数理解析研究所講究録
820 巻 1993 年 82-90

グラフの辺取りゲーム

(ニムとケイレスの一般化)

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1.Introduction.

We first define a new game played on graphs. Let G be a finite graph without loops or multiple edges, and $\mathcal{H}$ a set of graphs. This game is played by two players on the graph G. Each player in turn removes a set of edges which induces a graph isomorphic to a graph in $\mathcal{H}$. The winner is a player who removes edges such that the resulting graph contains no graph of $\mathcal{H}$, that is, the player who cannot move loses. We call this game an *edge-removing game of normal* $\mathcal{H}$ *type*. If we change the rule to one where the player who removes the last edges loses, then the game is called an *edge-removing game of misere* $\mathcal{H}$ *type*. In this paper, we shall discuss only games of normal type.

We call the complete bipartite graph $K_{1,n} = K(1,n)$ the *star* of order n+1, and denote by P_n the path of order n. If $\mathcal{H}$ is the set of all stars, then we call this game the *edge-removing game of normal star type*, or simply *ER-game of star type*. If we play ER-game of star type on a graph consisting of some stars, then this game is nothing but the game of Nim. Similarly ER-game of star type played on a graph consisting of some paths is equivalent to the game called Kayles. So ER-game of star type is a generalization of these two games. In this paper we give some results on ER-game of star type played on double stars, forks and trees.

2. ER-game of star type played on doubles stars

In order to solve ER-game of normal $\mathcal{H}$ type played on a graph G, it suffices to determine the Sprague-Grundy number $g(G)$ of G, which is often called the Grundy number [1,2,3]. The Grundy number is defined inductively as follows: If a graph G_1 contains no graph of $\mathcal{H}$, then $g(G_1) = 0$. Let $H_1, H_2, \dots, H_m$ be the set of all graphs which can be obtained from a graph G by one move. Then

$$g(G) = \min\{\{0,1,2,3,\dots\} - \{g(H_i) \mid 1 \leq i \leq m\}\}$$

By this definiton, we can easily show that $g(G) \leq |E(G)|$ by induction on $|E(G)|$. It is well-known that if a graph G consists of the components $D_1, \dots, D_r$, then

$$g(G) = \text{the nim-sum of } g(D_1), g(D_2), \dots g(D_r).$$
$$= g(D_1) \dot{+} g(D_2) \dot{+} \dots \dot{+} g(D_r).$$

Namely if

$$g(D_k) = \sum_{i \geq 0} x_k(i)\, 2^i, \qquad x_k(i) \in \{0,1\}$$

then

$$g(G) = \sum_{i \geq 0} y(i)\, 2^i, \qquad y(i) \equiv \sum_{k=1}^{r} x_k(i) \pmod 2 \qquad \text{and} \qquad y(i) \in \{0,1\}.$$

Moreover, it is easy to see that the player going second can win if and only if $g(G) = 0$.

We denote by $(\dots)$ an order set, that is, $(x_1, x_2, \dots, x_k) = (y_1, y_2, \dots, y_k)$ means that $x_i = y_i$ for all i, $1 \leq i \leq k$.

Theorem A. [1,2,3] The Grundy numbers of stars $K_{1,n}$ and paths P_n of ER-game of star type are given by the following statements.

(i) $g(K_{1,n}) = n$.

(ii) $g(P_{n+12}) = g(P_n)$ if $n \geqq 72$, and
$(\, g(P_k) \mid 72 \leqq k < 84) = (7,4,1,2,8,1,4,7,2,1,8,4)$.

For convenience, we denote by $K_{1,0}$ a graph with one vertex and no edge. The double star $DS(n,m)$ is a graph obtained from two stars $K_{1,n}$ and $K_{1,m}$ by joining their two centers by a new edge. Then the order of $DS(n,m)$ is $n+m+2$ and its size is $n+m+1$. We now give a conjecture on the grundy numbers of double stars.

Conjecture B. Suppose that ER-game of star type is played on a double star $DS(n,m)$). Then

(i) For every positive odd integer n, there exists an integer $M = M(n)$ for which $g(DS(n,m)) = n+m+1$ if $m \geqq M$.

(ii) For every positive even integer n, there exists integers $p = p(n)$ and $M = M(n)$ for which $g(DS(n,m+p))=g(DS(n,,m)) + p$ if $m \geqq M$.

We shall show that the conjecture is true if $n = 2^k - 1$, $n = 2^k$ or $1 \leqq n \leqq 10$. Moreover, by making use of computer, we observe that if $n < 50$ and $m \leqq 5000$ then the conjecture holds and that $M(n) < 800$ except $n = 33$ ($M(n) = 1953$), $n = 34$ ($M(n) = 2141$) and $n = 48$ ($M(n) = 2157$), furthermore, we may give a conjecture on $p(n)$ that $p(n) = 2^{k+1}$ if $2^k \leqq n < 2^{k+1}$ except $n = 24$ ($p(n) = 64$).

Theorem 1. Suppose that ER-game of star type is played on a double star. Then

(i) For every integers $k \geqq 1$ and $m \geqq 0$, we have

$$g(DS(2^k - 1,m)) = 2^k + m.$$

(ii) For every integers $k \geqq 1$ and $h \geqq 0$, we have

$$g(DS(2^k, h\,2^{k+1} + s)) = h\,2^{k+1} + 2^k + s + 1,$$

where $-1 \leqq s \leqq 2^k - 1$ and

$$g(DS(2^k, h\,2^{k+1} + 2^k + s) = h\,2^{k+1} + s + 1,$$

where $0 \leqq s \leqq 2^k - 2$. In particular,

$$g(DS(2^k, m + 2^{k+1})) = g(DS(2^k, m)) + 2^{k+1} \text{ for all } m \geqq 0.$$

Proof We first prove Statement (i). For convenience, let $n = 2^k - 1$. We shall prove that $g(DS(n,m)) = n + m + 1$ by induction on m. Since a double star $DS(n,0)$ is a star $K_{1,n+1}$, $g(DS(n,0)) = n + 1$ by Theorem A.

Suppose that $1 \leqq m \leqq n$. For every integer j, $0 \leqq j \leqq n$, let $r = m \dot{+} j$. Then $0 \leqq r \leqq n$ and $r \dot{+} m = j$. We can remove a star from the double star $DS(n,m)$ such that the resulting graph is $K_{1,r} \cup K_{1,m}$, whose grundy number is $r \dot{+} m = j$. By the induction hypothesis, we have that $g(DS(n,r)) = n + r + 1$ for every $0 \leqq r < m$. Therefore $g(DS(n,m)) \geqq n + m + 1$. Since $g(DS(n,m)) \leqq |E(DS(n,m))| = n + m + 1$, we can conclude that $g(DS(n,m)) = n + m + 1$.

Next assume that $n < m$. For every integer j, $0 \leqq j \leqq n$, let $r = n \dot{+} j$. Then $0 \leqq r \leqq n$ and $n \dot{+} r = j$. We can remove a star from the double star $DS(n,m)$ such that the resulting graph is $K_{1,n} \cup K_{1,r}$, whose grundy number is $n \dot{+} r = j$. By the same argument as above, we can also show that $g(DS(n,m)) = n + m + 1$.

For convenience, we denote the star $K_{1,l}$ by $K(1,l)$. In order to prove Statement (ii) we need to show that the following equation holds.

$$g(DS(h\,2^{k+1}-1,m)) = h\,2^{k+1}+m$$

for every integers $0 \leqq m \leqq 2^k$ and $h \geqq 1$. We prove the above equation by induction on m. Let $0 \leqq j < 2^k$ and $0 \leqq t \leqq h-1$. If $m < 2^k$, then $r = j \dotplus < 2^k$, and we can remove stars from $DS(h\,2^{k+1}-1,m)$ such that the resulting graphs are $K(1,t\,2^{k+1}+r) \cup K(1,m)$ and $K(1,t\,2^{k+1}+2^k+r) \cup K(1,m)$, whose grundy numbers are $t\,2^{k+1}+j$ and $t\,2^{k+1}+2^k+j$, respectively. If $m = 2^k$ then we can remove stars from $DS(h\,2^{k+1}-1,m)$ such that the resulting graphs are $K(1,t\,2^{k+1}+j) \cup K(1,2^k)$ and $K(1,t\,2^{k+1}+2^k+j) \cup K(1,2^k)$, whose grundy numbers are $t\,2^{k+1}+2^k+j$ and $t\,2^{k+1}+j$, respectively. By the induction hypothesis, we have that $g(DS(h\,2^{k+1}-1,y)) = h\,2^{k+1}+y$ for every $0 \leqq y < m$. Thus $g(DS(h\,2^{k+1}-1,m)) \geqq h\,2^{k+1}+m$, and so $g(DS(h\,2^{k+1}-1,m)) = h\,2^{k+1}+m$.

We now prove Statement (ii) by induction on $h\,2^{k+1}+s$ or $h\,2^{k+1}+2^k+s$. By Theorem A and the above statement, $g(DS(2^k,0)) = 2^k+1$ and $g(DS(2^k,h\,2^{k+1}-1)) = h\,2^{k+1}+2^k$. Consider a double star $DS(2^k,h\,2^{k+1}-1)$, $0 \leqq h$, $0 \leqq s \leqq 2^k-1$. For every integers $0 \leqq t \leqq r-1$, and $0 \leqq x < 2^k$, we have that

$$g(K(1,2^k)) \dotplus g(K(1,t\,2^{k+1}+2^k+x)) = t\,2^{k+1}+x$$

and

$$g(K(1,2^k)) \dotplus g(K(1,t\,2^{k+1}+x)) = t\,2^{k+1}+2^k+x.$$

Moreover,

$$g(K(1,x')) \dotplus g(K(1,h\,2^{k+1})) = h\,2^{k+1}+x' \text{ for } 0 \leqq x' \leqq 2^k,$$

and for every integer y, $0 \leqq y < s$, it follows from the induction hypothesis that

$$g(DS(2^k,h\,2^{k+1}+y)) = h\,2^{k+1}+2^k+y+1.$$

Therefore $g(DS(2^k,h\,2^{k+1}+s)) \geqq h\,2^k+2^k+s+1$, and thus $g(D(2^k,h\,2^{k+1}+s)) = h\,2^k+2^k+s+1$.

We next consider a double star $DS(2^k, h\,2^{k+1} + 2^k + s)$, $0 \leqq s \leqq 2^k - 2$. By the same argument as above, we can easily show that

$$\{g(K(1,2^k)) \dot{+} g(K(1,y)) \mid 0 \leqq y \leqq h\,2^{k+1} + 2^k + s\}$$
$$= \{0,1,2, \dots , h\,2^{k+1} + s\} \cup \{h\,2^{k+1} + 2^k, \dots , (h + 1)2^{k+1} - 1\}.$$

Thus $g(DS(2^k, h\,2^{k+1} + 2^k + s)) \geqq h\,2^{k+1} + s + 1$. It is obvious that for every $0 < t < 2^k$, $DS(t, h\,2^{2k+1} + 2^k + s)$ contains $K(1,t) \cup K(1, h\,2^{k+1} + r)$, $r = t \dot{+} (s + 1)$, whose grundy number is $h\,2^{k+1} + s + 1$. Therefore $g(DS(t, h\,2^{k+1} + 2^k + s)) \neq h\,2^{k+1} + s + 1$. Consequently we can conclude that $g(DS(2^k, h\,2^{k+1} + 2^k + s)) = h\,2^{k+1} + s + 1$.

Theorem 2. The Grundy numbers of double stars $DS(n,m)$, $n \leqq 10$, of ER-game of star type are given by the following statemtents.

(i) If $n = 0$, $n = 1$, $n = 3$, $n = 5$ and $m \geqq 15$, $n = 7$ or $n = 9$ and $m \geqq 95$ then

$$g(\ DS(n,m)\ ) = n + m + 1.$$

(ii) Let $p = 4, 8$ or 16 according as $n = 2$, $n = 4, 6$, or $n = 8, 10$. Suppose that $m \geqq 15$ if $n = 6$, and $m \geqq 110$ if $n = 10$. Then

$$g(\ DS(n,m + p)\ ) = g(\ DS(n,m)\ ) + p.$$

Note that Theorem 2 holds for $n = 0, 1, 2, 3, 4, 7, 8$ by Theorem A and Theorem 1. We shall prove the following proposition instead of remaining Statement (ii) of of Theorem 2.

Proposition 3. Consider $g(DS(n,m))$ with $n = 2, 4, 6, 8$ or 10. Let t and s be integers such that $0 \leqq t$, and $0 \leqq s < 4$, $0 \leqq s < 8$, or $0 \leqq s < 16$ according as $n = 2$, $n = 4, 6$ or $n = 8, 10$. Then the following statements hold.

(i) $g(\ DS(2,4t+s)\ ) = 4t+3,\ 4t+4,\ 4t+1,\ 4t+6$ if $s = 0,\ 1,\ 2,\ 3$, respectively.

(ii) $g(\ DS(4,8t+s)\ ) = 8t+5,\ 8t+6,\ 8t+7,\ 8t+8,\ 8t+1,\ 8t+2,\ 8t+3,\ 8t+12$ if $s = 0,\ 1,\ 2,\ \ldots,\ 7$, respectively.

(iii) $g(\ DS(6,15+8t+s)\ ) = 8t+22,\ 8t+23,\ 8t+24,\ 8t+21,\ 8t+26,\ 8t+25,\ 8t+28,\ 8t+27$ if $s = 0, 1,\ 2,\ \ldots,\ 7$, respectively.

(iv) $g(\ DS(8,16t+s)\ ) = 8t+s+9,\ 8t+s-7$ or $8t+24$ if $0 \leqq s \leqq 7$, $8 \leqq s \leqq 14$ or $s = 15$, respectively.

(v) $g(\ DS(10,110+8t+s)\ ) = 8t+117, 8t+122, 8t+123, 8t+124, 8t+121, 8t+126, 8t+127,\ 8t+128,\ 8t+125,\ 8t+130,\ 8t+119,\ 8t+132,\ 8t+129$ if $s = 0,\ 1,\ \ldots,\ 15$, respectivley.

2. Grundy Numbers of Forks and trees

A *fork* $F(n,m)$ is defined to be a graph which is obtained from a star $K_{1,n}$ and a path P_m by joining the center of the star to one of the end vertices of the path by a new edge. Then the order of $F(n,m)$ is $n + m + 1$ and its size is $n + m$. Note that $F(0,m) = P_{m+1}$, $F(1,m) = P_{m+2}$, $F(n,0) = K_{1,n}$ and $F(n,1) = K_{1,n+1}$ and these Grundy numbers are given by Theorem A.

Theorem 2. The Grundy numbers of forks $F(n,m)$ with $n \leqq 10$ or $m \leqq 10$ of ER-game of star type are given by the following statements.

(i) If $n = 2$ and $m \geqq 152$, $n = 3$ and $m \geqq 141$, $n = 4$ and $m \geqq 142$, $n = 5$ and $m \geqq 286$, $n = 6$ and $m \geqq 286$, $n = 7$ and $m \geqq 215$, $n = 8$ and $m \geqq 112$, $n = 9$ and $m \geqq 141$, or $n = 10$ and $m \geqq 190$ then

$$g(\ F(n,m+12)\ ) = g(\ F(n,m)\ ).$$

(ii) If $m = 2$, $m = 3$ and $n \geqq 2$, $m = 4$ and $n \geqq 5$, $m = 6$ and $n \geqq 15$, or $m = 10$ and $n \geqq 30$ then

$$g(\ F(n,m)\ ) = n + m.$$

(iii) Let $p = 4$, 16 or 8 according as $m = 5$, $m = 7, 8$ or $m = 9$. Suppose that $n \geqq 8$, 9, 10 or 15 if $m = 5$, 7, 8, or 9, respectively. Then

$$g(\ F(n+p,m)\ ) = g(\ F(n,m)\ ) + p.$$

Conjecture C. (i) For every positive integer n, there exists an integer $M = M(n)$ for which $g(F(n,m+12)) = g(F(n,m))$ if $m \geqq M$.

(ii) For every positive even integer m, there exists integers $p = p(m)$ and $M = M(m)$ for which $g(F(n+p,m)) = g(F(n,m)) + p$ if $n \geqq M$.

We finally give some remarks on ER-game of star type on trees and propose a related problem. The Grundy number of every tree with order less than 10 is non-zero, and there exist 16 trees of order 10 and seven trees of order 11 whose Grundy numbers are equal to 0. These trees are given below. Let T_i denote a tree of order 10 or 11 whose Grundy number is 0, and let $V(T_i) = \{1, 2, \dots ,9, a\}$ or $\{1, 2, \dots ,9, a, b\}$. If the order of T_i is 10, then T_i contains a set of edge $F = \{12, 23, 34, 45, 56\}$, and so we dnote only $F_i = E(T) - F$.
$F_1 = \{67,78,89,2a\}$, $F_2 = \{67,78,89,3a\}$, $F_3 = \{67,78,79,4a\}$, $F_4 = \{67,58,89,2a\}$, $F_5 = \{67,78,79,6a\}$, $F_6 = \{67,78,69,4a\}$, $F_7 = \{67,58,89,3a\}$, $F_8 = \{37,78,89,1a\}$, $F_9 = \{37,78,89,2a\}$, $F_{10} = \{67,48,29,3a\}$, $F_{11} = \{37,78,29,7a\}$, $F_{12} = \{47,78,29,3a\}$, $F_{13} = \{47,78,79,5a\}$, $F_{14} = \{67,78,79,7a\}$, $F_{15} = \{67,68,69,4a\}$ and $F_{16} = \{47,78,29,2a\}$.

If the order of T_i is 11, then the edge set E_i of T_i are given as follows:

E_1 = {12,23,34,45,56,67,58,89,4a,ab}, E_2 = {12,23,34,45,56,37,78,29,4a,4b}, E_3 = {12,23,34,45,56,57,78,59,4a,4b}, E_4 = {12,23,34,45,46,67,48,89,3a,3b}, E_5 = {12,23,34,45,56,57,58,59,4a,4b}, E_6 = {12,13,14,15,16,67,68,89,7a,7b} and E_7 = {12,23,35,56,34,37,78,29,2a,2b}.

Problem Characterize trees whose Grundy numbers are equal to 0.

ER-game of path type (i.e. $\mathcal{H}$ is the set of all paths) will be deal with other paper. Is it possible to solve ER-games of the following $\mathcal{H}$ type on certain class of graphs: $\mathcal{H}$ is the set of all cycles, $\mathcal{H}$ is the set of all trees, $\mathcal{H}$ is the set of all matchings, $\mathcal{H}$ is the set of all forests, and so on.

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