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Kyoto University
ON CERTAIN CLASSES OF MEROMORPHICALLY P-VALENT STARLIKE FUNCTIONS

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Abstract. Let $M_{n+p-1}(\alpha)$ denote the class of functions of the form

$$f(z) = \frac{1}{z^p} + \frac{a_0}{z^{p-1}} + \ldots + \frac{a_{k+p-1} z^k}{z^{k+p-1}} + \ldots \ (p \in \mathbb{N} = \{1, 2, 3, \ldots\})$$

that are regular in the annulus $D = \{z : 0 < |z| < 1\}$ and satisfy

$$\text{Re}\left\{\frac{z(D^{n+p-1}f(z))'}{D^{n+p-1}f(z)}\right\} < -\alpha$$

for $0 \leq \alpha < p$ and $|z| < 1$, where

$$D^{n+p-1}f(z) = \frac{1}{z^p} \left(\frac{z^{n+2p-1}f(z)}{(n+p-1)!}\right)^{(n+p-1)}.$$

We prove that $M_{n+p}(\alpha) \subset M_{n+p-1}(\alpha)$, where $n$ is any integer greater than $-p$. We also consider some integrals of functions in the class $M_{n+p-1}(\alpha)$.

1. Introduction

Let $\sum_p$ denote the class of functions of the form

$$f(z) = \frac{1}{z^p} + \frac{a_0}{z^{p-1}} + \ldots + \frac{a_{k+p-1} z^k}{z^{k+p-1}} + \ldots, \quad (1.1)$$

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which are regular in the annulus $D = \{z : 0 < |z| < 1\}$, where $p$ is a positive integer. The Hadamard product or convolution of two functions $f$ and $g$ in $\sum_p$ will be denoted by $f \ast g$. Let

$$D^{n+p-1}f(z) = \frac{1}{z^{p}(1-z)^{n+p}} \ast f(z), \quad (z \in D) \quad (1.2)$$

or, equivalently,

$$D^{n+p-1}f(z) = \frac{1}{z^{p}} \left( \frac{z^{n+2p-1}f(z)}{(n+p-1)!} \right)^{(n+p-1)}$$

$$= \frac{1}{z^{p}} + (n+p)a_{0}\frac{1}{z^{p-1}} + \frac{(n+p+1)(n+p)}{2!}a_{1}\frac{1}{z^{p-2}} + \ldots$$

$$\ldots + \frac{(n+k+2p-1)\ldots(n+p)}{(k+p)!}a_{k+p-1}z^{k} + \ldots \quad (z \in D),$$

where $n$ is any integer greater than $-p$.

In this paper, among other things, we shall show that a function $f(z)$ in $\sum_p$, which satisfies one of the conditions

$$\text{Re}\left\{ \frac{z(D^{n+p-1}f(z))'}{D^{n+p-1}f(z)} \right\} < -\alpha \quad (z \in U = \{z ; |z| < 1\}), \quad (1.3)$$

where $n$ is any integer greater than $-p$, is meromorphically $p$-valent starlike in $U$. More precisely, it is proved that, for the classes $M_{n+p-1}(\alpha)$ of functions in $\sum_p$ satisfying (1.3),

$$M_{n+p}(\alpha) \subset M_{n+p-1}(\alpha) \quad (0 \leq \alpha < p). \quad (1.4)$$

Since $M_0(\alpha)$ equals $\sum^*{\alpha}(\text{the class of meromorphically } p\text{-valent starlike functions of order } \alpha \text{ in } U \ [4])$, the starlikeness of members of $M_{n+p-1}(\alpha)$ is a consequence of (1.4).

2. Properties of the class $M_{n+p-1}(\alpha)$
In proving our main results, we shall need the following lemma due to Jack [3].

**Lemma.** Let \( w \) be non-constant regular in \( U = \{ z : |z| < 1 \} \), \( w(0) = 0 \). If \( |w| \) attains its maximum value on the circle \( |z| = r < 1 \) at \( z_0 \), we have \( z_0 w'(z_0) = kw(z_0) \), where \( k \) is a real number, \( k \geq 1 \).

**Theorem 1.** \( M_{n+p}(\alpha) \subset M_{n+p-1}(\alpha) \) for each integer \( n \) greater than \(-p\).

**Proof.** Let \( f(z) \in M_{n+p}(\alpha) \). Then

\[
\text{Re}\left\{ \frac{z(D^{n+p}f(z))'}{D^{n+p}f(z)} \right\} < -\alpha. \tag{2.1}
\]

We have to show that (2.1) implies the inequality

\[
\text{Re}\left\{ \frac{z(D^{n+p-1}f(z))'}{D^{n+p-1}f(z)} \right\} < -\alpha. \tag{2.2}
\]

Define \( w(z) \) in \( U \) by

\[
\frac{z(D^{n+p-1}f(z))'}{D^{n+p-1}f(z)} = \frac{-p + (2\alpha - p)w(z)}{1 + w(z)}. \tag{2.3}
\]

Clearly, \( w(z) \) is regular and \( w(0) = 0 \). Using the identity

\[
z(D^{n+p-1}f(z))' = (n + p)D^{n+p}f(z) - (n + 2p)D^{n+p-1}f(z), \tag{2.4}
\]

the equation (2.3) may be written as

\[
\frac{D^{n+p}f(z)}{D^{n+p-1}f(z)} = \frac{n + p + (n + 3p - 2\alpha)w(z)}{(n + p)(1 + w(z))}. \tag{2.5}
\]

Differentiating (2.5) logarithmically, we obtain

\[
\frac{z(D^{n+p}f(z))'}{D^{n+p}f(z)} = \frac{-p + (2\alpha - p)w(z)}{1 + w(z)} + \frac{2(p - \alpha)zw'(z)}{(1 + w(z))(n + p + (n + 3p - 2\alpha)w(z))}. \tag{2.6}
\]
We claim that $|w(z)| < 1$ in $U$. For otherwise (by Jack's lemma) there exists $z_0$ in $U$ such that
\[ z_0 w'(z_0) = kw(z_0), \]  
(2.7)
where $|w(z_0)| = 1$ and $k \geq 1$. The equation (2.6) in conjunction with (2.7) yields
\[ \frac{z_0(D^{n+p}f(z_0))'}{D^{n+p}f(z_0)} = -\frac{p + (2\alpha - p)w(z_0)}{1 + w(z_0)} + \frac{2(p - \alpha)kw(z_0)}{(1 + w(z_0))(n + p + (n + 3 - 2\alpha)w(z_0))}. \]  
(2.8)
Thus
\[ \text{Re}\left\{ \frac{z_0(D^{n+p}f(z_0))'}{D^{n+p}f(z_0)} \right\} \geq -\alpha + \frac{p - \alpha}{2(n + 2p - \alpha)} \geq -\alpha, \]  
(2.9)
which contradicts (2.1). Hence $|w(z)| < 1$ in $U$ and from (2.3) it follows that $f(z) \in M_{n+p-1}(\alpha)$.

**Theorem 2.** Let $f(z) \in \sum_p$ satisfy the condition
\[ \text{Re}\left\{ \frac{z(D^{n+p-1}f(z))'}{D^{n+p-1}f(z)} \right\} < -\alpha + \frac{p - \alpha}{2(c + p - \alpha)} (z \in U). \]  
(2.10)
for a given integer $n > -p$ and $c > 0$. Then
\[ F_c(z) = \frac{c}{z^{c+p}} \int_0^z t^{c+p-1}f(t)dt \]  
(2.11)
belongs to $M_{n+p-1}(\alpha)$.

**Proof.** Let $f(z) \in M_{n+p-1}(\alpha)$. Define $w(z)$ in $U$ by
\[ \frac{z(D^{n+p-1}F_c(z))'}{D^{n+p-1}F_c(z)} = -\frac{p + (2\alpha - p)w(z)}{1 + w(z)}. \]  
(2.12)
Clearly, $w(z)$ is regular and $w(0) = 0$. Using the identity
\[ z(D^{n+p-1}F_c(z))' = cD^{n+p-1}f(z) - (c + p)D^{n+p-1}F_c(z), \]  
(2.13)
the equation (2.12) may be written as

\[
\frac{z(D^{n+p-1}f(z))'}{D^{n+p-1}f(z)} = \frac{p + (2\alpha - p)w(z)}{1 + w(z)} + \frac{2(p - \alpha)zw'(z)}{(1 + w(z))(c + (c + 2p - 2\alpha)w(z))}.
\]

(2.14)

We claim that \(|w(z)| < 1\) in \(U\). For otherwise (by Jack’s lemma) there exists \(z_0\) in \(U\) such that

\[
z_0w'(z) = kw(z_0),
\]

(2.15)

where \(|w(z_0)| = 1\) and \(k \geq 1\). Combining (2.14) and (2.15), we obtain

\[
\frac{z_0(D^{n+p-1}f(z_0))'}{D^{n+p-1}f(z_0)} \geq -\alpha + \frac{p - \alpha}{2(c + p - \alpha)} \geq -\alpha,
\]

(2.16)

which contradicts (2.10). Hence \(|w(z)| < 1\) in \(U\) and from (2.12) it follows that \(F(z) \in M_{n+p-1}(\alpha)\).

Similarly, from Theorem 2, we have

**Corollary.** Let \(f(z) \in M_{n+p-1}(\alpha)\). Then \(F_c(z)\) defined by (2.11) belongs to the class \(M_{n+p-1}(\alpha)\).

**Remarks.** (1). A result of Bajpai[1] turns out to be a particular case of the above Theorem 2 when \(p = 1, n = 0, \alpha = 0\) and \(c = 1\).

(2). For \(p = 1, n = 0\) and \(\alpha = 0\), the above Theorem 2 extends a result of Goel and Sohi[2].

**Theorem 3.** Let \(f(z) \in M_{n+p-1}(\alpha)\). Then \(F_{n+p}(z)\) defined by (2.11) with \(c = n + p\) belongs to the class \(M_{n+p}(\alpha)\).

**Proof.** For the function \(F_{n+p}(z)\) defined by (2.11) with \(c = n + p\), we have

\[
cD^{n+p-1}f(z) = (n + p)D^{n+p}F_{n+p}(z) - (n + p - c)D^{n+p-1}F_{n+p}(z)
\]

(2.17)
Taking $c = n + p$ in the above relation (2.17), we obtain

$$D^{n+p-1}f(z) = D^{n+p}F_{n+p}(z).$$  \hspace{1cm} (2.18)

This implies that $F_{n+p}$ belongs to the class $M_{n+p-1}(\alpha)$.

**Theorem 4** Let $F_c(z) \in M_{n+p-1}(\alpha)$ and let $f(z)$ be defined as (2.11). Then $f(z) \in M_{n+p-1}(\alpha)$ in $|z| < R_c$, where

$$R_c = \frac{-(p - \alpha + 1) + \sqrt{(p - \alpha + 1)^2 + c(c + 2(p - \alpha))}}{c + 2(p - \alpha)}.$$  \hspace{1cm} (2.19)

**Proof.** Since $F_c(z) \in M_{n+p-1}(\alpha)$, we can write

$$\frac{z(D^{n+p-1}F_c(z))'}{D^{n+p-1}F_c(z)} = -(\alpha + (p - \alpha)u(z)),$$  \hspace{1cm} (2.20)

where $u(z) \in P$, the class of functions with positive real part in $U$ and normalized by $u(0) = 1$. Using the equation (2.13) and differentiating (2.20), we obtain

$$-\frac{z(D^{n+p-1}f(z))'}{D^{n+p-1}f(z)} + \frac{\alpha}{p - \alpha} = u(z) + \frac{zu'(z)}{(c + p) - (\alpha + (p - \alpha)u(z))}.$$  \hspace{1cm} (2.21)

Using the well known estimates, $\frac{|zu'(z)|}{Reu(z)} \leq \frac{2r}{1-r^2}(|z| = r)$ and $Reu(z) \leq \frac{1+r}{1-r^2}(|z| = r)$, the equation (2.21) yields

$$Re\left\{-\frac{z(D^{n+p-1}f(z))'}{D^{n+p-1}f(z)} + \frac{\alpha}{p - \alpha}\right\} \geq Reu(z)\left\{1 - \frac{2r}{(1-r^2)(c + p - (\alpha + (p - \alpha)\frac{1+r}{1-r})}\right\}.$$  \hspace{1cm} (2.22)

Now the right hand side of (2.22) is positive provided $r < R_c$. Hence $f(z) \in M_{n+p-1}(\alpha)$ for $|z| < R_c$. 

References


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