A generalization class of certain subclasses of \( p \)-valently analytic functions with negative coefficients*

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Abstract
Recently we [5] have discussed a new generalization class \( A(n, \alpha, \beta) \) of certain subclasses of analytic functions with negative coefficients in the unit disk and have proved some properties of functions belonging to the class \( A(n, \alpha, \beta) \). In the present paper we introduce a new generalization class \( A_p(n, \alpha, \beta) \) of certain subclasses of \( p \)-valently analytic functions with negative coefficients in the unit disk and discuss some properties of functions belonging to the class \( A_p(n, \alpha, \beta) \).

1. Introduction
Let \( p \) be a positive integer, and let \( A_p(n) \) denote the class of functions of the form

\[
(1.1) \quad f(z) = z^p - \sum_{h=n+p}^{\infty} a_h z^h \quad (a_h \geq 0, \ n \in N = \{1, 2, 3, \cdots\}),
\]

which are analytic in the unit disk \( U = \{z : |z| < 1\} \).
A function \( f(z) \) in the class \( A_p(n) \) is said to be a member of the class \( R_p(n, \alpha) \) if it satisfies

\[
(1.2) \quad \Re \left\{ \frac{pf(z)}{z^p} \right\} > \alpha \quad (z \in U)
\]

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for some $\alpha(0 \leq \alpha < p)$. Further, a function $f(z)$ in the class $A_p(n)$ is said to be in the class $P_p(n, \alpha)$ if it satisfies

\begin{equation}
\text{Re} \left\{ \frac{f'(z)}{z^{p-1}} \right\} > \alpha \quad (z \in U)
\end{equation}

for some $\alpha(0 \leq \alpha < p)$.

By generalization of some results due to Sarangi and Uralegaddi [2], we see that

**Lemma A.** A function $f(z) \in A_p(n)$ is in the class $R_p(n, \alpha)$ if and only if

\begin{equation}
\sum_{h=n+p}^{\infty} \frac{p}{p-\alpha} a_h \leq 1.
\end{equation}

**Lemma B.** A function $f(z) \in A_p(n)$ is in the class $P_p(n, \alpha)$ if and only if

\begin{equation}
\sum_{h=n+p}^{\infty} \frac{k}{p-\alpha} a_h \leq 1.
\end{equation}

Now, we define

**Definition.** Suppose that $f(z) \in A_p(n), 0 \leq \alpha < p$ and $\beta \geq 0$. Then the function $f(z)$ is said to be a member of the class $A_p(n, \alpha, \beta)$ if it satisfies

\begin{equation}
\text{Re} \left\{ (1 - \beta) \frac{pf(z)}{z^p} + \beta \frac{f'(z)}{z^{p-1}} \right\} > \alpha \quad (z \in U).
\end{equation}

We note that $A_p(n, \alpha, 0) = R_p(n, \alpha)$ and $A_p(n, \alpha, 1) = P_p(n, \alpha)$. We have
Lemma 1. Suppose that \( f(z) \in A_p(n) \), \( 0 \leq \alpha < p \) and \( \beta \geq 0 \). Then the function \( f(z) \) is in the class \( A_p(n, \alpha, \beta) \) if and only if

\[
\sum_{h=n+p}^{\infty} \left\{ \frac{(1-\beta)p + \beta k}{p - \alpha} \right\} a_h \leq 1.
\]

Proof: Let \( f(z) \in A_p(n, \alpha, \beta) \). Then we have, by (1.6),

\[
\text{Re} \left\{ (1-\beta) \frac{pf(z)}{z^p} + \beta \frac{f'(z)}{z^{p-1}} \right\} = \text{Re} \left\{ p - \sum_{h=n+p}^{\infty} \{(1-\beta)p + \beta k\} a_h z^{h-p} \right\} > \alpha \quad (z \in U).
\]

Letting \( z \to 1 \) through real values, we obtain (1.7). Conversely, let \( f(z) \in A_p(n) \) satisfy inequality (1.7). Then we have

\[
\left| \left\{ (1-\beta) \frac{pf(z)}{z^p} + \beta \frac{f'(z)}{z^{p-1}} \right\} - p \right| = \left| \sum_{h=n+p}^{\infty} \{(1-\beta)p + \beta k\} a_h z^{h-p} \right| \leq \sum_{h=n+p}^{\infty} \{(1-\beta)p + \beta k\} a_h |z|^{h-p} < p - \alpha \quad (z \in U).
\]

This proves that inequality (1.6) holds true.

The class \( A_1(n, \alpha, \beta) \) is a special case \( B_k = \frac{1+(k-1)\beta}{1-\alpha} \) of the class \( A(n, B_k) \) introduced by Sekine [3].

2. Distortion Theorem
**Theorem 1.** If \( f(z) \in A_p(n, \alpha, \beta) \) for \( 0 \leq \alpha < p \) and \( \beta \geq 0 \), then

\[
|z|^p - \frac{p - \alpha}{p + n\beta}|z|^{n+p} \leq |f(z)| \leq |z|^p + \frac{p - \alpha}{p + n\beta}|z|^{n+p} \quad (z \in U)
\]

for \( \beta \geq 0 \), and

\[
|f'(z)| \leq p|z|^{p-1} + \frac{(p - \alpha)(n + p)}{p + n\beta}|z|^{n+p-1} \quad (z \in U)
\]

\[
|f'(z)| \geq p|z|^{p-1} - \frac{(p - \alpha)(n + p)}{p + n\beta}|z|^{n+p-1} \quad (z \in U)
\]

for \( \beta \geq 1 \). The equalities in (2.1) and (2.2) are attained for the function

\[
f(z) = z^p - \frac{p - \alpha}{p + n\beta}z^{n+p}.
\]

**Proof:** Note that

\[
\sum_{k=n+p}^{\infty} a_k \leq \frac{p - \alpha}{p + n\beta} \quad (\beta \geq 0)
\]

and

\[
\frac{p + n\beta}{n + p} \sum_{k=n+p}^{\infty} ka_k \leq \sum_{k=n+p}^{\infty} \{(1 - \beta)p + \beta k\}a_k \leq p - \alpha \quad (\beta \geq 1)
\]

for \( f(z) \in A_p(n, \alpha, \beta) \). Therefore, we have (2.1) and (2.2). □

**Remark.** Putting \( p = 1 \) in Theorem 1, we have the corresponding result due to Yaguchi, Sekine, Saitoh, Owa, Nunokawa and Fukui [5].

3. Inclusion Relations

**Theorem 2.** If

\[
0 \leq \alpha_1 < p, \quad 0 \leq \alpha_2 < p,
\]

\[
0 \leq \beta_1, \quad 0 \leq \beta_2, \quad p(\beta_1 - \beta_2) < \alpha_2\beta_1 - \alpha_1\beta_2,
\]

\[
p\{\alpha_1 - \alpha_2 + (\beta_1 - \beta_2)n\} \leq n(\alpha_2\beta_1 - \alpha_1\beta_2),
\]
then we have

\[(3.2) \quad A_p(n, \alpha_2, \beta_2) \subsetneq A_p(n, \alpha_1, \beta_1).\]

**Proof:** Suppose \(f(z) \in A_p(n, \alpha_2, \beta_2).\) Since by Lemma 1

\[(3.3) \quad \sum_{k=n+p}^{\infty} \frac{(1-\beta_2) p + k\beta_2}{p-\alpha_2} a_k \leq 1,
\]
we have only to prove the inequality

\[(3.4) \quad \frac{(1-\beta_1) p + k\beta_1}{p-\alpha_1} \leq \frac{(1-\beta_2) p + k\beta_2}{p-\alpha_2} \quad (k \geq n+p),
\]
which is equivalent to the inequality

\[(3.5) \quad k \geq \frac{p\{(\beta_2 - \beta_1) p + \alpha_1 - \alpha_2 + \alpha_2\beta_1 - \alpha_1\beta_2\}}{(\beta_2 - \beta_1) p + \alpha_2\beta_1 - \alpha_1\beta_2} \quad (k \geq n+p).
\]
But conditions (3.1) lead to the inequality

\[(3.6) \quad \frac{p\{(\beta_2 - \beta_1) p + \alpha_1 - \alpha_2 + \alpha_2\beta_1 - \alpha_1\beta_2\}}{(\beta_2 - \beta_1) p + \alpha_2\beta_1 - \alpha_1\beta_2} \leq n+p,
\]
which proves (3.5). The function \(f_0(z)\) defined by

\[(3.7) \quad f_0(z) = z^p - \frac{p - \alpha_1}{p + (n+1)\beta_1} z^{p+n+1}
\]
belongs to the class \(A_p(n, \alpha_1, \beta_1) - A_p(n, \alpha_2, \beta_2),\) which proves

\[(3.8) \quad A_p(n, \alpha_1, \beta_1) \neq A_p(n, \alpha_2, \beta_2).\]
**Corollary 1.** If

\[(3.9) \ 0 \leq \alpha_1 \leq \alpha_2 < p, \quad 0 \leq \beta_1 \leq \beta_2, \quad (\beta_2 - \beta_1) + (\alpha_2 - \alpha_1) > 0,\]

then we have

\[(3.10) \ A_p(n, \alpha_2, \beta_2) \subsetneq A_p(n, \alpha_1, \beta_1)\]

**Proof:** By Theorem 2, we have

\[(3.11) \quad A_p(n, \alpha_2, \beta_1) \subsetneq A_p(n, \alpha_1, \beta_1) \quad (0 \leq \alpha_1 < \alpha_2 < p),\]

\[A_p(n, \alpha_2, \beta_2) \subsetneq A_p(n, \alpha_1, \beta_1) \quad (0 \leq \beta_1 < \beta_2),\]

which prove Corollary 1. 

**Corollary 2.** If \(0 < \beta_1 < 1 < \beta_2\), then

\[(3.12) \ A_p(n, \alpha, \beta_2) \subsetneq P_p(n, \alpha) \subsetneq A_p(n, \alpha, \beta_1) \subsetneq R_p(n, \alpha).\]

4. **Starlikeness**

A function \(f(z)\) in the class \(A_p(n)\) is said to be \(p\)-valently starlike of order \(\alpha\) if it satisfies

\[(4.1) \quad \text{Re} \frac{zf'(z)}{f(z)} > \alpha \quad (z \in U)\]

for some \(\alpha(0 \leq \alpha < p)\). We need the following lemma which is a generalization of a result due to Chatterjea [1] (also Srivastava, Owa and Chatterjea [4]).

**Lemma C.** A function \(f(z) \in A_p(n)\) is \(p\)-valently starlike of order \(\gamma\) if and only if

\[(4.2) \quad \sum_{h=n+P}^{\infty} \frac{k-\gamma}{p-\gamma} a_h \leq 1\]

for some \(\gamma(0 \leq \gamma < p)\).

Lemma C is proved by using the similar method as in Chatterjea [1]. Using Lemma C, we have
**Theorem 3.** If \( f(z) \in A_p(n, \alpha, \beta) \) for \( 0 \leq \alpha < p \) and \( \beta \geq 1 \), then \( f(z) \) is starlike of order \((1 - \frac{1}{\beta})p\).

**Proof:** It follows from \( f(z) \in A_p(n, \alpha, \beta) \) that

\[
(4.3) \quad \sum_{k=n+p}^{\infty} \left\{ k - \left(1 - \frac{1}{\beta}\right)p \right\} a_k \leq \frac{p - \alpha}{\beta} \leq p - \left(1 - \frac{1}{\beta}\right)p.
\]

Therefore, by Lemma C, we have the assertion of Theorem 3. \( \blacksquare \)

5. **Quasi-Hadamard product**

For functions \( f_1(z) \) and \( f_2(z) \) defined by

\[
(5.1) \quad f_j(z) = z^p - \sum_{h=n+p}^{\infty} a_{j,h} z^h \quad (a_{j,h} \geq 0, \ n \in N, \ j = 1, 2)
\]

in the class \( A_p(n) \), we denote by \( f_1 \ast f_2(z) \) the quasi-Hadamard product of functions \( f_1(z) \) and \( f_2(z) \), that is,

\[
(5.2) \quad f_1 \ast f_2(z) = z^p - \sum_{h=n+p}^{\infty} a_{1,h} a_{2,h} z^h.
\]

**Theorem 4.** If \( f_j(z) \in A_p(n, \alpha_j, \beta) \) for \( 0 \leq \alpha_j < p, \beta \geq 0 \) and \( j = 1, 2 \), then \( f_1 \ast f_2(z) \in A_p(n, \alpha, \beta) \), where

\[
(5.3) \quad \alpha = p - \frac{(p - \alpha_1)(p - \alpha_2)}{p + \beta n}.
\]

The result is sharp for functions \( f_1(z) \) and \( f_2(z) \) defined by

\[
(5.4) \quad f_j(z) = z^p - \frac{z}{p + \beta n} \quad (j = 1, 2).
\]

**Proof:** We have to find the largest \( \alpha \) such that

\[
(5.5) \quad \sum_{h=n+p}^{\infty} \frac{(1 - \beta)p + \beta k}{p - \alpha} a_{1,h} a_{2,h} \leq 1.
\]
For functions $f_j(z) \in A_p(n, \alpha_j, \beta)$, we have

\begin{equation}
\sum_{h=n+p}^{\infty} \left\{ \frac{(1-\beta)p+\beta k}{p-\alpha} \right\} a_{j,h} \leq 1 \quad (j = 1, 2).
\end{equation}

By the Cauchy-Schwarz inequality, inequality (5.6) lead to the inequality

\begin{equation}
\sum_{h=n+p}^{\infty} \frac{(1-\beta)p+\beta k}{\sqrt{(p-\alpha_1)(p-\alpha_2)}} \sqrt{a_{1,h}a_{2,h}} \leq 1.
\end{equation}

Therefore, it is sufficient to prove that

\begin{equation}
\frac{(1-\beta)p+\beta k}{p-\alpha} a_{1,h}a_{2,h} \leq \frac{(1-\beta)p+\beta k}{\sqrt{(p-\alpha_1)(p-\alpha_2)}} \sqrt{a_{1,h}a_{2,h}} \quad (k \geq n+p),
\end{equation}

that is, that

\begin{equation}
\sqrt{a_{1,h}a_{2,h}} \leq \frac{p-\alpha}{\sqrt{(p-\alpha_1)(p-\alpha_2)}} \quad (k \geq n+p).
\end{equation}

From (5.7), we need to show that

\begin{equation}
\frac{\sqrt{(p-\alpha_1)(p-\alpha_2)}}{(1-\beta)p+\beta k} \leq \frac{p-\alpha}{\sqrt{(p-\alpha_1)(p-\alpha_2)}} \quad (k \geq n+p)
\end{equation}

or

\begin{equation}
\alpha \leq p - \frac{(p-\alpha_1)(p-\alpha_2)}{(1-\beta)p+\beta k} \quad (k \geq n+p).
\end{equation}

Noting that the function

\begin{equation}
\phi(k) = p - \frac{(p-\alpha_1)(p-\alpha_2)}{(1-\beta)p+\beta k} \quad (k \geq n+p)
\end{equation}

is increasing on $k$, we have

\begin{equation}
\alpha \leq \phi(n+p) = p - \frac{(p-\alpha_1)(p-\alpha_2)}{p+\beta n}.
\end{equation}

Finally, we derive
Theorem 5. Let $f_j(z)(j = 1, 2)$ define by (5.1). If $f_j(z) \in A_p(n, \alpha_j, \beta)(j = 1, 2)$, then the function

$$\sum_{h=n+p}^{\infty} \left\{ (a_{1,h})^2 + (a_{2,h})^2 \right\} z^h$$

is in the class $A_p(n, \alpha, \beta)$, where

$$\alpha = p - \frac{2(p - \alpha_0)^2}{p + \beta n} \quad (\alpha_0 = \min\{\alpha_1, \alpha_2\}).$$

The result is sharp for the function $f(z)$ defined by

$$f_j(z) = z^p - \frac{p - \alpha_0}{p + \beta n} z^{n+p} \quad (j = 1, 2),$$

when $\alpha_0 = \alpha_1 = \alpha_2$.

Proof: Since

$$\sum_{h=n+p}^{\infty} \left\{ \frac{(1-\beta)p + \beta k}{p - \alpha_j} a_{j,h} \right\}^2 \leq \left\{ \sum_{h=n+p}^{\infty} \frac{(1-\beta)p + \beta k}{p - \alpha_j} a_{j,h} \right\}^2 \leq 1 \quad (j = 1, 2),$$

we obtain that

$$\sum_{h=n+p}^{\infty} \left\{ \frac{(1-\beta)p + \beta k}{p - \alpha_0} \right\}^2 \left\{ (a_{1,h})^2 + (a_{2,h})^2 \right\} \leq 2,$$

and

$$\sum_{h=n+p}^{\infty} \left\{ \frac{(1-\beta)p + \beta k}{p - \alpha_1} a_{1,h} \right\}^2 + \sum_{h=n+p}^{\infty} \left\{ \frac{(1-\beta)p + \beta k}{p - \alpha_2} a_{2,h} \right\}^2 \leq 2,$$
where $\alpha_0$ is defined by (5.15). This implies that we only find the largest $\alpha$ such that

\[(5.19) \quad \frac{(1-\beta)p+\beta k}{p-\alpha} \leq \frac{1}{2} \left\{ \frac{(1-\beta)p+\beta k}{p-\alpha_0} \right\}^2 \quad (k \geq n+p)\]

or

\[(5.20) \quad \alpha \leq p - \frac{2(p-\alpha_0)^2}{(1-\beta)p+\beta k} \quad (k \geq n+p).\]

Since the function

\[(5.21) \quad \phi(k) = p - \frac{2(p-\alpha_0)^2}{(1-\beta)p+\beta k} \quad (k \geq n+p).\]

is increasing on $k$, we have

\[(5.22) \quad \alpha \leq \phi(n+p) = p - \frac{2(p-\alpha_0)^2}{p+\beta n}. \]

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