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A generalization class of certain subclasses of $p$-valently analytic functions with negative coefficients*

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Abstract
Recently we [5] have discussed a new generalization class $A(n, \alpha, \beta)$ of certain subclasses of analytic functions with negative coefficients in the unit disk and have proved some properties of functions belonging to the class $A(n, \alpha, \beta)$. In the present paper we introduce a new generalization class $A_p(n, \alpha, \beta)$ of certain subclasses of $p$-valently analytic functions with negative coefficients in the unit disk and discuss some properties of functions belonging to the class $A_p(n, \alpha, \beta)$.

1. Introduction
Let $p$ be a positive integer, and let $A_p(n)$ denote the class of functions of the form

\[(1.1) \quad f(z) = z^p - \sum_{h=n+p}^{\infty} a_h z^h \quad (a_h \geq 0, \ n \in N = \{1, 2, 3, \cdots \}),\]

which are analytic in the unit disk $U = \{z : |z| < 1\}$. A function $f(z)$ in the class $A_p(n)$ is said to be a member of the class $R_p(n, \alpha)$ if it satisfies

\[(1.2) \quad \text{Re} \left\{ \frac{pf(z)}{z^p} \right\} > \alpha \quad (z \in U)\]

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for some $\alpha(0 \leq \alpha < p)$. Further, a function $f(z)$ in the class $A_p(n)$ is said to be in the class $P_p(n, \alpha)$ if it satisfies

$$\text{Re} \left\{ \frac{f'(z)}{z^{p-1}} \right\} > \alpha \quad (z \in U)$$

for some $\alpha(0 \leq \alpha < p)$.

By generalization of some results due to Sarangi and Uralegaddi [2], we see that

**Lemma A.** A function $f(z) \in A_p(n)$ is in the class $R_p(n, \alpha)$ if and only if

$$\sum_{h=n+p}^{\infty} \frac{p}{p-\alpha} a_h \leq 1.$$  

**Lemma B.** A function $f(z) \in A_p(n)$ is in the class $P_p(n, \alpha)$ if and only if

$$\sum_{h=n+p}^{\infty} \frac{k}{p-\alpha} a_h \leq 1.$$  

Now, we define

**Definition.** Suppose that $f(z) \in A_p(n), 0 \leq \alpha < p$ and $\beta \geq 0$. Then the function $f(z)$ is said to be a member of the class $A_p(n, \alpha, \beta)$ if it satisfies

$$\text{Re} \left\{ (1-\beta)\frac{pf(z)}{z^p} + \beta \frac{f'(z)}{z^{p-1}} \right\} > \alpha \quad (z \in U).$$

We note that $A_p(n, \alpha, 0) = R_p(n, \alpha)$ and $A_p(n, \alpha, 1) = P_p(n, \alpha)$. We have
**Lemma 1.** Suppose that \( f(z) \in A_p(n), 0 \leq \alpha < p \) and \( \beta \geq 0 \). Then the function \( f(z) \) is in the class \( A_p(n, \alpha, \beta) \) if and only if

\[
\sum_{h=n+p}^{\infty} \left\{ \frac{(1-\beta)p + \beta k}{p-\alpha} \right\} a_h \leq 1.
\]

**Proof:** Let \( f(z) \in A_p(n, \alpha, \beta) \). Then we have, by (1.6),

\[
\text{Re} \left\{ (1-\beta)\frac{pf(z)}{z^p} + \beta \frac{f'(z)}{z^{p-1}} \right\} = \text{Re} \left\{ p - \sum_{h=n+p}^{\infty} \{(1-\beta)p + \beta k\} a_h z^{h-p} \right\} > \alpha \quad (z \in U).
\]

Letting \( z \to 1 \) through real values, we obtain (1.7). Conversely, let \( f(z) \in A_p(n) \) satisfy inequality (1.7). Then we have

\[
\left| \left\{ (1-\beta)\frac{pf(z)}{z^p} + \beta \frac{f'(z)}{z^{p-1}} \right\} - p \right| = \left| \sum_{h=n+p}^{\infty} \{(1-\beta)p + \beta k\} a_h z^{h-p} \right| \leq \sum_{h=n+p}^{\infty} \{(1-\beta)p + \beta k\} a_h |z|^{h-p} < p - \alpha \quad (z \in U).
\]

This proves that inequality (1.6) holds true. \( \blacksquare \)

The class \( A_1(n, \alpha, \beta) \) is a special case \( B_k = \frac{1+(k-1)\beta}{1-\alpha} \) of the class \( A(n, B_k) \) introduced by Sekine [3].

2. Distortion Theorem
Theorem 1. If \( f(z) \in A_p(n, \alpha, \beta) \) for \( 0 \leq \alpha < p \) and \( \beta \geq 0 \), then

\[
|z|^p - \frac{p - \alpha}{p + n\beta}|z|^{n+p} \leq |f(z)| \leq |z|^p + \frac{p - \alpha}{p + n\beta}|z|^{n+p} \quad (z \in U)
\]

for \( \beta \geq 0 \), and

\[
|f'(z)| \leq p|z|^{p-1} + \frac{(p - \alpha)(n + p)}{p + n\beta}|z|^{n+p-1} \quad (z \in U)
\]

\[
(2.2)
\]

\[
|f'(z)| \geq p|z|^{p-1} - \frac{(p - \alpha)(n + p)}{p + n\beta}|z|^{n+p-1} \quad (z \in U)
\]

for \( \beta \geq 1 \). The equalities in (2.1) and (2.2) are attained for the function

\[
f(z) = z^p - \frac{p - \alpha}{p + n\beta}z^{n+p}.
\]

Proof: Note that

\[
(2.4)
\]

\[
\sum_{k=n+p}^{\infty} a_k \leq \frac{p - \alpha}{p + n\beta} \quad (\beta \geq 0)
\]

and

\[
(2.5)
\]

\[
\sum_{k=n+p}^{\infty} k a_k \leq \sum_{k=n+p}^{\infty} \{(1 - \beta)p + \beta k\} a_k \leq p - \alpha \quad (\beta \geq 1)
\]

for \( f(z) \in A_p(n, \alpha, \beta) \). Therefore, we have (2.1) and (2.2). \( \blacksquare \)

Remark. Putting \( p = 1 \) in Theorem 1, we have the corresponding result due to Yaguchi, Sekine, Saitoh, Owa, Nunokawa and Fukui [5].

3. Inclusion Relations

Theorem 2. If

\[
0 \leq \alpha_1 < p, \quad 0 \leq \alpha_2 < p,
\]

\[
0 \leq \beta_1, \quad 0 \leq \beta_2, \quad p(\beta_1 - \beta_2) < \alpha_2\beta_1 - \alpha_1\beta_2,
\]

\[
p\{\alpha_1 - \alpha_2 + (\beta_1 - \beta_2)n\} \leq n(\alpha_2\beta_1 - \alpha_1\beta_2),
\]

\[
(3.1)
\]
then we have

\[(3.2) \quad A_p(n, \alpha_2, \beta_2) \subset \neq A_p(n, \alpha_1, \beta_1).\]

**Proof:** Suppose \( f(z) \in A_p(n, \alpha_2, \beta_2) \). Since by Lemma 1

\[(3.3) \quad \sum_{h=n+p}^{\infty} \frac{(1-\beta_2)p+k\beta_2}{p-\alpha_2} a_h \leq 1,\]

we have only to prove the inequality

\[(3.4) \quad \frac{(1-\beta_1)p+k\beta_1}{p-\alpha_1} \leq \frac{(1-\beta_2)p+k\beta_2}{p-\alpha_2} \quad (k \geq n+p),\]

which is equivalent to the inequality

\[(3.5) \quad k \geq \frac{p\{((\beta_2-\beta_1)p+\alpha_1-\alpha_2+\alpha_2\beta_1-\alpha_1\beta_2)\}}{(\beta_2-\beta_1)p+\alpha_2\beta_1-\alpha_1\beta_2} \quad (k \geq n+p).\]

But conditions \((3.1)\) lead to the inequality

\[(3.6) \quad \frac{p\{((\beta_2-\beta_1)p+\alpha_1-\alpha_2+\alpha_2\beta_1-\alpha_1\beta_2)\}}{(\beta_2-\beta_1)p+\alpha_2\beta_1-\alpha_1\beta_2} \leq n+p,\]

which proves \((3.5)\). The function \( f_0(z) \) defined by

\[(3.7) \quad f_0(z) = z^p - \frac{p-\alpha_1}{p+(n+1)\beta_1} z^{p+n+1} \]

belongs to the class \( A_p(n, \alpha_1, \beta_1) - A_p(n, \alpha_2, \beta_2) \), which proves

\[(3.8) \quad A_p(n, \alpha_1, \beta_1) \neq A_p(n, \alpha_2, \beta_2). \]
Corollary 1. If

\[ 0 \leq \alpha_1 \leq \alpha_2 < p, \quad 0 \leq \beta_1 \leq \beta_2, \quad (\beta_2 - \beta_1) + (\alpha_2 - \alpha_1) > 0, \]

then we have

\[ A_p(n, \alpha_2, \beta_2) \subset A_p(n, \alpha_1, \beta_1) \quad (3.10) \]

Proof: By Theorem 2, we have

\[ A_p(n, \alpha_2, \beta_1) \subset A_p(n, \alpha_1, \beta_1) \quad \text{(0} \leq \alpha_1 < \alpha_2 < p), \]
\[ A_p(n, \alpha_2, \beta_2) \subset A_p(n, \alpha_2, \beta_1) \quad \text{(0} \leq \beta_1 < \beta_2), \quad (3.11) \]

which prove Corollary 1. \[ \square \]

Corollary 2. If \( 0 < \beta_1 < 1 < \beta_2 \), then

\[ A_p(n, \alpha, \beta_2) \subset P_p(n, \alpha) \subset A_p(n, \alpha, \beta_1) \subsetneqq R_p(n, \alpha). \quad (3.12) \]

4. Starlikeness

A function \( f(z) \) in the class \( A_p(n) \) is said to be \( p \)-valently starlike of order \( \alpha \) if it satisfies

\[ \Re \frac{zf'(z)}{f(z)} > \alpha \quad (z \in U) \quad (4.1) \]

for some \( \alpha (0 \leq \alpha < p) \). We need the following lemma which is a generalization of a result due to Chatterjea [1] (also Srivastava, Owa and Chatterjea [4]).

Lemma C. A function \( f(z) \in A_p(n) \) is \( p \)-valently starlike of order \( \gamma \) if and only if

\[ \sum_{h=n+p}^{\infty} \frac{k - \gamma}{p - \gamma} a_k \leq 1 \quad (4.2) \]

for some \( \gamma (0 \leq \gamma < p) \).

Lemma C is proved by using the similar method as in Chatterjea [1]. Using Lemma C, we have
THEOREM 3. If \( f(z) \in A_{p}(n, \alpha, \beta) \) for \( 0 \leq \alpha < p \) and \( \beta \geq 1 \), then \( f(z) \) is starlike of order \( (1 - \frac{1}{\beta})p \).

PROOF: It follows from \( f(z) \in A_{p}(n, \alpha, \beta) \) that

\[
\sum_{h=n+p}^{\infty} \{k - (1 - \frac{1}{\beta})p\}a_{h} \leq \frac{p - \alpha}{\beta} \leq p - (1 - \frac{1}{\beta})p.
\]

Therefore, by Lemma C, we have the assertion of Theorem 3. ■

5. Quasi-Hadamard product

For functions \( f_{1}(z) \) and \( f_{2}(z) \) defined by

\[
f_{j}(z) = z^{p} - \sum_{h=n+p}^{\infty} a_{j,h}z^{h} \quad (a_{j,h} \geq 0, \ n \in N, \ j = 1, 2)
\]
in the class \( A_{p}(n) \), we denote by \( f_{1} * f_{2}(z) \) the quasi-Hadamard product of functions \( f_{1}(z) \) and \( f_{2}(z) \), that is,

\[
f_{1} * f_{2}(z) = z^{p} - \sum_{h=n+p}^{\infty} a_{1,h}a_{2,h}z^{h}.
\]

THEOREM 4. If \( f_{j}(z) \in A_{p}(n, \alpha_{j}, \beta) \) for \( 0 \leq \alpha_{j} < p, \beta \geq 0 \) and \( j = 1, 2 \), then \( f_{1} * f_{2}(z) \in A_{p}(n, \alpha, \beta) \), where

\[
\alpha = p - \frac{(p - \alpha_{1})(p - \alpha_{2})}{p + \beta n}.
\]
The result is sharp for functions \( f_{1}(z) \) and \( f_{2}(z) \) defined by

\[
f_{j}(z) = z^{p} - \frac{p - \alpha_{j}}{p + \beta n}z^{n+p} \quad (j = 1, 2).
\]

PROOF: We have to find the largest \( \alpha \) such that

\[
\sum_{h=n+p}^{\infty} \frac{(1 - \beta)p + \beta k}{p - \alpha}a_{1,h}a_{2,h} \leq 1.
\]
For functions $f_j(z) \in A_p(n, \alpha_j, \beta)$, we have

$$
\sum_{h=n+p}^{\infty} \left\{ \frac{(1-\beta)p+\beta k}{p-\alpha} \right\} a_{j,h} \leq 1 \quad (j = 1, 2). 
$$

By the Cauchy-Schwarz inequality, inequality (5.6) lead to the inequality

$$
\sum_{h=n+p}^{\infty} \frac{(1-\beta)p+\beta k}{\sqrt{(p-\alpha_1)(p-\alpha_2)}} \sqrt{a_{1,h}a_{2,h}} \leq 1. 
$$

Therefore, it is sufficient to prove that

$$
\frac{(1-\beta)p+\beta k}{p-\alpha} a_{1,h}a_{2,h} \leq \frac{(1-\beta)p+\beta k}{\sqrt{(p-\alpha_1)(p-\alpha_2)}} \sqrt{a_{1,h}a_{2,h}} \quad (k \geq n+p),
$$

that is, that

$$
\sqrt{a_{1,h}a_{2,h}} \leq \frac{p-\alpha}{\sqrt{(p-\alpha_1)(p-\alpha_2)}} \quad (k \geq n+p).
$$

From (5.7), we need to show that

$$
\frac{\sqrt{(p-\alpha_1)(p-\alpha_2)}}{(1-\beta)p+\beta k} \leq \frac{p-\alpha}{\sqrt{(p-\alpha_1)(p-\alpha_2)}} \quad (k \geq n+p)
$$
or

$$
\alpha \leq p - \frac{(p-\alpha_1)(p-\alpha_2)}{(1-\beta)p+\beta k} \quad (k \geq n+p).
$$

Noting that the function

$$
\phi(k) = p - \frac{(p-\alpha_1)(p-\alpha_2)}{(1-\beta)p+\beta k} \quad (k \geq n+p)
$$
is increasing on $k$, we have

$$
\alpha \leq \phi(n+p) = p - \frac{(p-\alpha_1)(p-\alpha_2)}{p+\beta n}.
$$

Finally, we derive
THEOREM 5. Let \( f_j(z)(j = 1, 2) \) define by (5.1). If \( f_j(z) \in A_p(n, \alpha_j, \beta)(j = 1, 2) \), then the function

\[
f(z) = z^p - \sum_{k=n+p}^{\infty} \left\{ (a_{1,k})^2 + (a_{2,k})^2 \right\} z^k
\]
is in the class \( A_p(n, \alpha, \beta) \), where

\[
\alpha = p - \frac{2(p - \alpha_0)^2}{p + \beta n} \quad (\alpha_0 = \min\{\alpha_1, \alpha_2\}).
\]
The result is sharp for the function \( f(z) \) defined by

\[
f_j(z) = z^p - \frac{p - \alpha_0}{p + \beta n} z^{n+p} \quad (j = 1, 2),
\]
when \( \alpha_0 = \alpha_1 = \alpha_2 \).

PROOF: Since

\[
\sum_{k=n+p}^{\infty} \left\{ \frac{(1-\beta)p + \beta k}{p - \alpha_j} a_{j,k} \right\}^2 \leq \sum_{k=n+p}^{\infty} \left\{ \frac{(1-\beta)p + \beta k}{p - \alpha_0} a_{j,k} \right\}^2 \leq 1 \quad (j = 1, 2),
\]
we obtain that

\[
\sum_{k=n+p}^{\infty} \left\{ \frac{(1-\beta)p + \beta k}{p - \alpha_0} \right\}^2 \left\{ (a_{1,k})^2 + (a_{2,k})^2 \right\} \leq \sum_{k=n+p}^{\infty} \left\{ \frac{(1-\beta)p + \beta k}{p - \alpha_1} a_{1,k} \right\}^2 + \sum_{k=n+p}^{\infty} \left\{ \frac{(1-\beta)p + \beta k}{p - \alpha_2} a_{2,k} \right\}^2 \leq 2,
\]
where $\alpha_0$ is defined by (5.15). This implies that we only find the largest $\alpha$ such that

\[(5.19) \quad \frac{(1 - \beta)p + \beta k}{p - \alpha} \leq \frac{1}{2} \left\{ \frac{(1 - \beta)p + \beta k}{p - \alpha_0} \right\}^2 \quad (k \geq n + p)\]

or

\[(5.20) \quad \alpha \leq p - \frac{2(p - \alpha_0)^2}{(1 - \beta)p + \beta k} \quad (k \geq n + p).\]

Since the function

\[(5.21) \quad \phi(k) = p - \frac{2(p - \alpha_0)^2}{(1 - \beta)p + \beta k} \quad (k \geq n + p).\]

is increasing on $k$, we have

\[(5.22) \quad \alpha \leq \phi(n + p) = p - \frac{2(p - \alpha_0)^2}{p + \beta n}.\]

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