

Generalized Classes of Multivalent Functions  
with Negative Coefficients

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1. Introduction and Representation

Let  $\Lambda_n(p)$  denote the class of functions of the form

$$f(z) = z^p - \sum_{k=p+n}^{\infty} a_k z^k \quad (a_k \geq 0, \text{ and } p, n \in \mathbb{N})$$

which are analytic and  $p$ -valent in the unit disk  $U = \{z : |z| < 1\}$ .

Let  $\Lambda_n(p; \{B_k\})$  denote the subclass of  $\Lambda_n(p)$  consisting of functions which satisfy the following inequality

$$\sum_{k=p+n}^{\infty} B_k \leq 1 \quad (B_k > 0, n \in \mathbb{N}).$$

By means of  $\Lambda_n(p; \{B_k\})$  with particular values of  $n$  or  $B_k$ , we have the following representation for subclasses of analytic and  $p$ -valent functions with negative coefficients ([1], [2], [3] and [5]).

$$(1.1) \quad T_p^*(A, B) = A_1 \left( p ; \left\{ \frac{(1+B)(k-p) + (B-A)p}{(B-A)p} \right\} \right) \quad ([1])$$

$$(1.2) \quad C_p(A, B) = A_1 \left( p ; \left\{ \frac{k[(1+B)(k-p) + (B-A)p]}{(B-A)p^2} \right\} \right) \quad ([1])$$

$$(1.3) \quad T_p(a, b) = A_1\left(p; \left\{\frac{k(1+b)}{2b(1-a)p}\right\}\right) \quad ([2]).$$

$$(1.4) \quad C_p(a, b) = A_1\left(p; \left\{\frac{k^2(1+b)}{2b(1-a)p^2}\right\}\right) \quad ([2]).$$

$$(1.5) \quad T^*(p, \alpha) = A_1\left(p; \left\{\frac{k-\alpha}{p-\alpha}\right\}\right) \quad ([3]).$$

$$(1.6) \quad C(p, \alpha) = A_1\left(p; \left\{\frac{k(k-\alpha)}{p(p-\alpha)}\right\}\right) \quad ([3]).$$

$$(1.7) \quad N_{p,m} = A_1\left(p; \left\{(2k-2p+1) \binom{m-1+k}{k-p}\right\}\right) \quad ([5]).$$

## 2. Distortion theorems

Theorem 1. If  $f(z) \in A_n(p; \{B_k\})$  and  $B_k \leq B_{k+1}$ , then we have

$$(2.1) \quad \text{Max} \left\{0, |z|^p - \frac{1}{B_{p+n}} |z|^{p+n}\right\} \leq |f(z)| \leq |z|^p + \frac{1}{B_{p+n}} |z|^{p+n}.$$

The estimate is sharp for the function

$$(2.2) \quad f(z) = z^p - \frac{1}{B_{p+n}} z^{p+n}.$$

Let  $p=1$  in Theorem 1. Then we have

Corollary 1 ([4], Theorem 1). If  $f(z) \in A_n(1; \{B_k\})$  and  $B_k \leq B_{k+1}$  then we have

$$(2.3) \quad \text{Max} \left\{0, |z| - \frac{1}{B_{n+1}} |z|^{n+1}\right\} \leq |f(z)| \leq |z| + \frac{1}{B_{n+1}} |z|^{n+1}.$$

The estimate is sharp for the function

$$(2.4) \quad f(z) = z - \frac{1}{B_{n+1}} z^{n+1}.$$

Let  $n=1$  in Theorem 1. Then we have

Corollary 2. If  $f(z) \in A_1(p; \{B_k\})$  and  $B_k \leq B_{k+1}$ , then we have

$$(2.5) \quad \text{Max} \left\{ 0, |z|^p - \frac{1}{B_{p+1}} |z|^{p+1} \right\} \leq |f(z)| \leq |z|^p + \frac{1}{B_{p+1}} |z|^{p+1}.$$

The estimate is sharp for the function

$$(2.6) \quad f(z) = z^p - \frac{1}{B_{p+1}} z^{p+1}.$$

Theorem 2. If  $f(z) \in A_n(p; k\{B_k\})$  and  $B_k \leq B_{k+1}$ , then we have

$$(2.7) \quad \text{Max} \left\{ 0, p |z|^{p-1} - \frac{1}{B_{p+n}} |z|^{p+n-1} \right\} \leq |f'(z)| \\ \leq p |z|^{p-1} + \frac{1}{B_{p+n}} |z|^{p+n-1}.$$

The estimate is sharp for the function

$$(2.8) \quad f(z) = z^p - \frac{1}{(p+n) B_{p+n}} z^{p+n}.$$

Let  $p=1$  in Theorem 2. Then we have

Corollary 3 ([4], Theorem 2). If  $f(z) \in A_n(1; k\{B_k\})$  and  $B_k \leq B_{k+1}$ , then we have

$$(2.9) \quad \text{Max} \left\{ 0, 1 - \frac{1}{B_{n+1}} |z|^n \right\} \leq |f'(z)| \leq 1 + \frac{1}{B_{n+1}} |z|^n.$$

The estimate is sharp for the function

$$(2.10) \quad f(z) = z - \frac{1}{(n+1)B_{n+1}} z^{n+1}.$$

Let  $n=1$  in Theorem 2. Then we have

Corollary 4. If  $f(z) \in \Lambda_1(p; k\{B_k\})$  and  $B_k \leq B_{k+1}$ , then we have

$$(2.11) \quad \text{Max} \left\{ 0, p|z|^{p-1} - \frac{1}{B_{p+1}} |z|^p \right\} \leq |f'(z)| \\ \leq p|z|^{p-1} + \frac{1}{B_{p+1}} |z|^{p+1}.$$

The estimate is sharp for the function

$$(2.12) \quad f(z) = z^p - \frac{1}{(p+1)B_{p+n}} z^{p+1}.$$

We now show the distortion theorems for  $f^{(j)}(z)$  ( $2 \leq j \leq n+1$ ). First we need the following lemma.

Lemma 1. Let

$$\prod_{i=1}^j (k-1+i) = \sum_{i=1}^j A_i k^i \quad (j \geq 2, k \in N).$$

Then

$$\sum_{i=1}^j A_i (n+p)^{i-1} = \prod_{i=2}^j (n+p-1+i) \quad (n, p \in N).$$

The proof is the same as that of Lemma 1([4]).

Putting  $p=1$  in above Lemma 1, we get Lemma 1([4]).

Using Lemma 1, we have

Theorem 3. If  $f(z) \in \Lambda_n(p; k^q\{B_k\})$ ,  $B_k \leq B_{k+1}$  and  $2 \leq q \leq p+n$ , then we have

$$(2.13) \text{ (I) } |f^{(j)}(z)| \leq \prod_{i=1}^j (p+1-i) |z|^{p-j} + \frac{\prod_{i=2}^j (n+p-1+i)}{(p+n)^{q-1} B_{p+n}} |z|^{p+n-j},$$

$$|f^{(j)}(z)| \geq \prod_{i=1}^j (p+1-i) |z|^{p-j} - \frac{\prod_{i=2}^j (n+p-1+i)}{(p+n)^{q-1} B_{p+n}} |z|^{p+n-j},$$

(2 \leq j \leq p),

$$(2.14) \text{ (II) } |f^{(j)}(z)| \leq \frac{\prod_{i=2}^j (n+p-1+i)}{(p+n)^{q-1} B_{p+n}} |z|^{p+n-j}$$

(p+1 \leq j \leq p+n).

Let  $p=1$  in Theorem 3 (II). Then we have

Corollary 5 ([4], Theorem 3). If  $f(z) \in A_n \in (1; k^q \{B_k\})$ ,  $B_k \leq B_{k+1}$  and  $2 \leq q \leq n+1$ , then we have

$$(2.15) \quad q |f^{(j)}(z)| \leq \frac{\prod_{i=2}^j (n+i)}{(n+1)^{q-1} B_{n+1}} |z|^{n-j+1} \quad (2 \leq j \leq n+1).$$

Let  $n=1$  in Theorem 3. Then we have

Corollary 6. If  $f(z) \in A_1(p; k^q \{B_k\})$ ,  $B_k \leq B_{k+1}$  and  $2 \leq q \leq p+1$ , then we have

$$(2.16) \quad |f^{(j)}(z)| \leq \prod_{i=1}^j (p+1-i) |z|^{p-j} + \frac{\prod_{i=2}^j (p+i)}{(p+1)^{q-1} B_{p+1}} |z|^{p-j+1},$$

$$|f^{(j)}(z)| \geq \prod_{i=1}^j (p+1-i) |z|^{p-j} - \frac{\prod_{i=2}^j (p+i)}{(p+1)^{q-1} B_{p+1}} |z|^{p-j+1},$$

(2 \leq j \leq p),

$$|f(j)(z)| \leq \frac{\prod_{i=2}^j (p+i)}{(p+1)^{j-1} B_{p+1}} \quad (j=p+1).$$

Remark 1. Substituting some values for  $B_{p+1}$  in Corollary 2, we have the following estimates of  $f(z)$ .

(2.17) If  $f(z) \in T_p^*(A, B)$   $\left( B_{p+1} = \frac{1+B+(B-A)p}{(B-A)p} \right)$ , then

$$|z|^p - \frac{(B-A)p}{1+B+(B-A)p} |z|^{p+1} \leq |f(z)|$$

$$\leq |z|^p + \frac{(B-A)p}{1+B+(B-A)p} |z|^{p+1}$$

([1], Theorem 3).

(2.18) If  $f(z) \in C_p(A, B)$   $\left( B_{p+1} = \frac{(p+1)\{1+B+(B-A)p\}}{(B-A)p^2} \right)$ ,

then

$$|z|^p - \frac{(B-A)p^2}{(p+1)\{1+B+(B-A)p\}} |z|^{p+1} \leq |f(z)|$$

$$\leq |z|^p + \frac{(B-A)p^2}{(p+1)\{1+B+(B-A)p\}} |z|^{p+1}$$

(cf. [1], Theorem 4).

(2.19) If  $f(z) \in T_p(a, b)$   $\left( B_{p+1} = \frac{(p+1)(1+b)}{2b(1-a)p} \right)$ , then

$$|z|^p - \frac{2b(1-b)p}{(p+1)(1+b)} |z|^{p+1} \leq |f(z)|$$

$$\leq |z|^p + \frac{2b(1-b)p}{(p+1)(1+b)} |z|^{p+1}$$

([2], Theorem 6).

(2.20) If  $f(z) \in C_p(a, b)$   $\left( B_{p+1} = \frac{(p+1)^2(1+b)}{2b(1-a)p^2} \right)$ , then

$$\begin{aligned} |z|^p - \frac{2b(1-b)p^2}{(p+1)^2(1+b)} |z|^{p+1} &\leq |f(z)| \\ &\leq |z|^p + \frac{2b(1-b)p^2}{(p+1)^2(1+b)} |z|^{p+1} \end{aligned}$$

([2], Theorem 7).

(2.21) If  $f(z) \in T^*(p, \alpha)$   $\left( B_{p+1} = \frac{p+1-\alpha}{p-\alpha} \right)$ , then

$$|z|^p - \frac{p-\alpha}{p+1-\alpha} |z|^{p+1} \leq |f(z)| \leq |z|^p + \frac{p-\alpha}{p+1-\alpha} |z|^{p+1}$$

([3], Theorem 3.1).

(2.22) If  $f(z) \in C(p, \alpha)$   $\left( B_{p+1} = \frac{(p+1)(p+1-\alpha)}{p(p-\alpha)} \right)$ , then

$$\begin{aligned} |z|^p - \frac{p(p-\alpha)}{(p+1)(p+1-\alpha)} |z|^{p+1} &\leq |f(z)| \\ &\leq |z|^p + \frac{p(p-\alpha)}{(p+1)(p+1-\alpha)} |z|^{p+1} \end{aligned}$$

([3], Theorem 3.2).

(2.23) If  $f(z) \in N_{p,m}$   $(B_{p+1} = 3(m+p))$ , then

$$|z|^p - \frac{1}{3(m+p)} |z|^{p+1} \leq |f(z)| \leq |z|^p + \frac{1}{3(m+p)} |z|^{p+1} \quad ([5]).$$

Remark 2. Substituting some values for  $B_{p+1}$  in Corollary 4, we have the following estimates of  $f'(z)$

$$(2.24) \text{ If } f(z) \in T_p^*(A, B) \left( B_{p+1} = \frac{1+B+(B-A)p}{(p+1)(B-A)p} \right)$$

$$\begin{aligned} p|z|^{p-1} - \frac{(p+1)(B-A)p}{1+B+(B-A)p} |z|^p &\leq |f'(z)| \\ &\leq p|z|^{p-1} + \frac{(p+1)(B-A)p}{1+B+(B-A)p} |z|^p \end{aligned}$$

([1], Theorem 3).

$$(2.25) \text{ If } f(z) \in C_p(A, B) \left( B_{p+1} = \frac{1+B+(B-A)p}{(B-A)p^2} \right), \text{ then}$$

$$\begin{aligned} p|z|^{p-1} - \frac{(B-A)p^2}{1+B+(B-A)p} |z|^p &\leq |f'(z)| \\ &\leq p|z|^{p-1} + \frac{(B-A)p^2}{1+B+(B-A)p} |z|^p \end{aligned}$$

(cf. [1], Theorem 4).

$$(2.26) \text{ If } f(z) \in T_p(a, b) \left( B_{p+1} = \frac{1+b}{2b(1-a)p} \right), \text{ then}$$

$$\begin{aligned} p|z|^{p-1} - \frac{2b(1-a)p}{1+b} |z|^p &\leq |f'(z)| \\ &\leq p|z|^{p-1} + \frac{2b(1-a)p}{1+b} |z|^p \end{aligned}$$

([2], Theorem 6).

$$(2.27) \text{ If } f(z) \in C_p(a, b) \left( B_{p+1} = \frac{(p+1)(1+b)}{2b(1-a)p^2} \right), \text{ then}$$

$$\begin{aligned} p|z|^{p-1} - \frac{2b(1-a)p^2}{(p+1)(1+b)} |z|^p &\leq |f'(z)| \\ &\leq p|z|^{p-1} + \frac{2b(1-a)p^2}{(p+1)(1+b)} |z|^p \end{aligned}$$



([2], Theorem 7).

(2.28) If  $f(z) \in T^*(p, \alpha)$   $\left( B_{p+1} = \frac{p+1-\alpha}{(p+1)(p-\alpha)} \right)$ , then

$$\begin{aligned} p|z|^{p-1} - \frac{(p+1)(p-\alpha)}{p+1-\alpha} |z|^p &\leq |f'(z)| \\ &\leq p|z|^{p-1} + \frac{(p+1)(p-\alpha)}{p+1-\alpha} |z|^p. \end{aligned}$$

([3], Theorem 3.1).

(2.29) If  $f(z) \in C(p, \alpha)$   $\left( B_{p+1} = \frac{p+1-\alpha}{p(p-\alpha)} \right)$ , then

$$\begin{aligned} p|z|^{p-1} - \frac{p(p-\alpha)}{p+1-\alpha} |z|^p &\leq |f'(z)| \\ &\leq p|z|^{p-1} + \frac{p(p-\alpha)}{p+1-\alpha} |z|^p. \end{aligned}$$

([3], Theorem 3.2).

(2.30) If  $f(z) \in N_{p,m}$   $\left( B_{p+1} = \frac{3(m+p)}{p+1} \right)$ , then

$$p|z|^{p-1} - \frac{p+1}{3(m+p)} |z|^p \leq |f'(z)| \leq p|z|^{p-1} + \frac{p+1}{3(m+p)} |z|^p$$

([6]).

**Remark 3.** Substituting some values for  $B_{p+1}$  in Corollary 6, we have the following estimates of  $f''(z)$

(2.31) If  $f(z) \in C_p(A, B)$   $\left( B_{p+1} = \frac{1+B+(B-A)p}{(p+1)(B-A)p^2} \right)$ , then

$$p(p-1)|z|^{p-2} - \frac{(p+2)(B-A)p^2}{1+B+(B-A)p}|z|^{p-1} \leq |f''(z)|$$

$$\leq p(p-1)|z|^{p-2} + \frac{(p+2)(B-A)p^2}{1+B+(B-A)p}|z|^{p-1} \quad (p \geq 2),$$

$$|f''(z)| \leq \frac{3(B-A)}{1+2B-A} \quad (p=1).$$

(2.32) If  $f(z) \in C_p(a, b)$   $\left( B_{p+1} = \frac{1+b}{2b(1-a)p^2} \right)$ , then

$$p(p-1)|z|^{p-2} - \frac{2(p+2)b(1-a)p^2}{(p+1)(1+b)}|z|^{p-1} \leq |f''(z)|$$

$$\leq p(p-1)|z|^{p-2} + \frac{2(p+2)b(1-a)p^2}{(p+1)(1+b)}|z|^{p-1} \quad (p \geq 2)$$

(cf. [2], Theorem 7).

$$|f''(z)| \leq \frac{3b(1-a)}{1+b} \quad (p=1).$$

(2.33) If  $f(z) \in C(p, \alpha)$   $\left( B_{p+1} = \frac{p+1-\alpha}{(p+1)p(p-\alpha)} \right)$ , then

$$p(p-1)|z|^{p-2} - \frac{p(p+2)(p-\alpha)}{p+1-\alpha}|z|^{p-1} \leq |f''(z)|$$

$$\leq p(p-1)|z|^{p-2} + \frac{p(p+2)(p-\alpha)}{p+1-\alpha}|z|^{p-1} \quad (p \geq 2)$$

(cf. [3], Theorem 3.3).

$$|f''(z)| \leq \frac{3(1-\alpha)}{2-\alpha} \quad (p=1) \quad ([4], (3.4)).$$

#### References

- [1]. R.M. Goel and N.S. Sohi, Multivalent functions with negative coefficients, Indian j. pure appl. Math., 12(1981), 844-853.
- [2]. S. Owa, On certain subclasses of analytic p-valent functions J. Korean Math. Soc., 20(1983), 41-58.
- [3]. S. Owa, On certain classes of p-valent functions with negative coefficients, SIMON STEVIN, 4(1985), 385-402.
- [4]. T. Sekine, On new generalized classes of analytic functions with negative coefficients, Report Res. Inst. Sci. Tech. Nihon Univ., 32(1987), 1-26.
- [5]. T. Sekine, On certain classes of p-valent functions with negative coefficients(負の係数をもつあるp-valentな関数の族について), Presented at the 1992 Annual Meeting of the Mathematical Society of Japan, Fukuoka, Japan 1992.

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