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ON CERTAIN CLASSES OF MEROMORPHICALLY P-VALENT STARLIKE FUNCTIONS

Nak Eun Cho (釜山水産大学)

Shigeyoshi Owa (近畿大・理工 尾和重義)

Abstract. Let $M_{n+p-1}(\alpha)$ denote the class of functions of the form

$$f(z)=\frac{1}{z^{p}}+\frac{a_{0}}{z^{p-1}}+\ldots+a_{k+p-1}z^{k}+\ldots\ (p\in N=\{1,2,3,\cdots\})$$

that are regular in the annulus $D=\{z:0<|z|<1\}$ and satisfy

$$\mathrm{Re}\{\frac{z(D^{n+p-1}f(z))'}{D^{n+p-1}f(z)}\}<-\alpha$$

for $0\leq\alpha<p$ and $|z|<1$, where

$$D^{n+p-1}f(z)=\frac{1}{z^{p}}(\frac{z^{n+2p-1}f(z)}{(n+p-1)!})^{(n+p-1)}.$$

We prove that $M_{n+p}(\alpha)\subset M_{n+p-1}(\alpha)$, where n is any integer greater than $-p$. We also consider some integrals of functions in the class $M_{n+p-1}(\alpha)$.

1. Introduction

Let $\sum_{p}$ denote the class of functions of the form

$$f(z)=\frac{1}{z^{p}}+\frac{a_{0}}{z^{p-1}}+\ldots+a_{k+p-1}z^{k}+\ldots, \tag{1.1}$$

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which are regular in the annulus $D = \{z : 0 < |z| < 1\}$, where p is a positive integer. The Hadamard product or convolution of two functions f and g in $\sum_p$ will be denoted by $f * g$. Let

$$D^{n+p-1}f(z) = \frac{1}{z^p(1-z)^{n+p}} * f(z), \quad (z \in D) \tag{1.2}$$

or, equivalently,

$$\begin{aligned} D^{n+p-1}f(z) &= \frac{1}{z^p}\Big(\frac{z^{n+2p-1}f(z)}{(n+p-1)!}\Big)^{(n+p-1)} \\ &= \frac{1}{z^p} + (n+p)a_0\frac{1}{z^{p-1}} + \frac{(n+p+1)(n+p)}{2!}a_1\frac{1}{z^{p-2}} + \dots \\ &\dots + \frac{(n+k+2p-1)\dots(n+p)}{(k+p)!}a_{k+p-1}z^k + \dots \ (z \in D), \end{aligned}$$

where n is any integer greater than $-p$.

In this paper, among other things, we shall show that a function $f(z)$ in $\sum_p$, which satisfies one of the conditions

$$\mathrm{Re}\Big\{\frac{z(D^{n+p-1}f(z))'}{D^{n+p-1}f(z)}\Big\} < -\alpha \ (z \in U = \{z : |z| < 1\}), \tag{1.3}$$

where n is any integer greater than $-p$, is meromorphically p-valent starlike in U. More precisely, it is proved that, for the classes $M_{n+p-1}(\alpha)$ of functions in $\sum_p$ satisfying (1.3),

$$M_{n+p}(\alpha) \subset M_{n+p-1}(\alpha) \ (0 \le \alpha < p). \tag{1.4}$$

Since $M_0(\alpha)$ equals $\sum^*(\alpha)$(the class of meromorphically p-valent starlike functions of order α in U [4]), the starlikeness of members of $M_{n+p-1}(\alpha)$ is a consequence of (1.4).

2. Properties of the class $M_{n+p-1}(\alpha)$

In proving our main results, we shall need the following lemma due to Jack [3].

Lemma. *Let w be non-constant regular in $U = \{z : |z| < 1\}$, $w(0) = 0$. If $|w|$ attains its maximum value on the circle $|z| = r < 1$ at z_0, we have $z_0 w'(z_0) = kw(z_0)$, where k is a real number, $k \geq 1$.*

Theorem 1. *$M_{n+p}(\alpha) \subset M_{n+p-1}(\alpha)$ for each integer n greater than $-p$.*

Proof. Let $f(z) \in M_{n+p}(\alpha)$. Then

$$\mathrm{Re}\left\{\frac{z(D^{n+p}f(z))'}{D^{n+p}f(z)}\right\} < -\alpha. \tag{2.1}$$

We have to show that (2.1) implies the inequality

$$\mathrm{Re}\left\{\frac{z(D^{n+p-1}f(z))'}{D^{n+p-1}f(z)}\right\} < -\alpha. \tag{2.2}$$

Define $w(z)$ in U by

$$\frac{z(D^{n+p-1}f(z))'}{D^{n+p-1}f(z)} = -\frac{p + (2\alpha - p)w(z)}{1 + w(z)}. \tag{2.3}$$

Clearly, $w(z)$ is regular and $w(0) = 0$. Using the identity

$$z(D^{n+p-1}f(z))' = (n+p)D^{n+p}f(z) - (n+2p)D^{n+p-1}f(z), \tag{2.4}$$

the equation (2.3) may be written as

$$\frac{D^{n+p}f(z)}{D^{n+p-1}f(z)} = \frac{n + p + (n + 3p - 2\alpha)w(z)}{(n+p)(1 + w(z))}. \tag{2.5}$$

Differentiating (2.5) logarithmically, we obtain

$$\frac{z(D^{n+p}f(z))'}{D^{n+p}f(z)} = -\frac{p + (2\alpha - p)w(z)}{1 + w(z)} + \frac{2(p-\alpha)zw'(z)}{(1 + w(z))(n + p + (n + 3p - 2\alpha)w(z))}. \tag{2.6}$$

We claim that $|w(z)| < 1$ in U. For otherwise (by Jack's lemma) there exists z_0 in U such that

$$z_0 w'(z_0) = k w(z_0), \tag{2.7}$$

where $|w(z_0)| = 1$ and $k \geq 1$. The equation (2.6) in conjuction with (2.7) yields

$$\frac{z_0(D^{n+p}f(z_0))'}{D^{n+p}f(z_0)} = -\frac{p+(2\alpha-p)w(z_0)}{1+w(z_0)} + \frac{2(p-\alpha)kw(z_0)}{(1+w(z_0))(n+p+(n+3p-2\alpha)w(z_0))}. \tag{2.8}$$

Thus

$$\operatorname{Re}\left\{\frac{z_0(D^{n+p}f(z_0))'}{D^{n+p}f(z_0)}\right\} \geq -\alpha + \frac{p-\alpha}{2(n+2p-\alpha)} \geq -\alpha, \tag{2.9}$$

which contradicts (2.1). Hence $|w(z)| < 1$ in U and from (2.3) it follows that $f(z) \in M_{n+p-1}(\alpha)$.

Theorem 2. *Let $f(z) \in \sum_p$ satisfy the condition*

$$\operatorname{Re}\left\{\frac{z(D^{n+p-1}f(z))'}{D^{n+p-1}f(z)}\right\} < -\alpha + \frac{p-\alpha}{2(c+p-\alpha)} \quad (z \in U). \tag{2.10}$$

for a given integer $n > -p$ and $c > 0$. Then

$$F_c(z) = \frac{c}{z^{c+p}} \int_0^z t^{c+p-1} f(t)dt \tag{2.11}$$

belongs to $M_{n+p-1}(\alpha)$.

Proof. Let $f(z) \in M_{n+p-1}(\alpha)$. Define $w(z)$ in U by

$$\frac{z(D^{n+p-1}F_c(z))'}{D^{n+p-1}F_c(z)} = -\frac{p+(2\alpha-p)w(z)}{1+w(z)}. \tag{2.12}$$

Clearly, $w(z)$ is regular and $w(0) = 0$. Using the identity

$$z(D^{n+p-1}F_c(z))' = cD^{n+p-1}f(z) - (c+p)D^{n+p-1}F_c(z), \tag{2.13}$$

the equation (2.12) may be written as

$$\frac{z(D^{n+p-1}f(z))'}{D^{n+p-1}f(z)} = -\frac{p+(2\alpha-p)w(z)}{1+w(z)} + \frac{2(p-\alpha)zw'(z)}{(1+w(z))(c+(c+2p-2\alpha)w(z))}. \tag{2.14}$$

We claim that $|w(z)| < 1$ in U. For otherwise (by Jack's lemma) there exists z_0 in U such that

$$z_0 w'(z) = kw(z_0), \tag{2.15}$$

where $|w(z_0)| = 1$ and $k \geq 1$. Combining (2.14) and (2.15), we obtain

$$\frac{z_0(D^{n+p-1}f(z_0))'}{D^{n+p-1}f(z_0)} \geq -\alpha + \frac{p-\alpha}{2(c+p-\alpha)} \geq -\alpha, \tag{2.16}$$

which contradicts (2.10). Hence $|w(z)| < 1$ in U and from (2.12) it follows that $F(z) \in M_{n+p-1}(\alpha)$.

Similarly, from Theorem 2, we have

Corollary. *Let $f(z) \in M_{n+p-1}(\alpha)$. Then $F_c(z)$ defined by* (2.11) *belongs to the class $M_{n+p-1}(\alpha)$.*

Remarks. (1). A result of Bajpai[1] turns out to be a particular case of the above Theorem 2 when $p = 1, n = 0, \alpha = 0$ and $c = 1$.
(2). For $p = 1$, $n = 0$ and $\alpha = 0$, the above Theorem 2 extends a result of Goel and Sohi[2].

Theorem 3. *Let $f(z) \in M_{n+p-1}(\alpha)$. Then $F_{n+p}(z)$ defined by* (2.11) *with $c = n+p$ belongs to the class $M_{n+p}(\alpha)$.*

Proof. For the function $F_{n+p}(z)$ defined by (2.11) with $c = n+p$, we have

$$cD^{n+p-1}f(z) = (n+p)D^{n+p}F_{n+p}(z) - (n+p-c)D^{n+p-1}F_{n+p}(z) \tag{2.17}$$

Taking $c = n + p$ in the above relation (2.17), we obtain

$$D^{n+p-1}f(z) = D^{n+p}F_{n+p}(z). \tag{2.18}$$

This implies that F_{n+p} belongs to the class $M_{n+p-1}(\alpha)$.

Theorem 4 *Let $F_c(z) \in M_{n+p-1}(\alpha)$ and let $f(z)$ be defined as (2.11). Then $f(z) \in M_{n+p-1}(\alpha)$ in $|z| < R_c$, where*

$$R_c = \frac{-(p-\alpha+1) + \sqrt{(p-\alpha+1)^2 + c(c+2(p-\alpha))}}{c+2(p-\alpha)}. \tag{2.19}$$

Proof. Since $F_c(z) \in M_{n+p-1}(\alpha)$, we can write

$$\frac{z(D^{n+p-1}F_c(z))'}{D^{n+p-1}F_c(z)} = -(\alpha + (p-\alpha)u(z)), \tag{2.20}$$

where $u(z) \in P$, the class of functions with positive real part in U and normalized by $u(0) = 1$. Using the equation (2.13) and differentiating (2.20), we obtain

$$-\frac{\frac{z(D^{n+p-1}f(z))'}{D^{n+p-1}f(z)} + \alpha}{p-\alpha} = u(z) + \frac{zu'(z)}{(c+p) - (\alpha + (p-\alpha)u(z))}. \tag{2.21}$$

Using the well known estimates, $\frac{|zu'(z)|}{Reu(z)} \leq \frac{2r}{1-r^2}(|z| = r)$ and $Reu(z) \leq \frac{1+r}{1-r}(|z| = r)$, the equation (2.21) yields

$$\mathrm{Re}\Big\{-\frac{\frac{z(D^{n+p-1}f(z))'}{D^{n+p-1}f(z)} + \alpha}{p-\alpha}\Big\} \geq \mathrm{Re}u(z)\Big\{1 - \frac{2r}{(1-r^2)(c+p-(\alpha+(p-\alpha)\frac{1+r}{1-r})}\Big\}. \tag{2.22}$$

Now the right hand side of (2.22) is positive provided $r < R_c$. Hence $f(z) \in M_{n+p-1}(\alpha)$ for $|z| < R_c$.

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Nak Eun Cho
Department of Applied Mathematics
College of Natural Sciences
National Fisheries University of Pusan
Pusan 608-737
Korea

Shigeyoshi Owa
Department of Mathematics
Kinki University
Higashi-Osaka, Osaka 577
Japan