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BOUNDS ON THE EFFICIENCY OF THE RESIDUAL DESIGN OF EXTENDED BIB DESIGNS

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Abstract. This gives a complete proof for bounds of the efficiency, conjectured by Das and Kageyama (1992), on robustness of extended balanced incomplete block designs against the unavailability of any number of observations in a block.

1. Introduction

The robustness problem of block designs against the unavailability of data has been considered in various ways. Das and Kageyama (1992) showed the robustness of extended balanced incomplete block (BIB) designs when all the observations or any one observation in a block are lost. Furthermore, by a *computer search* they also observed the robustness of extended BIB designs against the unavailability of any number of observations in a block, *within some range of design parameters*.

In this note, we shall prove bounds of the efficiency to show generally the robustness of extended BIB designs against the unavailability of any number of observations in a block. This gives a complete answer to the problem presented in Section 3.3 of Das and Kageyama (1992).

2. The bounds

An extended BIB design d with v treatments and b+1 blocks of size k each is a design obtained by juxtaposing one binary block of size k to a BIB design with parameters v, b, r, k and λ denoted by a BIBD(v,b,r,k,λ).

In the extended BIB design d, suppose that s ($1 \le s \le k$) observations in any one block of d are lost. Let p be the number of treatments common to the missing treatments and k treatments in the added block, and let q be the number of treatments common to the remaining treatments (in a block containing the missing treatments) and k treatments in the added block. The parameters should satisfy

(2.1) $\quad 0 \le p \le s, \quad \max\{0, 2k-v-p\} \le q \le k-s.$

Let d^* be the design obtained by deleting s observations for $1 \le s \le k$ in any one block in d. Assume d^* to be connected. As shown in Das and Kageyama (1992), the efficiency of residual design d^* is given by

$$e_s(p,q) = \frac{\phi_2}{\phi_1(s)} \quad (= B, \text{ say})$$

where

$$\phi_2 = k\{\lambda v(v-1)+k(v-k)\}/\{\lambda v(\lambda v+k)\},$$

$$\phi_1(s) = k\{v-k-s+2(p-1)\}/(\lambda v) + k(k-p-2)/(\lambda v+k)$$
$$+ k(s-p-1)/(\lambda v-k) + 4\lambda vk(k-s)(1/D+1/E)$$
$$\text{for } 1 \le s \le k-1,$$

$$D = 2\lambda^2 v^2(k-s)-\alpha+\sqrt{\alpha^2-\beta}\ ,$$
$$E = 2\lambda^2 v^2(k-s)-\alpha-\sqrt{\alpha^2-\beta}\ ,$$
$$\alpha = (k-s)\{k(k+2p)-p(p+2q)\}-sq(k-q),$$
$$\beta = 4pk^2(k-s)(k-s-q)(2k-p-q); \quad \text{or}$$
$$\phi_1(s) = 2\lambda vk(k-p-1)/(\lambda^2 v^2-k^2) + k\{v-2(k-p)-1\}/(\lambda v)$$
$$+ 2\lambda vk/(\lambda^2 v^2-2kp+p^2)$$
$$\text{for } s = k.$$

Remark 2.1. The above expressions of $\phi_1(s)$ for $1 \le s \le k-1$ and $s = k$ are derived through Lemmas 3.1 and 3.3 of Das and Kageyama (1992), respectively. In fact, Lemma 3.1 is given under $p \ge 1$, $k-p-q \ge 1$, $q \ge 1$, $k-s-q \ge 1$, $s-p \ge 1$ and $v-2k+p+q \ge 1$, while Lemma 3.3 is given under $p \ge 1$, $k-p \ge 1$ and $v-2k+p \ge 1$. These restrictions occur from some patterns of the C-matrix of the residual design d^*. These parametric restrictions form a subset of ranges (2.1). To cover all the ranges as in (2.1), we have to consider special cases of such restrictions. Simplified expressions of $\phi_1(s)$ for these special cases can be given by Lemmas 3.2.1, 3.2.2, 3.2.3 and 3.2.4 of Das and Kageyama (1992). {Incidentally, expressions derived in such a way may coincide with those given by formally omitting eigenvalues with negative values of multiplicity, as in the results of Lemma 3.1 of Das and Kageyama (1992), used to derive $\phi_1(s)$.} For such special cases Das and Kageyama (1992) implicitly have shown that our theorem described later holds. Thus our theorem is expressed under (2.1).

The following bound is now conjectured by Das and Kageyama (1992;Section 3.3). The result is described in terms of BIB designs, because an extended BIB design is obtained from a BIB design. Recall that B is the efficiency of residual designs.

THEOREM: In a BIB design with parameters v, b, r, k and λ,

$$A \geq B \geq C$$

holds for all s such that $1 \leq s \leq k$, where

$$A = \frac{\lambda v(v-1)+k(v-k)}{\lambda v(v-1)+k(v-k+1)},$$

$$C = \frac{(\lambda v-k)\{\lambda v(v-1)+k(v-k)\}}{\lambda^2 v^2(v-1)-k^2(v-2k+1)} \quad \text{(when } v \geq 2k\text{)}$$

or

$$(\lambda v-k)\{\lambda v(v-1)+k(v-k)\}(\lambda^2 v^2-2vk+v^2)/[2\lambda^2 v^2\{(v-k-1)\times(\lambda^2 v^2-2vk+v^2)+\lambda^2 v^2-k^2\}+(2k-v-1)(\lambda^2 v^2-k^2)(\lambda^2 v^2-2vk+v^2)]$$

(when $k+1 \leq v \leq 2k-1$)

with

$$0 \leq p \leq s, \quad \max\{0,2k-v-p\} \leq q \leq k-s.$$

In fact, $A = e_1(1,k-1)$ and $C = e_k(p^*,0)$ with $p^* = \max\{0,2k-v\}$.

The theorem shall be proved by separating the range of s into two cases as $1 \leq s \leq k-1$ and $s = k$, $k-1$.

3. Proof of the theorem for s = k, k-1

Proposition 3.1. In a BIBD(v,b,r,k,λ), an inequality $A \geq B \geq C$ holds for $s = k$ and $k-1$.

Case: s = k. Note that $q = 0$ in this case.

(I) *A proof of* $B \geq C$. Since $\max\{0,2k-v\} \leq p$, $0 \leq p$ and $2k-v \leq p$. Hence Theorem 3.4 of Das and Kageyama (1992) shows $B \geq C$.

(II) *A proof of* $A \geq B$.

(i) *Case* $p = k$: Here we get $\phi_1(k) = k(v-1)/(\lambda v)$. Hence $B = \{\lambda v(v-1)+k(v-k)\}/\{(\lambda v+k)(v-1)\}$. Thus, $A - B = k(k-2)\{\lambda v(v-1)+k(v-k)\}/[(\lambda v+k)(v-1)\{\lambda v(v-1)+k(v-k+1)\}] \geq 0$.

(ii) *Case* $\max\{0,2k-v\} \leq p \leq k-1$: Now

$$A - B = [\{\lambda v(v-1)+k(v-k)\}\{2(\lambda^2 v^2-k^2)p(2k-p)+(\lambda^2 v^2-2kp+p^2)\times\{\lambda vk(k-2)-k^2(2p-k)\}\}]/[\{\lambda v(v-1)+k(v-k+1)\}N]$$

where

$$N = (\lambda^2 v^2-2kp+p^2)(\lambda^2 v^3-3\lambda^2 v^2+2k^3-vk^2-2k^2 p+k^2)+2\lambda^2 v^2(\lambda^2 v^2-k^2).$$

It follows that the denominator of A-B is positive. Because (1) $\lambda v(v-1)+k(v-k+1) > 0$, (2) $\lambda^2 v^2-2kp+p^2 > 0$, and (3) $\lambda^2 v^3-3\lambda^2 v^2+2k^3-vk^2-2k^2 p+k^2 > 0$. Furthermore, the fact that the numerator of A-B is positive can be shown by the following relations:

(4) $2(\lambda^2v^2-k^2)p(2k-p) \geq 0$, $\lambda v(v-1)+k(v-k) > 0$;

(5) $\lambda^2v^2-2kp+p^2 = \lambda^2v^2-p(2k-p) > p(v-2k+p) \geq 0$;

(6) $\lambda vk(k-2)-k^2(2p-k) > k^2(k-2)-k^2(2p-k) = 2k^2(k-p-1) \geq 0$.

Thus, we get $A \geq B$ when $\max\{0,2k-v\} \leq p \leq k-1$. Therefore, by cases (i) and (ii), $A \geq B$ holds for $\max\{0,2k-v\} \leq p \leq k$.

Case: $s = k-1$. As noted in Remark 3.2 of Das and Kageyama (1992), the case of $s = k-1$ is actually equivalent to the case of $s = k$. In fact, we only have $q = 0$ (for $s = k$) and $q = 0$ or 1 (for $s = k-1$). Then, in B, $e_{k-1}(p,0) = e_k(p,0)$ and $e_{k-1}(p,1) = e_k(p+1,0)$ for $p = 0,1,...,k-1$. This implies that $A \geq B \geq C$ holds for $s = k-1$. The proof is thus completed. □

4. Proof of the theorem for $1 \leq s \leq k-2$

Lemma 4.1. In the following BIB designs, $A \geq B \geq C$ holds for all s such that $1 \leq s \leq k-2$: BIBD(v,b,r,k,λ) = (4,4,3,3,2), (5,5,4,4,3), (6,6,5,5,4), (7,7,3,3,1), (7,7,4,4,2), (11,11,6,6,3).

This can be checked by calculation of factors A, maximum and minimum of B, and C. Note that the existence of these BIB designs is well-known.

Lemma 4.2. In a BIBD(v,b,r,k,λ), $F > 0$ and $G > 0$ hold for all s such that $1 \leq s \leq k-2$, where

$$F = \lambda^4v^4(k-s)-\lambda^2v^2k(k-s)(k+2p)+\lambda^2v^2p(k-s)(p+2q) + \lambda^2v^2sq(k-q) + pk^2(k-s-q)(2k-p-q),$$

$$G = 2(\lambda^2v^2-k^2)\{\lambda^4v^4(k-s)-pk^2(k-s-q)(2k-p-q)\}$$

with $0 \leq p \leq s$ and $\max\{0,2k-v-p\} \leq q \leq k-s$.

Proof. It is clear that

$$\lambda^2v^2p(k-s)(p+2q)+\lambda^2v^2sq(k-q)+pk^2(k-s-q)(2k-p-q) \geq 0. \tag{4.1}$$

Some calculation shows that $\lambda^4v^4(k-s)-\lambda^2v^2k(k-s)(k+2p) > \lambda^2v^2(k-s)[k^2\{k(k-2)-2\}+2k] > 0$, since $k \geq 3$. This with (4.1) implies $F > 0$. Next,

$$\begin{aligned}&\lambda^4v^4(k-s)-pk^2(k-s-q)(2k-p-q)\\ &\geq (k-s)\{\frac{v^4r^4(k-1)^4}{(v-1)^4} - 2pk^3\}+pk^2(p+q)+pqk^2(2k-p-q)\\ &> (k-s)k^3(k-1)\{(k-1)^3-2\} > 0,\end{aligned}$$

since $r \geq k$, $p \leq k-1$, $2k-p-q \geq 0$ and $v \geq 3$. This with $\lambda^2v^2-k^2>0$ implies $G > 0$. □

Lemma 4.3. In a BIBD(v,b,r,k,λ) with $v \leq 2k-1$, an inequality $k \leq r \leq 2\lambda$ holds.

Proof. By $\lambda(v-1) = r(k-1)$, $\lambda v = rk+\lambda-r \le \lambda(2k-1)$, i.e. $(r-2\lambda)(k-1) \le 0$. Hence $r \le 2\lambda$. The other $r \ge k$ is well-known. □

Lemma 4.4. In a BIBD(v,b,r,k,λ), $H > 0$ holds for all s such that $1 \le s \le k-2$, where

$$H = 4(k-s)\{(\lambda^2v^3-3\lambda^2v^2-vk^2+k^3+k^2s-2k^2p-\lambda vk^2+\lambda vks+\lambda vk)F + G\}.$$

Proof. This is given by separating into two cases.

Case (I): $v \ge 2k$. Note that in this case $v \ge 6$. At first, by $k-s>0$ and Lemma 4.2, $F > 0$ and $G > 0$. Next, it holds that

$$\begin{aligned}&\lambda^2v^3-3\lambda^2v^2-vk^2+k^3+k^2s-2k^2p-\lambda vk^2+\lambda vks+\lambda vk\\ &\ge k^2(k-p)+k^2(s-p)+v[(\lambda v-k)\{\lambda(v-3)+k\}-\lambda k(k+1)] \quad (\text{by } s \ge 1)\\ &\ge k^2(k-p)+k^2(s-p)+(\lambda v-k)v\{\lambda(v-3)-1\} \quad (\text{by } v \ge 2k \text{ and } \lambda \ge 1)\end{aligned}$$

which is positive. These relations show that $H > 0$.

Case (II): $k+1 \le v \le 2k-1$. Note that $v \ge 4$. Since $k \ge 3$, at first we have $k-s > 0$ and, by Lemma 4.2, $F > 0$ and $G > 0$. Next,

$$\begin{aligned}&\lambda^2v^3-3\lambda^2v^2-vk^2+k^3+k^2s-2k^2p-\lambda vk^2+\lambda vks+\lambda vk\\ &= k^2(k-p)+k^2(s-p)+v\{\lambda^2v(v-3)-\lambda k^2+(\lambda s+\lambda-k)k\}\end{aligned}$$

$$(4.2)\quad > \lambda^2v(v-3)-\lambda k^2+(2\lambda-k)k \quad (\text{by } k > p,\ s \ge p,\ s \ge 1)$$

$$(4.3)\quad \ge 2\lambda^2[\{k(k-3)-2\}/2]. \quad (\text{by Lemma 4.3 with } k \le 2\lambda \text{ and } k+1 \le v)$$

When $k \ge 4$, (4.3) is positive. When $k = 3$ and $v = 4$, $(4.2) = \lambda(4\lambda-3)-9 > 0$. When $k = 3$ and $v = 5$, $(4.2) = \lambda(10\lambda-3)-9 > 0$. Thus, $H > 0$ when $k+1 \le v \le 2k-1$. Hence the proof is completed. □

Lemma 4.5. In a BIBD(v,b,r,k,λ) with $k+1 \le v \le 2k-1$, $K > 0$ holds, where $K = 2\lambda^2v^2\{(v-k-1)(\lambda^2v^2-2vk+v^2)+\lambda^2v^2-k^2\}+(2k-v-1)\times(\lambda^2v^2-k^2)(\lambda^2v^2-2vk+v^2)$.

Proof. Since $\lambda \ge 1$ and $v > k$, $\lambda^2v^2-2vk+v^2 = (\lambda^2+1)v^2-2vk > 0$ and $\lambda^2v^2-k^2 > 0$. Hence $K > 0$. □

Thus, since in the six BIB designs as in Lemma 4.1 our bounds hold, such six BIB designs are, in particular, excluded to prove the following Propositions 4.1, 4.2 and 4.3.

Proposition 4.1. In a BIBD(v,b,r,k,λ), an inequality $A \ge B$ holds for all s such that $1 \le s \le k-2$.

Proof. Since with H as in Lemma 4.4

$$(A-B)/\{\lambda v(v-1)+k(v-k)\}$$

$$= \frac{H - (\lambda v-k)\{\lambda v(v-1)+k(v-k+1)\}DE}{\{\lambda v(v-1)+k(v-k+1)\}H} \quad (= \alpha,\ \text{say}),$$

if $\alpha \ge 0$, then $A \ge B$. It is clear from Lemma 4.4 that the denominator of α is positive. Now denote the numerator of α by α'. So

$$\alpha'/\{4(k-s)\} = (-2\lambda^2v^2+k^2s-2k^2p+k^2+\lambda vks-\lambda vk)F + G.$$

(a) *Case $s = 1$* (and hence $p = 0$ or 1 and $q \le k-1$):

(1) When $p = 0$, $\alpha'/\{4(k-s)\} = 2\lambda^2v^2(\lambda^2v^2-k^2)(k^3-k^2-kq+q^2) \ge 2\lambda^2v^2(\lambda^2v^2-k^2)\{k^2(k-2)+k+q^2\} > 0$.

(2) When $p = 1$, $\alpha'/\{4(k-s)\} = (2k-q-1)(k-q-1)[2\lambda^2v^2\{v^2r^2(k-1)^2/(v-1)^2-2k^2\}+2k^4] \ge (2k-q-1)(k-q-1)[2\lambda^2v^2k^2\{k(k-2)-1\}+2k^4]$

which is non-negative, since $k \ge 3$ and $v \ge 4$. Thus $\alpha' \ge 0$.

(b) *Case $2 \le s \le k-2$* (and hence $k \ge 4$): Let

$$\alpha'/\{4(k-s)\} = 2(\lambda^2v^2-k^2)L + MF$$

where

$$L = \lambda^2v^2[\{(k-s)(k+p)-sq\}(k-q)+(k-s)(k-p)(p+q)] - 2pk^2(k-s-q)(2k-p-q),$$

$$M = k^2s-2k^2p-k^2+\lambda vks-\lambda vk.$$

(3) Since $\lambda^2v^2-k^2 > 0$, it follows from Lemma 4.2 that $F > 0$.

(4)
$$\begin{aligned} L &\ge \lambda^2v^2[\{(k-s)(k+p)-sq\}(k-q)+(k-s)(k-p)(p+q) - (k-s-q)(2k-p-q)] \\ &\ge \lambda^2v^2\{(k-s)(k+p)-sq+(k-p)(p+q)-(2k-p-q)\}(k-s-q) \\ &= \lambda^2v^2\{(k+p)(k-s-1)-k+(k+1)(p+q)-sq+p(1-p-q)\}(k-s-q) \\ &\ge 0. \end{aligned}$$

(5) $M \ge 0$ can be shown by considering four cases, $s \ge p+1$, $s = p$, $s \ge 3$ and $s = 2$, separately, after some algebra.

Thus, $\alpha' \ge 0$. Hence the proof is completed. □

Proposition 4.2. In a BIBD(v,b,r,k,λ) with $v \ge 2k$, an inequality $B \ge C$ holds for all s such that $1 \le s \le k-2$.

Proof. Since

$$(B-C)/[(\lambda v-k)\{\lambda v(v-1)+k(v-k)\}] = \frac{\{\lambda^2v^2(v-1)-k^2(v-2k+1)\}DE - H}{\{\lambda^2v^2(v-1)-k^2(v-2k+1)\}H} \quad (= \beta,\ \text{say})$$

if $\beta \ge 0$, then $B \ge C$. It follows from Lemma 4.4 that the denominator of β is positive. For, $\lambda^2v^2(v-1)-k^2(v-2k+1) > 2\lambda^2v^2(k-1) > 0$. Now denote the numerator of β by β'. Then, since $k \ge 3$ and hence $v \ge 6$,

$$\begin{aligned} \beta'/\{4(k-s)\} &= \lambda^4v^4(k^3+k^2-k^2s+2k^2p+\lambda vk^2-\lambda vks-\lambda vk)(k-s) + \{(\lambda^2v^2-k^2)+ \\ &\quad k^2(k-s)+2k^2p+\lambda vk(k-s)+\lambda v(\lambda v-k)\}\{\lambda^2v^2p(k-s)(p+2q) \\ &\quad -\lambda^2v^2k(k-s)(k+2p)+\lambda^2v^2sq(k-q)\} + pk^2\{3(\lambda^2v^2-k^2) \\ &\quad +k^2(k-s)+2k^2p+\lambda vk(k-s)+\lambda v(\lambda v-k)\}(k-s-q)(2k-p-q) \\ &\ge \lambda^4v^4(k^3+k^2-k^2s+2k^2p+\lambda vk^2-\lambda vks-\lambda vk)(k-s) - \lambda^2v^2k\{(\lambda^2v^2- \end{aligned}$$

$k^2)+k^2(k-s)+2k^2p+\lambda vk(k-s)+\lambda v(\lambda v-k)\}(k-s)(k+2p)$

$\geq \lambda^2v^2k\{\lambda^2v^2-k(k+2p)\}(\lambda v+k)(k-s-1)+2\lambda^2v^2kp\{\lambda^2v^2(k-2)-$

$k^2(k+2p)\}$

which is non-negative, since $\lambda^2v^2-k(k+2p) = v^2r^2(k-1)^2/(v-1)^2 - k(k+2p) > k^2\{k(k-2)-2\}+2k > 0$, and $\lambda^2v^2(k-2)-k^2(k+2p) = v^2r^2(k-1)^2(k-2)/(v-1)^2 - k^2(k+2p) > k^3\{(k-2)^2-2\}$ which can be shown to be positive, because six BIB designs as in Lemma 4.1 are excluded from consideration. Thus, $\beta' \geq 0$. Hence the result. □

Proposition 4.3. In a BIBD(v,b,r,k,λ) with $k+1 \leq v \leq 2k-1$, an inequality $B \geq C$ holds for all s such that $1 \leq s \leq k-2$.

Proof. Since with K as in Lemma 4.5

$$(B-C)/[(\lambda v-k)\{\lambda v(v-1)+k(v-k)\}] = \frac{KDE - (\lambda^2v^2-2vk+v^2)H}{HK} \quad (= \beta,\ \text{say}),$$

if $\beta \geq 0$, then $B \geq C$. It follows from Lemmas 4.4 and 4.5 that the denominator of β is positive. Now denote the numerator of β by β'. Then, since $k \geq 3$,

$$\begin{aligned}
&\beta'/\{4(k-s)\}\\
&= \lambda^4v^4[\{k(\lambda v-k)(k-s-1)+2k^2(v-k+p-s)\}(\lambda^2v^2-2vk+v^2)\\
&\quad +2v(\lambda^2v^2-k^2)(2k-v)](k-s) + [\{k(\lambda v-k)(k-s-1)\\
&\quad +2k^2(v-k+p-s)\}(\lambda^2v^2-2vk+v^2)+2\lambda^2v^2(\lambda^2v^2-k^2)]\{\lambda^2v^2p\\
&\quad \times(k-s)(p+2q)-\lambda^2v^2k(k-s)(k+2p)+\lambda^2v^2sq(k-q)\}\\
&\quad +pk^2[\{k(\lambda v-k)(k-s-1)+2k^2(v-k+p-s)\}(\lambda^2v^2-2vk+v^2)\\
&\quad +2(\lambda^2v^2-k^2)(2\lambda^2v^2-2vk+v^2)](k-s-q)(2k-p-q)\\
&\geq 2\lambda^2v^2(\lambda^2v^2-k^2)\{v(2k-v)-k(k+2p)+p(p+2q)\}\\
&\quad +k(\lambda v-k)(\lambda^2v^2-2vk+v^2)\{\lambda^2v^2-k(k+2p)\}\\
&= \lambda^2v^2(\lambda v-k)\{\lambda v(\lambda vk-2k^2-4kp)-2k^2(k+2p)\}\\
&\quad +\lambda^2v^2(\lambda v-k)\{2p(2\lambda vq-k^2)+2\lambda vp^2+2kp^2+4kpq\\
&\quad +vk(2k-v)\} + \lambda^2v^2(\lambda v-k)\{2\lambda v^2(2k-v)-k^3\}\\
&\quad +vk^2(\lambda v-k)(k+2p)(2k-v)
\end{aligned}$$

which can be shown to be positive, by noting the following relations through Lemma 4.3: (i) $\lambda^2v^2 > 0$, $\lambda v-k > 0$, $2k-v \geq 1$, (ii) $\lambda v(\lambda vk-2k^2-4kp)-2k^2(k+2p) > 0$, (iii) $2p(2\lambda vq-k^2)+2\lambda vp^2+2kp^2+4kpq+vk(2k-v) > 0$. Because the six BIB designs in Lemma 4.1 are excluded from our consideration. Thus, $\beta' \geq 0$. Hence the result. □

Therefore, Propositions 3.1, 4.1, 4.2 and 4.3 can show the validity of the theorem in Section 2. □□

5. Conclusion

In Sections 3 and 4, the conjecture of Das and Kageyama (1992) has been proved mathematically. This means, through Section 3.3 of Das and Kageyama (1992), that extended BIB designs are fairly robust against the unavailability of any number of observations in any one block.

Reference

Das, A. and Kageyama, S. (1992). Robustness of BIB and extended BIB designs against the unavailability of any number of observations in a block. Computational Statistics & Data Analysis **14** (to appear)