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Some characterizations of generalized complex ellipsoids in $\mathbb{C}^n$ and related problems

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Let D be a domain in $\mathbb{C}^n$, Aut(D) the group of all biholomorphic transformations of D and x a point on the boundary ∂D of D. Assume that there is a local holomorphic peaking function for D at x, and that x is an accumulation point of an Aut(D)-orbit. Can we then determine the global structure of D from the local shape of the boundary ∂D near x ? There are several papers closely related to this problem [1, 2, 6, 8, 9, 10, 12, 13, 14, 17, 18]. For instance, in 1977 it was shown by Wong [18] that if D is a bounded strictly pseudoconvex domain in $\mathbb{C}^n$ with C^∞-smooth boundary and Aut(D) is noncompact, then D is biholomorphically equivalent to the open unit B^n in $\mathbb{C}^n$. It was later extended by Rosay to the following :

Theorem R (Rosay [17]). Let D be a bounded domain in $\mathbb{C}^n$ with a C^2-smooth strictly pseudoconvex boundary point $x \in \partial D$. Assume that there exist a point $b \in D$ and a sequence $\{\varphi_\nu\} \subset \mathrm{Aut}(D)$ such that $\varphi_\nu(b) \to x$ as $\nu \to \infty$. Then D is biholomorphically equivalent to the unit ball B^n in $\mathbb{C}^n$.

Here it seems natural to ask what happens when the point x is a weakly pseudoconvex boundary point of D. The first result in this direction was the following theorem of Greene and Krantz [8], which gives a characterization of the pseudoconvex domain

$$E(m) = \{ z \in \mathbb{C}^n ; \sum_{i=1}^{n-1} |z_i|^2 + |z_n|^{2m} < 1 \}, \quad 1 < m \in \mathbb{Z} :$$

Theorem G-K (Greene and Krantz [8]). Let D be a bounded domain in $\mathbb{C}^n$ with a C^{n+1}-smooth boundary such that $x = (1, 0, \ldots, 0) \in \partial D$. Assume that there are neighborhoods U, V of x in $\mathbb{C}^n$ such that, up to a local biholomorphism, $U \cap D$ and $V \cap E(m)$ coincide. Assume further that there exists a point $b \in D$ and a sequence $\{\varphi_\nu\} \subset \mathrm{Aut}(D)$ such that $\varphi_\nu(b) \to x$ as $\nu \to \infty$. Then D is biholomorphically equivalent to the model domain $E(m)$.

Note that the point $x = (1, 0, \ldots, 0)$ is a C^ω-smooth weakly pseudoconvex boundary point of D. Their proof was based on normal families argument, combined with some complicated uniform estimates for the $\bar\partial$-equation on D. Without using $\bar\partial$ - methods, it was later extended by Kodama [13, 14] to the more general domain

$$E(p_1, \ldots, p_n) = \{ z \in \mathbb{C}^n ; \sum_{i=1}^{n} |z_i|^{2p_i} < 1 \}, \quad 0 < p_i \in \mathbb{R},$$

as follows :

Theorem K (Kodama [13, 14]). Let D be a bounded domain in $\mathbb{C}^n$ $(n > 1)$ with a point $x = (x_1, \ldots, x_n) \in \partial D$. After renumbering the coordinates if necessary, we assume that the following three conditions are satisfied :

(1) There exist an integer $k \geq 0$, real numbers p_i with $0 < p_i \neq 1$ $(k+1 \leq i \leq n)$ and an open neighborhood U of x in $\mathbb{C}^n$ such that

(i) $x \in \partial E(1,\dots,1,p_{k+1},\dots,p_n)$ and

(ii) $D \cap U = E(1,\dots,1,p_{k+1},\dots,p_n) \cap U$;

here it is understood that $E(1,\dots,1,p_{k+1},\dots,p_n) = B^n$ if $k=n$.

(2) $\#\{ i \in \mathbb{Z} ; x_i \neq 0, 1 \le i \le n \} = j$, where # denotes the number of elements contained in the set.

(3) There exist a point $b \in D$ and a sequence $\{\varphi_\nu\} \subset \mathrm{Aut}(D)$ such that $\varphi_\nu(b) \to x$ as $\nu \to \infty$.

Then we have $1 \le j \le k$ and $D = E(1,\dots,1,p_{k+1},\dots,p_n)$ as sets ; hence x is necessarily of the form $x = (x_1,\dots,x_k,0,\dots,0)$.

Here, it shoud be remarked that a glance at the proof of Theorem K tells us Theorem G-K follows immediately from ours.

The main purpose of this short note is to announce the following Theorems I, II and III due to Kodama, Krantz and Ma [15]. From these we see that the analogue of Theorem K is still valid for the generalized complex ellipsoid

$$E(n;n_1,\dots,n_s;p_1,\dots,p_s) = \{ (z_1,\dots,z_s) \in \mathbb{C}^{n_1} \times \cdots \times \mathbb{C}^{n_s} ; \sum_{i=1}^{s} \|z_i\|^{2p_i} < 1 \}$$

in $\mathbb{C}^n = \mathbb{C}^{n_1} \times \cdots \times \mathbb{C}^{n_s}$, where $p_1,\dots, p_s$ are positive real numbers and $n_1,\dots, n_s$ are positive integers with $n = n_1 + \quad + n_s$. In general this domain is not geometrically convex, and its boundary is not smooth. For convenience and without no loss of generality, in the following we will always assume $p_1 = 1$, $p_2,\dots,p_s \neq 1$, $n_2,\dots,n_s > 0$, and write a generalized complex ellipsoid in the form $E(n;n_1,n_2,\dots,n_s;1,p_2,\dots,p_s)$. Here it is understood that 1 does not appear if $n_1 = 0$, and also this domain is the unit ball B^n if $s = 1$. Under this assumption, we can prove the following :

Theorem I. Let D be a bounded domain in $\mathbb{C}^n$ and E a generalized complex ellipsoid in $\mathbb{C}^n$. Let $x \in \partial D$ and $\tilde{x} \in \partial E$. Assume that the following two conditions are satisfied :

(1) There exist open neighborhoods Q of x and $\tilde{Q}$ of $\tilde{x}$ in $\mathbb{C}^n$ and a biholomorphic mapping $\Gamma : Q \longrightarrow \tilde{Q}$ such that $\Gamma(x) = \tilde{x}$ and $\Gamma(D \cap Q) = E \cap \tilde{Q}$.

(2) There exist points $b \in D$, $\tilde{b} \in E$ and sequences $\{\varphi_\nu\} \subset \mathrm{Aut}(D)$, $\{\tilde{\varphi}_\nu\} \subset \mathrm{Aut}(E)$ such that $\varphi_\nu(b) \to x$, and $\tilde{\varphi}_\nu(\tilde{b}) \to \tilde{x}$ as $\nu \to \infty$.

Then D is biholomorphically equivalent to E.

In the special case where all the p_i are integers, it follows that E is a geometrically convex bounded domain with C^ω-smooth boundary; hence, all of its boundary points are of finite type in the sense of D'Angelo [3]. Thus, our Theorem I is an immediate consequence of Kim [10]. If all $n_i = 1$, this reduces to Theorem K above. Moreover, under the strong assumption that one can choose Γ, b, $\tilde{b}$, $\{\varphi_\nu\}$, $\{\tilde{\varphi}_\nu\}$ in such a way that $\Gamma(\varphi_\nu(b)) = \tilde{\varphi}_\nu(\tilde{b})$ for all sufficiently large ν, our theorem may be proved by the arguments in a recent paper by Lin and Wong [16].

Before proceeding, we need to introduce some terminology.

Definition 1. Let

$$E_1 = E(n; n_1, n_2, \ldots, n_s; 1, p_2, \ldots, p_s),$$
$$E_2 = E(n; m_1, m_2, \ldots, m_t; 1, q_2, \ldots, q_t)$$

be two generalized complex ellipsoids of the same dimension. Then we say that E_1 precedes E_2 if $s \leq t$ and there is a permutation σ of the $t-1$ numbers $\{2, \ldots, t\}$ such that $(p_j, n_j) = (q_{\sigma(j)}, m_{\sigma(j)})$ for $j = 2, \ldots, s$.

Definition 2. Let M and N be complex manifolds. We say that N exhausts M, or M is exhausted by N, if for every compact subset K of M, there is an injective holomorphic mapping $f_K : N \longrightarrow M$ such that $f_K(N) \supset K$.

It would be natural to ask :

(*) If N exhausts M, then how is M related to N ?

In this connection, Fornaess and Stout [4] showed that M has to be biholomorphically equivalent to N, provided that M is hyperbolic in the sense of Kobayashi [11] and N is the unit ball or the unit polydisk. This result was extended by Fornaess and Sibony [5] to the case where N is a hyperbolic manifold with compact quotient $N/\mathrm{Aut}(N)$. On the other hand, Fridman [7] proved that if a complete hyperbolic manifold M can be exhausted by a bounded C^3-smooth strictly pseudoconvex domain D in $\mathbb{C}^n$, then M is biholomorphically equivalent either to D or to B^n. In view of these results, we shall restrict ourselves to the case where M is hyperbolic and we want to ask the following : What happens when N is a generalized complex ellipsoid in (*) ? The answer is remarkable : Generalized complex ellipsoids can exhaust only generalized complex ellipsoids (possibly different). More precisely, we have :

Theorem II. Let M be a hyperbolic manifold of complex dimension

n. Assume that M can be exhausted by a generalized complex ellipsoid E. Then M is biholomorphically equivalent to a generalized complex ellipsoid $\tilde{E}$ that precedes E.

In the special case $E = E(n;n;1) = B^n$, this is just the result of Fornaess and Stout [4] mentioned above.

We remark that in general the property that N exhausts M does not imply that M and N are biholomorphically equivalent. Indeed, we can prove the following Theorem III that gives an explicit example in the class of generalized complex ellipsoids. This theorem also indicates that in Theorem II the two ellipsoids E and $\tilde{E}$ are not biholomorphically equivalent, in general.

Theorem III. For each $p > 0$, the unit ball B^n in $\mathbb{C}^n$ can be exhausted by the generalized complex ellipsoid

$$E(n;n-1,1;1,p) = \{ (z,w) \in \mathbb{C}^{n-1} \times \mathbb{C} \ ; \ \|z\|^2 + |w|^{2p} < 1 \ \}.$$

For the complete proofs of Theorem I, II and III, see our joint paper Kodama, Krantz and Ma [15].

We finish this note by some problems concerning the characterization of generalized complex ellipsoids. Recall that, in Theorem I, we assumed the following :

(**) There exist a point $\tilde{b} \in E$ and a sequence $\{\tilde{\varphi}_\nu\} \subset \mathrm{Aut}(E)$ such that $\tilde{\varphi}_\nu(\tilde{b}) \to \tilde{x}$ as $\nu \to \infty$.

However, in the case of $n_i = 1$ for all $i = 1,\ldots,s$, we do not need to assume the condition (**). In fact, by Theorem K this follows automatically from the other. In view of this fact, we would like to ask the following :

Problem 1. In Theorem I, can we drop the condition (**)? ;namely, let D be a bounded domain in $\mathbb{C}^n$ and E a generalized complex ellipsoid in $\mathbb{C}^n$. Assume that there exist a common boundary point $x \in \partial D \cap \partial E$ and its open neighborhood U in $\mathbb{C}^n$ such that $D \cap U = E \cap U$. Assume further that there are a point $b \in D$ and a sequence $\{\varphi_\nu\} \subset \mathrm{Aut}(D)$ such that $\varphi_\nu(b) \to x$ as $\nu \to \infty$. Can we then choose a point $\tilde{b} \in E$ and a sequence $\{\tilde{\varphi}_\nu\} \subset \mathrm{Aut}(E)$ such that $\tilde{\varphi}_\nu(\tilde{b}) \to x$ as $\nu \to \infty$?

As the proof of Theorem I shows, we can prove at least the following Proposition, which is a natural generalization of Berteloot [2]:

Proposition. Let D be a bounded domain in $\mathbb{C}^n$ and E a generalized complex ellipsoid in $\mathbb{C}^n$. Let $x \in \partial D$ and $\tilde{x} \in \partial E$. Assume that the following two conditions are satisfied :

(1) There exist open neighborhoods Q of x and $\tilde{Q}$ of $\tilde{x}$ in $\mathbb{C}^n$ and a biholomorphic mapping $\Gamma : Q \longrightarrow \tilde{Q}$ such that $\Gamma(D \cap Q) = E \cap \tilde{Q}$, $\Gamma(x) = \tilde{x}$.

(2) There exist a point $b \in D$ and a sequence $\{\varphi_\nu\} \subset \mathrm{Aut}(D)$ such that $\varphi_\nu(b) \to x$ as $\nu \to \infty$.

Then D is biholomorphically equivalent to a generalized complex ellipsoid $\tilde{E}$ that precedes E.

At this moment, we have not succeeded in proving that $\tilde{E}$ is biholomorphically equivalent to E. Even in the case where ∂D is C^ω-

smooth near x (and hence, so is ∂E near $\tilde{x}$), it seems difficult to verify this.

Next, recall the proof of Theorem R in the case of C^2-smooth strictly pseudoconvex boundary points x. Then one can see that the following fact is a crucial point for the proof : Up to a local biholomorphism, the boundary ∂D of D and the boundary ∂B^n of the unit ball B^n are very close in the sense that the defining functions of D and B^n have the same Taylor expansions up to order 2 at x. So it is natural to attempt to prove an analogue of Theorem I in the case where D and E are very close; so that we ask the following :

Problem 2. Does Theorem I remain true in the case where, up to a local biholomorphism, ∂D near x and ∂E near $\tilde{x}$ are very close "in some sense" ? (Recall that the boundaries ∂D and ∂E are not necessarily smooth in our case.)

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