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Order of a holomorphic curve with maximal deficiency sum for moving targets

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1. Introduction. Nevanlinna's defect relation remains valid for mutually distinct meromorphic target functions $g_1,\ldots,g_q$ on $\mathbb{C}$ which grow more slowly than a given meromorphic function f on $\mathbb{C}$ (slow moving targets), that is, the Nevanlinna characteristic functions of those functions satisfy $T_{g_j}(r) = o(T_f(r))$ as $r \longrightarrow \infty$, $(j=1,\ldots,q)$. (See N.Steinmetz [7]) On the other hand, in higher dimensional case, M.Ru - W.Stoll [4] [5] and Shirosaki [6] proved a defect relation with defect bound $n+1$ for slow moving targets to the case of nondegenerate holomorphic curve.

While, A.Edrei - W.H.J.Fuchs [1] proved that a finite order meromorphic function with $\delta(\infty, f) = 1$ is of positive integral order and regular growth if it has the maximal deficiency sum 2.

In this note, we investigate the order of holomorphic curve in some class with maximal deficiency sum for slow moving targets.

Let $f : \mathbb{C} \longrightarrow \mathbb{P}^n(\mathbb{C})$ be a finite order and nondegenerate holomorphic curve and $\tilde{f} := (f_0,\ldots,f_n)$ a reduced representation of f. Let g^j $(j=0,\ldots,q)$ be slowly moving targets for f in general position, We show that if there exists an f_{i_0} such that

$N_1(r, 1/f_{i_0}) = o(T_f(r))$ and $T_1(r, f_{i_j}/f_{i_0}) = o(T_f(r))$ $(j=0,\ldots,n-1)$

and $\sum_{j=0}^{q} \delta(f, g^j) = n + 1$, then f is of positive integral order and ofregular growth . In the case $n = 1$, the theorem is sharp by F.Nevanlinna's example. But in the case $n > 1$, I could not find an example to show the shrapness of the theorem.

2. Preliminaries and statesment of result.

Let $f : \mathbb{C} \longrightarrow \mathbb{P}^n(\mathbb{C})$ be a holomorphic mapping of $\mathbb{C}$ into $\mathbb{P}^n(\mathbb{C})$, and $\tilde{f} := (f_0, \ldots, f_n) : \mathbb{C} \longrightarrow \mathbb{C}^{n+1} - \{0\}$ a reduced representation of f. Set $\|f(z)\|^2 := \sum_{i=1}^{n} |f_i(z)|^2$.

We define the characteristic function $T_f(r)$ of f by

$$T_f(r) := \frac{1}{2\pi} \int_0^{2\pi} \log \|f(re^{i\theta})\| \, d\theta.$$

We define the order λ_f and the lower order μ_f of f as follows:

$$\lambda_f := \limsup_{r \longrightarrow \infty} \frac{\log T_f(r)}{\log r} \quad \text{and} \quad \mu_f := \liminf_{r \longrightarrow \infty} \frac{\log T_f(r)}{\log r}.$$

We say that f is of regular growth if $\lambda_f = \mu_f$.

For a meromorphic function $\varphi(z) : \mathbb{C} \longrightarrow \mathbb{C} \cup \{\infty\}$, its proximity function $m_1(r, \varphi)$, counting function $N_1(r, \varphi)$ and the characteristic function $T_1(r, \varphi)$ are defined by

$$m_1(r, \varphi) := \frac{1}{2\pi} \int_0^{2\pi} \log^+ |\varphi(re^{i\theta})| \, d\theta, \qquad N_1(r, \varphi) := \int_0^r n_1(t, \varphi) \, dt/t$$

and

$$T_1(r, \varphi) := m_1(r, \varphi) + N_1(r, \varphi),$$

respectively, where $n_1(t, \varphi)$ is the number of poles of φ in $|z| < t$

counting multiplicities and $\log^+ x := \max(\log x, 0)$.

Let $\mathfrak{G}$ be a finite set of holomorphic mappings $g : \mathbb{C} \longrightarrow \mathbb{P}^n(\mathbb{C})^*$ with $n+2 \le q := {}^{\#}\mathfrak{G} < \infty$. Here we say that g is a moving target.

Assume that

(A 1) $\mathfrak{G}$ is in general position. (cf. [5])

This means that at least one point $z_0 \in \mathbb{C}$ exists, such that ${}^{\#}\mathfrak{G}(z_0) = q$ and $\mathfrak{G}(z_0)$ is in general position, that is,

$$\det \left(g_\ell^{j_k}\right)_{0 \le k, \ell \le n} \ne 0, \quad \text{where } \mathfrak{G} = \{g^j : \mathbb{C} \longrightarrow \mathbb{P}^n(\mathbb{C}), \quad (j=0,\dots,q)\}$$

and $(g^j_0, \dots, g^j_n)$ a reduced representation of g^j.

Let $(f_0, \dots, f_n)$ and $(g_0, \dots, g_n)$ be reduced representations of f and g, respectively. Define $N_{f,g}(r) := N_1(r, 1/h)$ and

$$m_{f,g}(r) := \frac{1}{2\pi}\int_0^{2\pi} \log \frac{\|f(re^{i\theta})\|\|g(re^{i\theta})\|}{|h(re^{i\theta})|}\, d\theta \ \ge 0,$$

where $h(z) := \sum_{i=0}^{n} f_i(z)\, g_i(z) \not\equiv 0$. Then it is known that

$$T_f(r) + T_g(r) = N_{f,g}(r) + m_{f,g}(r) + O(1) \quad (r \longrightarrow \infty).$$

If f or g is nonconstant, then $T_f(r) + T_g(r) \longrightarrow \infty$ as $r \longrightarrow \infty$ and the defect $\delta(f,g)$ for the moving target g is defined by

$$0 \le \delta(f,g) := \liminf_{r \to \infty} \frac{m_{f,g}(r)}{T_f(r)+T_g(r)} = 1 - \limsup_{r \to \infty} \frac{N_{f,g}(r)}{T_f(r)+T_g(r)} \le 1.$$

Assume that

(A 2) $T_{g^j}(r) = o(T_f(r)) \quad (r \longrightarrow \infty)$, for all $g^j \in \mathfrak{G}$.

Then the moving target g^j is said to grow more slowly than f, and the defect $\delta(f, g^j)$ is written as

$$\delta(f,\ g^j) = \liminf_{r \to \infty} m_{f,g^j}(r)/T_f(r) = 1 - \limsup_{r \to \infty} N_{f,g^j}(r)/T_f(r).$$

Let $\mathcal{R}_{\mathfrak{G}}$ be the field generated by $\mathfrak{G}$ over $\mathbb{C}$, that is, the field generated by elements of the form $\zeta_{ji} = g_i^j / g_0^j$, $(i=0,\ldots,n;$ $j=0,\ldots,q)$ over $\mathbb{C}$, where $(g_0^j,\ldots,g_n^j)$ is a reduced representation of g^j. By assumption (A 2), $T_\psi(r) = o(T_f(r))$ as $r \longrightarrow \infty$, for any $\psi \in \mathcal{R}_{\mathfrak{G}}$. Assume that

(A 3) f is linearly nondegenerate over $\mathcal{R}_{\mathfrak{G}}$,

that is, $f_0,\ldots,f_n$ are linearly independent over $\mathcal{R}_{\mathfrak{G}}$. Then we have the following:

<u>Theorem</u>. Let $f : \mathbb{C} \longrightarrow \mathbb{P}^n(\mathbb{C})$ be a finite order holomorphic curve and linearly nondegenerate over $\mathcal{R}_{\mathfrak{G}}$, and $(f_0,\ldots,f_n)$ a reduced representation of f. Let $\mathfrak{G}$ be a finite set of slowly growing moving targets as above. Assume that there exists an $f_{i_0} \equiv 0$ such that $N_1(r,1/f_{i_0}) = o(T_f(r))$ and $T_1(r,f_{i_j}/f_{i_0}) = o(T_f(r))$ $(r \longrightarrow \infty)$, $(j=0,\ldots,n-1)$. Then if $\sum_{j=0}^{q} \delta(f,\ g^j) = n+1$, f is of positive integral order and of regular growth.

3. Proof of the theorem.

We may assume that $i_j = j$, $(j=0,\ldots,n)$. We may assume that $g_n^j \equiv 0$ $(j=0,\ldots,n-1)$, by adding some constant targets in general position, if necessary, and also may assume that $g_0^j \equiv 0$ $(j=0,\ldots,q)$, by a unitary transformation of $\mathbb{P}^n(\mathbb{C})$. Set

$\zeta_{jk} = g_k^j/g_0^j$, $(j=0,\ldots,q;\ k=0,\ldots,n)$, so $\zeta_{j0} = 1$ $(j=0,\ldots,q)$,

and

$$h_j = g_0^j \cdot f_0 + \cdots + g_n^j \cdot f_n, \qquad (j=0,\ldots,q).$$

Then the assumption (A 3) yields $h_j(z) \neq 0$. Let $\mathcal{L}(p)$ be the vector space over $\mathbb{C}$ spanned by the set

$$\{ \prod_{\substack{0 \le j \le q \\ 0 \le i \le n}} \xi_{ji}^{p_{ji}} \mid p_{ji} \text{ are non-negative integers with } \sum_{\substack{0 \le j \le q \\ 0 \le i \le n}} p_{ji} = p\},$$

and $\{b_1,\ldots,b_t\}$ be a basis of $\mathcal{L}(p+1)$ such that $\{b_1,\ldots,b_s\}$ a basis of $\mathcal{L}(p)$ $(s \le t)$. Then $\mathcal{L}(p) \subset \mathcal{L}(p+1)$ and $\{b_j f_{\alpha_k}\}$ $(j=1,\ldots,t;\ k=0,\ldots,n)$ are linearly independent over $\mathbb{C}$. Let

$$F_j = h_j/g_0^j = \sum_{i=0}^{n} \xi_{ji} f_i, \qquad (j=0,\ldots,q).$$

Then $b_j F_{\alpha_k}$ $(j=1,\ldots,s;\ k=0,\ldots,n)$ are linearly independent over $\mathbb{C}$. Since $b_j F_k$ $(j=1,\ldots,s; k=0,\ldots,n)$ are written as linear combination of $b_j f_k$ $(j=1,\ldots,t; k=0,\ldots,n)$ over $\mathbb{C}$, there exsist $\beta_{mj}^{k\ell} \in \mathbb{C}$ and $C \in GL((n+1)\cdot t, \mathbb{C})$ such that

$$\left(b_j F_k\ (1 \le j \le s,\ 0 \le k \le n);\ h_{mj} (s+1 \le j \le t,\ 0 \le m \le n)\right) = \left(b_j f_k\ (1 \le j \le t,\ 0 \le k \le n)\right) \cdot C,$$

where $h_{mj} = \sum_{\substack{1 \le k \le t \\ 0 \le \ell \le n}} \beta_{mj}^{k\ell}\, b_k f_\ell$, $(j=s+1,\ldots,t;\ m=0,\ldots,n)$. Then we have

$$W\left(b_j F_k\ \ (1 \le j \le s,\ 0 \le k \le n);\ h_{mj}\ \ (s+1 \le j \le t,\ 0 \le m \le n)\right)$$
$$= W\left(b_j F_k\ \ (1 \le j \le t,\ 0 \le k \le n)\right) \cdot \det C.$$

Let $\alpha = (\alpha_0,\ldots,\alpha_n)$ $(\alpha_k \in \{0,\ldots,q\})$ be multi-indices. Put

$$W_\alpha := W\left(b_j F_{\alpha_k}\ \ (1 \le j \le s,\ 0 \le k \le n);\ h_{m_j}^{\alpha}\ \ (s+1 \le j \le t,\ 0 \le m \le n)\right)$$

and

$$W := W\left(b_j f_k\ \ (1 \le j \le t,\ 0 \le k \le n)\right) \neq 0.$$

Then from a similar argument as above by using $F_{\alpha_0},\dots,F_{\alpha_n}$ instead of $F_0,\dots,F_n$, we have $W_\alpha = C_\alpha W$, where $C_\alpha \in GL((n+1)\cdot t, \mathbb{C})$.

For any fixed $z \in \mathbb{C}$, we arrange F_{j_k}'s in order that

$$|F_{j_1}(z)| \le |F_{j_2}(z)| \le \cdots \le |F_{j_n}(z)| \le \cdots \le |F_{j_{q+1}}(z)| \le \infty.$$

Then we have

$$\|f(z)\| \le A_1(z)\cdot|F_{j_k}(z)|, \qquad (k=n+1,\dots,q+1),$$

where $\int_0^{2\pi}\log^+ A_1(re^{i\theta})\,d\theta = o(T_f(r))$ $(r \to \infty)$, since A_1 can be represented by a combination of ξ_{jk}'s. Hence we have

$$\prod_{j=0}^{q}\left(\frac{\|f(z)\|}{|F_j(z)|}\right) \le A_1(z)^{q-n+1}\prod_{k=1}^{n}\left(\frac{\|f(z)\|}{|F_{j_k}(z)|}\right).$$

Thus we obtain that for any $z \in \mathbb{C}$,

$$\prod_{j=0}^{q}\left(\frac{\|f(z)\|^s}{|F_j(z)|^s}\right) \le A_2(z,s)\cdot\sum_{(\ )}\frac{\|f(z)\|^{ns}}{\prod_{k=1}^{n}|F_{j_k}(z)|^s},$$

where $\int_0^{2\pi}\log^+ A_2(re^{i\theta})\,d\theta = o(T_f(r))$ $(r \longrightarrow \infty)$. Therefore we have

$$\prod_{j=0}^{q}\left(\frac{\|f\|^s}{|F_j|^s}\right) \le A_2(z,s)\cdot\|f\|^{sn}\cdot\left[1+\sum_{(\ )'}\left(\prod_{i=0}^{n-1}|F_i|^s/\prod_{k=1}^{n}|F_{j_k}|^s\right)\right]\left(1/\prod_{i=0}^{n-1}|F_i|^s\right)$$

Here the summation $\sum_{(\ )} = \sum_{(j_k)}$ is taken over all combinations of $(j_1,\dots,j_n)$ and $\sum_{(\ )'} = \sum_{(j_k)'}$ is taken over all combinations without $(0,\dots,n-1)$. Hence we have

$$\log\prod_{j=0}^{q}\left(\frac{\|f\|^s}{|F_j|^s}\right) \le \log^+\sum_{(\ )'}\left(\prod_{i=0}^{n-1}|F_i|^s \Big/ \prod_{k=1}^{n}|F_{j_k}|^s\right)$$

$$+ \log\left(\|f\|^{sn} \Big/ \prod_{i=0}^{n-1}|F_i|^s\right) + \log^+ A_2(z,s) + O(1)$$

and

$$\log^+ \sum_{(\ )'} \left(\prod_{i=0}^{n-1} |F_i|^s / \prod_{k=1}^{n} |F_{j_k}|^s \right)$$

$$\leq \log^+ \sum_{(\ )'} [|W_\alpha| / (\prod_{k=1}^{n} |F_{j_k}|^s \cdot (|F_0| + \cdots + |F_{n-1}|)^s \cdot \|f\|^{(n+1)(t-s)})]$$

$$+ \log^+ (\prod_{i=0}^{n-1} |F_i|^s \cdot (|F_0| + \cdots + |F_{n-1}|)^s \cdot \|f\|^{(n+1)(t-s)} / |W_\alpha|)$$

$$= \sum_{(\ )'} \log^+ D_\alpha + \log^+ (\prod_{i=0}^{n-1} |F_i|^s (|F_0| + \cdots + |F_{n-1}|)^s \|f\|^{(n+1)(t-s)} / |W| \cdot |C_\alpha|).$$

$$\leq \sum_{(\ )'} \log^+ D_\alpha + \log^+ 1/|\tilde{W}| + \log^+ \|\tilde{f}\|^{(n+1)(t-s)}$$

$$+ \log^+ (\prod_{i=0}^{n-1} |\tilde{F}_i|^s (|\tilde{F}_0| + \cdots + |\tilde{F}_{n-1}|)^s) + \log^+ 1/|C|,$$

where $D_\alpha = |W_\alpha| / (\prod_{k=1}^{n} |F_{j_k}|^s (|F_0| + \cdots + |F_n|^s) \|f\|^{(n+1)(t-s)})$ and

we write $\tilde{u}(z) = u(z)/ f_0^m$ for a function $u(z)$ with homogeneous form of degree m in $f_0, \ldots, f_n$. Thus we obtain that

$$(1) \quad \log \prod_{j=0}^{q} \left(\frac{\|f\|^s}{|F_j|^s} \right) \leq \sum_{(\ )'} \log^+ D_\alpha + \log^+ 1/|\tilde{W}| + \log^+ \|\tilde{f}\|^{(n+1)(t-s)}$$

$$+ \log^+ \prod_{i=0}^{n-1} |\tilde{F}_i|^s (|\tilde{F}_0| + \cdots + |\tilde{F}_{n-1}|)^s + \log^+ 1/|C_\alpha|$$

$$+ \log (\|f\|^{sn} / \prod_{i=0}^{n-1} |F_i|^s) + \log^+ A_2(z,s) + O(1).$$

By integrating both side of (1) on a circle $|z| = r$, we obtain

$$s \cdot \sum_{j=0}^{q} \frac{1}{2\pi} \int_0^{2\pi} \log \frac{\|f\|}{|F_j|} \, d\theta \leq o(T_f(r)) + \frac{1}{2\pi} \int_0^{2\pi} \log^+ 1/|\tilde{W}| \, d\theta$$

$$+ (n+1)(t-s) \cdot T_f(r) + 2s \cdot \sum_{i=0}^{n-1} \frac{1}{2\pi} \int_0^{2\pi} \log^+ |\tilde{F}_i| \, d\theta$$

$$+ s\cdot\sum_{i=0}^{n-1}\frac{1}{2\pi}\int_0^{2\pi}\log\frac{\|f\|}{|F_i|}\,d\theta + \int_0^{2\pi}\log^+ A_2(z,s)\,d\theta + O(1)\quad (r\longrightarrow\infty),$$

by the lemma on logarithmic derivatives and the assumption $T_1(r, f_j/f_0) = o(T_f(r))$ as $r\longrightarrow\infty$, $(j=0,\dots,n-1)$. Hence we have

$$(2)\qquad \sum_{j=0}^{q}\frac{1}{2\pi}\int_0^{2\pi}\log\frac{\|f\|}{|F_j|}\,d\theta \le \{n + (n+1)(\frac{t}{s} - 1) + o(1)\}\cdot T_f(r) + \frac{1}{2\pi s}\int_0^{2\pi}\log^+ 1/|\widetilde{W}|\,d\theta,\quad (r\longrightarrow\infty).$$

We note that

$$m_{f,g^j}(r) = \frac{1}{2\pi}\int_0^{2\pi}\log\frac{\|f\|\ \|g^j\|}{\|h_j\|}\,d\theta = \frac{1}{2\pi}\int_0^{2\pi}\log\frac{\|f\|}{|F_j|}\,d\theta + o(T_f(r))$$

$(r\longrightarrow\infty)$, $(j=0,\dots,q)$. Therefore dividing both side of (2) by $T_f(r)$ and taking a limit infimum as $r\longrightarrow\infty$, we obtain

$$\sum_{j=0}^{q}\delta(f,g^j) \le n + (n+1)(\frac{t}{s} - 1) + \liminf_{r\longrightarrow\infty}\left(\frac{1}{2\pi}\int_0^{2\pi}\log^+\frac{1}{|\widetilde{W}|}\,d\theta\right)/s\cdot T_f(r).$$

From Steinmetz' lemma [7, p.138], we see $\inf_{p\longrightarrow\infty} t/s = 1$, so we obtain that for any small $\varepsilon > 0$ there exists p such that $t_\varepsilon/s_\varepsilon < (1 + \frac{\varepsilon}{n+1})$. Hence we have

$$\sum_{j=0}^{q}\delta(f,g^j) \le (n + \varepsilon) + \liminf_{r\longrightarrow\infty}\left(\frac{1}{2\pi}\int_0^{2\pi}\log^+\frac{1}{|\widetilde{W}|}\,d\theta\right)/s_\varepsilon\cdot T_f(r),$$

Thus if $\sum_{j=0}^{q}\delta(f,g^j) = n + 1$, we have

$$1 - \varepsilon \le \liminf_{r\longrightarrow\infty}\left(\frac{1}{2\pi}\int_0^{2\pi}\log^+\frac{1}{|\widetilde{W}|}\,d\theta\right)/s_\varepsilon\cdot T_f(r) \le \liminf_{r\longrightarrow\infty} T_1(r,\widetilde{W})/s_\varepsilon\cdot T_f(r),$$

where $\widetilde{W} = \widetilde{W}(z,s_\varepsilon)$ depends on s_ε. Also, we have

$$T_1(r,\widetilde{W}) \le \frac{1}{2\pi}\int_0^{2\pi}\log^+(\ \widetilde{W}(b_1\tilde{f}_0,\dots,b_{t_\varepsilon}\tilde{f}_0,\dots,b_1\tilde{f}_n,\dots,b_{t_\varepsilon}\tilde{f}_n)/\prod_{k=0}^{n}|\tilde{f}_k|^{t_\varepsilon})\,d\theta + \frac{1}{2\pi}\int_0^{2\pi}\log^+\prod_{k=0}^{n}|\tilde{f}_k|^{t_\varepsilon}\,d\theta + m\cdot N_1(r,0,f_0)$$

$$= o(T_f(r)) + t_\varepsilon \cdot \sum_{k=0}^{n} T_1(r,\tilde{f}_k) = (t_\varepsilon + o(1))\, T_f(r) \quad (r \longrightarrow \infty).$$

Thus we have

$$1 - \varepsilon \le \liminf_{r \longrightarrow \infty} T_1(r,\tilde{W})/s_\varepsilon T_f(r) \le \limsup_{r \longrightarrow \infty} T_1(r,\tilde{W})/s_\varepsilon T_f(r) \le t_\varepsilon/s_\varepsilon.$$

This yields

$$1 - \varepsilon \le (t_\varepsilon/s_\varepsilon) \cdot \liminf_{r \longrightarrow \infty} \frac{1}{2\pi} \int_0^{2\pi} \log^+ \frac{1}{|\tilde{W}|}\, d\theta \,/\, T_1(r,\tilde{W}).$$

Therefore we deduce that

$$0 \le \limsup_{r \longrightarrow \infty} (N_1(r,\tilde{W}) + N_1(r,1/\tilde{W}))/T_1(r,\tilde{W}) \le 2\varepsilon.$$

From Edrei-Fuchs' theorem [1, p.298], if

$$\kappa := \limsup_{r \longrightarrow \infty} \{N_1(r,\tilde{W}) + N_1(r,1/\tilde{W})\}/T_1(r,\tilde{W}) < 2\varepsilon,$$

then there is an integer γ such that

$$\text{(3)} \qquad \gamma - 10e(\gamma + 1)\cdot\varepsilon \le \mu_{\tilde{W}} \le \lambda_{\tilde{W}} < \gamma + e(\gamma + 1)\cdot\varepsilon .$$

Thus, we deduce that $\tilde{W} = \tilde{W}(z,s_\varepsilon)$ is a meromorphic function of order $\lambda_{\tilde{W}}$ and of lower order $\mu_{\tilde{W}}$ satisfying (3). On the other hand, since

$$(1 - \varepsilon)s_\varepsilon T_f(r) \le T_1(r,\tilde{W}) \le (1 + \varepsilon)s_\varepsilon T_f(r).$$

Hence we obtain that the order and lower order of f are equal to the order and lower order of $\tilde{W}$, respectively. Now taking $p \longrightarrow \infty$ and $\varepsilon \longrightarrow 0$, we obtain that f is of positive integral order and regular growth. This completes the proof of the theorem.

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