<table>
<thead>
<tr>
<th>Title</th>
<th>Block designs with nested rows and columns for symmetric parallel line assays (Optimal Combinatorial Structures on Discrete Mathematical Models)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Gupta, Sudhir</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (1993), 820: 24-32</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1993-02</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/83179">http://hdl.handle.net/2433/83179</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Text version</td>
<td>publisher</td>
</tr>
</tbody>
</table>

Kyoto University
Block designs with nested rows and columns for symmetric parallel line assays

By Sudhir Gupta
Division of Statistics, Northern Illinois University
DeKalb, IL 60115, U.S.A.

Summary

Symmetric parallel lines bioassays with even number of doses of each preparation are considered when a block design with nested rows and columns is used for the experiment. Thus, it is assumed that two sources of nuisance variability are to be controlled in the experiment. Nested row-column designs which estimate the preparation contrast, the combined regression contrast and the parallelism contrast with full efficiency are characterized and some methods of construction are provided.

1 Introduction

Suppose that a symmetric parallel line (SPL) assay involving two preparations, standard and test, each at $m$ equi-spaced doses is to be conducted using a block design with nested rows and columns. Singh and Dey (1979) defined variance balanced incomplete block designs with nested rows and columns. Since then several authors have given methods of constructing variance balanced and partially variance balanced designs with nested rows and columns, e.g. Agrawal and Prasad (1982, 1983), Sreenath (1989), Uddin and
Morgan (1990). The purpose of this paper is to consider these designs for SPL assays when \( m \), the number of doses of each preparation, is even.

The preparation contrast \( L_p \), the combined regression contrast \( L_1 \), and the parallelism contrast \( L_1^1 \) are of main importance in SPL assays. The \( L_p \) and \( L_1 \) provide an estimate of relative potency and \( L_1^1 \) is important for testing deviation from parallelism of the regression lines for the standard and the test preparations. Therefore it is desired to estimate these three contrasts with full efficiency. Designs which estimate \( L_p, L_1 \) and \( L_1^1 \) with full efficiency will be referred to as \( L \)-designs in this paper.

\( L \)-designs with one dimensional blocks have been considered by several authors, see for example Kyi Win and Dey (1980), Nigam and Boopathy (1985), Gupta (1989), Gupta and Mukerjee (1991). Das and Kulkarni (1966) gave some designs which estimate \( L_p \) and \( L_1 \) with full efficiency. The reader is referred to Kshirsagar and Yuan (1992) for a unified theory of parallel line bioassays in incomplete block designs.

\( L \)-designs with nested rows and columns are defined in Section 2. Then some characterization and construction aspects are considered in Section 3.

### 2 \( L \)-designs with nested rows and columns

Let \( s_1 < s_2 < \cdots < s_u < s_{u+1} < \cdots < s_m \) and \( t_1 < t_2 < \cdots < t_u < t_{u+1} < \cdots < t_m \), where \( m = 2u \), denote the doses of the standard and the test preparations respectively. Let these \( v = 2m \) treatments be coded as \( 1, 2, \cdots, 2m \) respectively. Also, let \( \tau = (\tau_1 \tau_2 \cdots \tau_v)' \) be the vector of treatment parameters where \( \tau_i \) and \( \tau_{m+i} \) denote the effects of \( s_i \) and \( t_i \) respectively, \( i = 1, 2, \cdots, m \). Let \( \epsilon_i, i = 1, 2, \cdots, m-1 \), denote all possible contrast vectors of size \( m \) as given by orthogonal polynomials. Let

\[
\ell_p = (1_m' - 1_m')' \tag{2.1}
\]

\[
\ell_{i1} = (\epsilon_i' \epsilon_i'), \quad \ell_{i2} = (\epsilon_i' - \epsilon_i')', \quad i = 1, 2, \cdots, m-1,
\]
where \(1_q\) denotes a column vector of 1's of size \(q\). Then

\[
L_p = \ell_p' \tau, \\
L_i = \ell_i' \tau, \quad L_i^1 = \ell_i' \tau, \quad i = 1, 2, \cdots, m - 1.
\]

(2.2)

It should be noted that \(e_i, i = 1, 2, \cdots, m - 1\), have the following structure,

\[
e_i' = \begin{cases} 
  f_i' [I_u - I_{cu}] & , \quad i = 1, 3, 5, \cdots, m - 1 \\
  f_i' [I_u I_{cu}] & , \quad i = 2, 4, 6, \cdots, m - 2
\end{cases}
\]

(2.3)

where \(f_i = [f_{i1} f_{i2} \cdots f_{iu}]'\) is a column vector of size \(u\), \(i = 1, 2, \cdots, m - 1\), \(I_u\) is the identity matrix and \(I_{cu}\) denotes the matrix containing unity in the \((i, u+1-i)\) positions, \(i = 1, 2, \cdots, u\), and zero elsewhere, both being of order \(u \times u\). Since \(e_i'1 = e_i'e_j = 0\), using (2.3) it follows that \(f_i'1_u = f_i'f_j = 0\), \(i \neq j = 2, 4, 6, \cdots, m - 2\).

First consider a SPL assay conducted using an incomplete block design having \(b\) blocks of \(k\) plots each, with every treatment replicated a constant number of times denoted by \(r\). Then, following Kyi Win and Dey (1980) and Gupta (1989), it can be proved that \(L_p, L_i\) and \(L_i^1, i = 1, 3, 5, \cdots, m - 1\), are estimated with full efficiency if the design satisfies the following conditions:

(a) The total number of concurrences of each of the pairs of treatments \((g, m + 1 - g)\), 
\((m + g, 2m + 1 - g), g = 1, 2, \cdots, u\), is equal to \(r\).

(b) Each block of the design contains \(k/2\) treatments belonging to \(\{1, 2, \cdots, m\}\) where \(k\) is necessarily even.

Let us now consider the case of block designs with nested rows and columns for SPL assays. Thus, let each block be arranged in \(p\) rows and \(q\) columns with \(k = pq\). Let \(N_1, N_2\) and \(N\) denote the \(v \times pb\) treatment versus row, \(v \times qb\) treatment versus column, and \(v \times b\) treatment versus blocks incidence matrices respectively. The component designs corresponding to \(N_1, N_2, N\) will be denoted by \(D_1, D_2\) and \(D\) respectively. The reduced
normal equations for estimating the vector of treatment parameters are then given by 
\[ C \tau = Q \]
where 
\[ C = rI_v - \frac{1}{q}N_1N'_1 - \frac{1}{p}N_2N'_2 + \frac{1}{pq}NN', \quad (2.4) \]
and \( Q \) is the vector of adjusted treatment totals. Along the lines of a \( L \)-design with one dimensional blocks considered above, we have the following definition for the case of designs with nested rows and columns.

**Definition 2.1.** A design with nested rows and columns is defined to be a \( L \)-design if it satisfies the following conditions,

(a) the two treatments \( g + (j-1)m \) and \( mj + 1 - g \) do not fall in different blocks of either of the two component designs \( D_1 \) or \( D_2 \), \( j = 1, 2; g = 1, 2, \cdots, u \), and

(b) each of the blocks of the component designs \( D_1 \) and \( D_2 \) contains an equal number of treatments belonging to \( \{1, 2, \cdots, m\} \) and \( \{m+1, m+2, \cdots, 2m\} \), where both \( p, q \) are necessarily even.

Using (2.1), (2.3) and (2.4), it can be verified that \( L \)-designs with nested rows and columns of Definition 2.1 estimate \( L_p, L_i \) and \( L^1_i \) with full efficiency, \( i = 1, 3, 5, \cdots, m-1 \).

### 3 Characterizations and constructions of \( L \)-designs

Definition 2.1 implies that a typical block of the component design \( D_1 \) or \( D_2 \) is of the form,

\[
\begin{bmatrix}
  a_{i(1)} & m - a_{i(1)} + 1 & a_{i(2)} & m - a_{i(2)} + 1 & \cdots & a_{i(h)} & m - a_{i(h)} + 1 \\
  b_{j(1)} & 3m - b_{j(1)} + 1 & b_{j(2)} & 3m - b_{j(2)} + 1 & \cdots & b_{j(h)} & 3m - b_{j(h)} + 1
\end{bmatrix} \quad (3.1)
\]

where \( 4h = q \) if the block belongs to \( D_1 \) and \( 4h = p \) if it belongs to the component design \( D_2 \), and
Each block of $D$ constitutes of $p$ rows and $q$ columns. Consider a typical block of the component design $D_1$ given by (3.1). The $q$ treatments which are contained in this block belong to $q$ different blocks of the component design $D_2$. These $q$ blocks of $D_2$ will be referred to as being associated with that particular block of $D_1$. From Definition 2.1, if the two treatments $g + (j - 1)m$ and $mj + 1 - g$ occur together $\lambda$ times in some block of the component design $D_1$, then they must occur together in at least $2\lambda$ blocks of $D_2$ associated with this particular block of $D_1$. If we now reverse the roles of $D_1$ and $D_2$ in the above discussion then it follows that if the pair of treatments $\{g + (j - 1)m, mj + 1 - g\}$ occurs together a certain number of times in the blocks of $D_1$ associated with a block of $D$, then it also occurs together the same number of times in the blocks of $D_2$ associated with that particular block of $D$. This means that the rows and columns within each block of $D$ can be permuted to yield an arrangement of the following type for each block of $D_1$ and $D_2$,

$$[A_{i(1)} A_{i(2)} \cdots A_{i(h)} B_{j(1)} B_{j(2)} \cdots B_{j(h)}]$$ (3.3)

where

$$A_{i(\ell)} = \begin{bmatrix} a_{i(\ell)} & m - a_{i(\ell)} + 1 \\ m - a_{i(\ell)} + 1 & a_{i(\ell)} \end{bmatrix},$$

$$B_{j(\ell)} = \begin{bmatrix} b_{j(\ell)} & 3m - b_{j(\ell)} + 1 \\ 3m - b_{j(\ell)} + 1 & b_{j(\ell)} \end{bmatrix},$$ (3.4)

$\ell = 1, 2, \cdots, h,$
and \(a_{t\ell}, \ b_{j\ell}, \ \ell = 1,2,\ldots, h,\) are as defined in equation (3.2). Let

\[
A_g = \begin{bmatrix}
g & m+1-g \\
m+1-g & g
\end{bmatrix}
\]

\[
B_g = \begin{bmatrix}
m+g & 2m+1-g \\
2m+1-g & m+g
\end{bmatrix} = A_g + mJ_2, \quad g = 1,2,\ldots, u,
\tag{3.5}
\]

where \(J_2\) is a \(2 \times 2\) matrix of 1's. Then \(A_{t\ell}, B_{j\ell}\) of equation (3.4) belong to \(\{A_1, A_2, \ldots, A_u\}\) and \(\{B_1, B_2, \ldots, B_u\}\) respectively.

We now present some methods for obtaining \(L\)-designs with nested rows and columns. The case of \(q = 2\) will be considered in some detail. Designs for \(q \geq 4\) can be derived using designs with \(q = 2\). A typical block of a \(L\)-design with nested rows and columns having \(q = 2\) is given by (3.3). If \(u\) is a multiple of \(h\) then the blocks,

\[
[A_{t-1}A_{(t-1)+h+1}\cdots A_{t+h} B_{(t-1)+h+1} B_{(t-1)+h+2}\cdots B_{t+h}]
\]

\[
\ell = 1,2,\ldots, w
\]

where \(w = u/h\), yield a \(L\)-design having parameters \(v = 2m, b = u/h = w, r = 2, k = 8h, p = 4h, q = 2\). Designs with \(q = 2, r = 2\) are necessarily disconnected. For \(r = 4\), the following blocks constitute a connected \(L\)-design with parameters \(v = 2m, b = 2u/h, r = 4, k = 8h, p = 4h, q = 2,\)

\[
[A_{(t-1)+h}A_{(t-1)+h+2}\cdots A_{t+h} B_{th+1} B_{th+2}\cdots B_{(t+1)+h}]
\]

\[
\ell = 1,2,\ldots, w-1,
\]
\[
\begin{aligned}
&[A_{(w-1)h+1} A_{(w-1)h+2} \cdots A_{wh} B_{(w-1)h+1} B_{(w-1)h+2} \cdots B_{wh}], \\
&[A_{(w-\ell)h+1} A_{(w-\ell)h+2} \cdots A_{(w-1+1)h} B_{(w-\ell-1)h+1} B_{(w-\ell-1)h+2} \cdots B_{(w-\ell)h}], \\
&\ell = 1, 2, \ldots, w - 1, \\
&[A_1 A_2 \cdots A_h B_1 B_2 \cdots B_h].
\end{aligned}
\]

Example 3.1. Suppose \( u = 4, h = 1, r = 4, q = 2 \). Then \( m = 8 \), and using (3.5),

\[
\begin{aligned}
A_1 & = \begin{bmatrix} 1 & 8 \\ 8 & 1 \end{bmatrix},
A_2 = \begin{bmatrix} 2 & 7 \\ 7 & 2 \end{bmatrix},
A_3 = \begin{bmatrix} 3 & 6 \\ 6 & 3 \end{bmatrix},
A_4 = \begin{bmatrix} 4 & 5 \\ 5 & 4 \end{bmatrix},
\end{aligned}
\]

\[
B_g = A_g + 8J_2 , \quad g = 1, 2, 3, 4.
\]

A \( L \)-design with nested rows and columns having \( v = 16, b = 8, r = 4, k = 8, p = 4, q = 2 \) is given by

\[
[A_1 B_2], \; [A_2 B_3], \; [A_3 B_4], \; [A_4 B_4],
\]

\[
[A_4 B_3], \; [A_3 B_2], \; [A_2 B_1], \; [A_1 B_1].
\]

When \( u \) is a multiple of \( b \), the number of blocks, let \( w_1 = u/b \). Then a \( L \)-design with parameters \( v = 4u, b, r = h/w_1, k = 8h, p = 4h, q = 2 \) is given by the following blocks,

\[
\begin{aligned}
&[A_{(i-1)w_1+1} A_{(i-1)w_1+2} \cdots A_{iw_1} B_{(i-1)w_1+1} B_{(i-1)w_1+2} \cdots B_{iw_1}] \\
&\quad \text{\( a \) times}
\ra
\]

\[
[A_{iw_1+1} A_{iw_1+2} \cdots A_{(i+1)w_1} B_{iw_1+1} B_{iw_1+2} \cdots B_{(i+1)w_1}] \\
\quad \text{\( (h/w_1-a) \) times}
\ra
\]

\[
i = 1, 2, \ldots, b - 1,
\]
\[
\left[ A_{(b-1)w_1+1} A_{(b-1)w_1+2} \ldots A_{bw_1} B_{(b-1)w_1+1} B_{(b-1)w_1+2} \ldots B_{bw_1} \right]^{a \text{ times}}
\]

\[
\left[ A_1 A_2 \ldots A_{w_1} B_1 B_2 \ldots B_{w_1} \right]^{(h/w_1-a) \text{ times}}
\]

$L$-designs for $q \geq 4$ can be obtained by some juxtaposition of the blocks of $L$-designs with $q = 2$. Latin square type arrangements for each of the blocks of $D$ can also be used which result in $L$-designs with $p = q$.

References


