smooth quadric and D' is tangent to C_2 at p. In either case we apply the preceding argument to the point $q \in C_3 \cap C_4$. Since C_4 is a smooth quadric we must have the situation where C_3 is a line and D' is tangent to C_4 at q. Thus we have the two exceptional cases:

(b) C_1, C_2 and C_3 are lines and C_4 is a smooth quadric and D' intersects C at the point $p \in C_1 \cap C_2$ and at the point $q \in C_3 \cap C_4$ and D' is tangent to C_4 at q;

(c) C_1, C_3 are lines and C_2, C_4 are smooth quadrics, D' intersects C at the point $p \in C_1 \cap C_2$ and at the point $q \in C_3 \cap C_4$ and D' is tangent to C_2 at p and also to C_4 at q.

In all other cases every irreducible component of D intersects C in at least 3 points and hence can only contain finitely many S-integral points. QED

References

[B] Brody, R., *Compact manifolds and hyperbolicity*, Tran. Amer. Math. Soc., **235** (1978), 213-219

[DSW] Dethloff, G., Schumacher, G. and Wong, P.M., *Hyperbolicity of Complement of 3 Generic Quadrics*, to appear in Göttengensis

[ES] Eremenko, A.E. and Sodin, M.L., *The value distribution of meromorphic functions and meromorphic curves from the point of view of potential theory*, St. Petersberg math. J. **3** (1992), 109-136

[F] Faltings, G., *Diophantine approximation on abelian varieties*, Ann. of Math. **133** (1991), 549-576

[Gm1] Green, M., *Holomorphic maps into complex projective space omitting hyperplanes*, Tran. Amer. Math. Soc., **169** (1972), 89-103

[Gm2] Green, M., *Some Picard theorems for holomorphic maps to algebraic varieties*, Amer. J. Math., **97** (1975), 43-75

[K] Kobayashi, S., *Hyperbolic Manifolds and Holomorphic Mappings*, Marcel Dekker, New York, (1970)

[L1] Lang, S., *Hyperbolic and diophantine analysis*, Bull. Amer. Math. Soc., **14** (1986), 159-205

[L2] Lang, S., *Number Theory III*, Ency. Math. Sc., vol **60**, Springer-Verlag, Berlin (1991)

[O] Osgood, C. F., *Sometimes effective Thue-Siegel Roth-Schmidt Nevanlinna bounds, or better*, J. Number Theory, **21** (1985), 347-389

[R] Ru, M., *The Geometric and Arithmetic Aspects of* $\mathbf{P}^n$ *minus Hyperplanes*, preprint (1992)

[RW] Ru, M. and Wong, P.M., *Integral points of* $\mathbf{P}^n(K)$ - $\{2n + 1$ *hyperplanes in general position*$\}$, Invent. Math., **106** (1991), 195-216

[Sc] Schmidt, W.M., *Diophantine Approximations*, Springer Lecture Notes **785**, Berlin (1980)

[Schl] Schlickewei, H.P., *Über die diophantische Gleichung* $X_1 + ... + X_n = 0$, Acta Arith. **33** (1977), 183-185

[Si] Silverman, J.H., "*Integral Points on Curves and Surfaces*" in Proc. Journees Arithmetiques (conference held in Ulm), Springer Lecture Notes **1380**, 202-241, Springer-Verlag, Berlin (1989)

[vdP] van der Poorten, A.J., *The growth conditions for recurrences*, unpublished (1982)

[V1] Vojta, P., *Diophantine approximations and value distribution theory*, Springer Lecture Notes **1239** (1987)

[V2] Vojta, P., *Siegel's theorem in compact case*, Ann. of Math. **133** (1991), 509-548

[Z] Zaidenberg, M., *Stability of hyperbolic imbeddedness and construction of examples*, Math. USSR Sbornik **63** (1989), 351-361