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Integral Transforms with Fox's H-function in Spaces of Summable Functions

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Abstract

Integral transforms involving Fox's H-functions as kernels are studied on the space $\mathfrak{L}_{\nu,2}$ of functions f such that

$$\int_0^\infty |t^\nu f(t)|^2 \frac{dt}{t} < \infty, \quad \nu \in R.$$

Mapping properties such as the boundedness, the repesentation and the range of these transforms $\boldsymbol{H}$ are given.

1. Introduction

In this paper we deal with the integral transforms of the form

$$(1.1) \qquad (\boldsymbol{H}f)(x) = \int_0^\infty H_{p,q}^{m,n}\left[xt \,\middle|\, \begin{matrix}(a_p,\alpha_p)\\ \\ (b_q,\beta_q)\end{matrix}\right] f(t)dt,$$

where $H_{p,q}^{m,n}\left[s \,\middle|\, \begin{matrix}(a_p,\alpha_p)\\ (b_q,\beta_q)\end{matrix}\right]$ is the Fox H-function. This function of general hypergeometric type was introduced by Fox [7]. For integers m,n,p,q such that $0 \leqq m \leqq q, 0 \leqq n \leqq p$ and $a_i, b_j \in \boldsymbol{C}$ with $\boldsymbol{C}$ of the field of complex numbers, and $\alpha_i, \beta_j \in \boldsymbol{R}_+ = (0,\infty)$ $(1 \leqq i \leqq p$, $1 \leqq j \leqq q)$ it can be written by

$$(1.2) \qquad H_{p,q}^{m,n}\left[s \,\middle|\, \begin{matrix}(a_p,\alpha_p)\\ \\ (b_q,\beta_q)\end{matrix}\right] = H_{p,q}^{m,n}\left[s \,\middle|\, \begin{matrix}(a_1,\alpha_1), & \cdots,(a_p,\alpha_p)\\ \\ (b_1,\beta_1), & \cdots,(b_q,\beta_q)\end{matrix}\right]$$

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$$= \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^{m} \Gamma(b_j + \beta_j t) \prod_{i=1}^{n} \Gamma(1 - a_i - \alpha_i t)}{\prod_{i=n+1}^{p} \Gamma(a_i + \alpha_i t) \prod_{j=m+1}^{q} \Gamma(1 - b_j - \beta_j t)} s^{-t} dt,$$

the contour L being specially chosen and an empty product, if it occurs, being taken to be one. The theory of this function may be found in [2], [23, Chapter 1], [41, Chapter 2] and [28, §8.3]. We abbreviate the Fox H-function (1.2) to $H^{m,n}_{p,q}(s)$ or $H(s)$ when no confusion can occur.

Most of the known integral transforms can be put into the form (1.1). When $\alpha_1 = \cdots = \alpha_p = \beta_1 = \cdots = \beta_q = 1$ then (1.2) is the Meijer G-function [6, Chapter 5.3] and (1.1) is reduced to the so-called integral transforms with G-function kernels or G-transforms. Such transforms include the classical Laplace and Hankel transforms. The Riemann-Liouville fractional integrals, the even and odd Hilbert transforms, the integral transforms with the Gauss hypergeometric function, etc. can be reduced to these G-transforms, for whose theory and historical notices see [34, §§36, 39]. There are other transforms which cannot be reduced to G-transforms but can put into the transforms $\boldsymbol{H}$ given in (1.1). These are the modified Laplace and Hankel transforms [39], [31], [34, §§18, 23, 39], the Erdélyi-Kober type fractional integration operators [20], [5], [39], [34, §18], the transforms with the Gauss hypergeometric function as kernel [27], [24], [32], [33], [34, §§23, 39], the Bessel-type integral transforms [21], [30], [16], [17], etc.

The integral transforms (1.1) with Fox's H-function kernels or transforms $\boldsymbol{H}$ were first considered by Fox [7] while investigating G- and H-functions as symmetrical Fourier kernels. This paper such as the ones [15], [35], [8], [9], [38], [12], [3], [22] and [26] was devoted to finding inversion formulae for the transforms $\boldsymbol{H}$ in the spaces $L_1(0,\infty)$ and $L_2(0,\infty)$. Some properties of transforms $\boldsymbol{H}$ such as their Mellin transform, the relation of fractional integration by parts, compositional formulae, etc. were considered in [10], [11], [40], [36] and [13]. In [37], [1] and [4] the integral operators of the form (1.1) with Fox's H-function in the kernels were represented as the compositions of the Erdélyi-Kober type operators and integral operators (1.1) with the H-functions of the less order. Factorization properties of (1.1) in special functional spaces L_2^{Φ} were investigated in [42] and the properties of such operators in McBride spaces $F_{p,\mu}$ and $F'_{p,\mu}$ (see [25] and [34, §8]) are studied in [29].

Our paper is devoted to studying the transform $\boldsymbol{H}$ on the weighted spaces $\mathfrak{L}_{\nu,2}, \nu \in \boldsymbol{R} = (-\infty,\infty)$, of those Lebesgue measurable complex valued functions f for which

$$\|f\|_{\nu,2} = \int_0^\infty |t^\nu f(t)|^2 \frac{dt}{t} < \infty. \tag{1.3}$$

For $f \in \mathfrak{L}_{\nu,2}$ the Mellin transform $\mathfrak{M}$ of f is defined [31] by

$$(\mathfrak{M}f)(\nu + it) = \int_{-\infty}^{\infty} e^{(\nu+it)\tau} f(e^\tau) d\tau, \tag{1.4}$$

for $\nu \in \boldsymbol{R}$ and $t \in \boldsymbol{R}$.

We also write $(\mathfrak{M}f)(s)$ with $\mathrm{Re}(s)=\nu$ for $(\mathfrak{M}f)(\nu+it)$. In particular, if $f\in\mathcal{L}_{\nu,2}\cap\mathcal{L}_{\nu,1}$, where

$$\mathcal{L}_{\nu,1}=\left\{f:\int_0^\infty t^{\nu-1}\left|f(t)\right|dt<\infty\right\},$$

then the Mellin transform $(\mathfrak{M}f)(s)$ is given by the usual expression

$$(\mathfrak{M}f)(s)=\int_0^\infty f(t)t^{s-1}dt,\qquad \mathrm{Re}(s)=\nu.$$

The Mellin transform (1.4) has the following properties [31]:

$$f(x)=\frac{1}{2\pi i}\lim_{R\to\infty}\int_{\nu-iR}^{\nu+iR}(\mathfrak{M}f)(s)x^{-s}ds,\ f\in\mathcal{L}_{\nu,2},\tag{1.5}$$

where the limit is taken in the topology of $\mathcal{L}_{\nu,2}$ and the integral is understood as

$$\int_{\nu-iR}^{\nu+iR}F(s)ds=i\int_{-R}^{R}F(\nu+it)dt\tag{1.6}$$

for $F(\nu+it)\in L_1(-R,R)$;

$$\int_0^\infty f(x)g(x)dx=\frac{1}{2\pi i}\lim_{R\to\infty}\int_{\nu-iR}^{\nu+iR}(\mathfrak{M}f)(s)(\mathfrak{M}g)(1-s)ds\tag{1.7}$$

for $f\in\mathcal{L}_{\nu,2}$ and $g\in\mathcal{L}_{1-\nu,2}$.

If we formally take the Mellin transform of (1.1) we obtain

$$(\mathfrak{M}\boldsymbol{H}f)(s)=\mathcal{H}^{m,n}_{p,q}\left[\begin{array}{c|c}(a_p,\alpha_p) & \\ & s\\ (b_q,\beta_q) & \end{array}\right](\mathfrak{M}f)(1-s)\tag{1.8}$$

in view of (1.7), where

$$\mathcal{H}^{m,n}_{p,q}\left[\begin{array}{c|c}(a_p,\alpha_p) & \\ & s\\ (b_q,\beta_q) & \end{array}\right]=\frac{\prod\limits_{j=1}^{m}\Gamma(b_j+\beta_js)\prod\limits_{i=1}^{n}\Gamma(1-a_i-\alpha_is)}{\prod\limits_{i=n+1}^{p}\Gamma(a_i+\alpha_is)\prod\limits_{j=m+1}^{q}\Gamma(1-b_j-\beta_js)}\tag{1.9}$$

is the ratio of the products of gamma functions in the integrand of (1.2) and an empty product, if it occurs, being taken to be one as in (1.2). When there is no possibility to confusion, we denote (1.9) simply by $\mathcal{H}^{m,n}_{p,q}(s)$ or $\mathcal{H}(s)$. For certain ranges of parameters in $\mathcal{H}(s)$, (1.8) can be used to define the transform $\boldsymbol{H}$ in $\mathcal{L}_{\nu,2}$ more precisely. Namely, for certain $h\in\boldsymbol{R}\setminus\{0\}$ and $\lambda\in\boldsymbol{C}$, we have

$$(\boldsymbol{H}f)(x)\tag{1.10}$$

$$=hx^{1-(\lambda+1)/h}\frac{d}{dx}x^{(\lambda+1)/h}\int_0^\infty H^{m,n+1}_{p+1,q+1}\left[xt\,\middle|\begin{array}{c}(-\lambda,h),(a_1,\alpha_1),\cdots,(a_p,\alpha_p)\\ \\ (b_1,\beta_1),\cdots,(b_q,\beta_q),(-\lambda-1,h)\end{array}\right]f(t)dt.$$

According to [28, (8.3.2.16) with $k = 1$ and $\sigma = -1$, and (8.3.2.6)] the formal differentiation under the integral sign in (1.10) yields (1.1).

The paper is organized as follows. Section 2 is devoted to finding certain asymptotic estimates as $|t| \to \infty$ for $\mathcal{H}(\sigma + it)$ and for its derivative $\mathcal{H}'(\sigma + it)$, and an integral representation and an estimate for $H(s)$. Section 3 deals with studying the boundedness and the range of $\boldsymbol{H}$ on the space $\mathcal{L}_{\nu,2}$.

The results obtained are the extensions of those by Rooney [31] from G-transforms to transforms $\boldsymbol{H}$. Some of them were announced in [18].

2. Some properties of functions $H^{m,n}_{p,q}$ and $\mathcal{H}^{m,n}_{p,q}$

First we define a number of parameters connected with H- and $\mathcal{H}$-functions given in (1.2) and (1.9). Let m, n, p, q be integers such that $0 \leqq m \leqq q, 0 \leqq n \leqq p$ and let $a_1, \cdots, a_p, b_1, \cdots, b_q$ be complex numbers and $\alpha_1, \cdots, \alpha_p, \beta_1, \cdots, \beta_q$ be real positive numbers. We define, see [28, §8.3],

$$\alpha = \begin{cases} \max\left[-\dfrac{\mathrm{Re}\,(b_1)}{\beta_1}, \cdots, -\dfrac{\mathrm{Re}\,(b_m)}{\beta_m}\right] & \text{if } m > 0, \\ -\infty & \text{if } m = 0, \end{cases} \tag{2.1}$$

$$\beta = \begin{cases} \min\left[\dfrac{1 - \mathrm{Re}(a_1)}{\alpha_1}, \cdots, \dfrac{1 - \mathrm{Re}(a_n)}{\alpha_n}\right] & \text{if } n > 0, \\ \infty & \text{if } n = 0, \end{cases} \tag{2.2}$$

$$a^* = \sum_{i=1}^{n} \alpha_i - \sum_{i=n+1}^{p} \alpha_i + \sum_{j=1}^{m} \beta_j - \sum_{j=m+1}^{q} \beta_j, \tag{2.3}$$

$$c^* = m + n - \frac{p+q}{2}, \tag{2.4}$$

$$\delta = \prod_{i=1}^{p} \alpha_i^{-\alpha_i} \prod_{j=1}^{q} \beta_j^{\beta_j}, \tag{2.5}$$

$$\Delta = \sum_{j=1}^{q} \beta_j - \sum_{i=1}^{p} \alpha_i, \tag{2.6}$$

$$\mu = \sum_{j=1}^{q} b_j - \sum_{i=1}^{p} a_i + \frac{p-q}{2}, \tag{2.7}$$

$$\xi = \sum_{j=1}^{m} b_j - \sum_{i=m+1}^{q} b_j + \sum_{i=1}^{n} a_i - \sum_{i=n+1}^{p} a_i. \tag{2.8}$$

We begin from the estimate of the function $\mathcal{H}(s)$ and its derivative.

Lemma 1. *Let $\sigma, t \in \boldsymbol{R}$, then the estimate*

$$|\mathcal{H}^{m,n}_{p,q}(\sigma+it)| \sim \delta^{\sigma} \prod_{i=1}^{p} \alpha_i^{1/2-\mathrm{Re}(a_i)} \prod_{j=1}^{q} \beta_j^{\mathrm{Re}(b_j)-1/2} (2\pi)^{c^*} |t|^{\Delta\sigma+\mathrm{Re}(\mu)} e^{-\pi|t|a^*/2-\pi \mathrm{Im}(\xi)\mathrm{sign}(t)/2} \tag{2.9}$$

holds as $|t| \to \infty$ uniformly in σ for σ in any bounded interval in $\boldsymbol{R}$. Further, as $|t| \to \infty$,

$$\left\{\mathcal{H}^{m,n}_{p,q}\right\}'(\sigma+it) = \mathcal{H}^{m,n}_{p,q}(\sigma+it)\left[\log\delta + \left(\sum_{j=1}^{m}\beta_j - \sum_{i=n+1}^{p}\alpha_i\right)\log(it)\right.$$
$$\left. - \left(\sum_{i=1}^{n}\alpha_i - \sum_{j=m+1}^{q}\beta_j\right)\log(-it) + \frac{\mu+\Delta\sigma}{it} + O(t^{-2})\right]. \tag{2.10}$$

Proof. According to the Stirling formula [6, 1.18(2)]

$$\Gamma(z) \sim \sqrt{2\pi} z^{z-1/2} e^{-z}, \qquad (|z| \to \infty), \tag{2.11}$$

we have for a complex constant $k = c + id$ and a variable $s = \sigma + it$,

$$\Gamma(k+s) = \Gamma(c+\sigma+i(d+t)) \sim (2\pi)^{1/2}|t|^{c+\sigma-1/2}e^{-\pi|t|/2-\pi d \mathrm{sign}(t)/2}. \tag{2.12}$$

as $|t| \to \infty$. Substituting this into (1.9) and using (2.3) - (2.8), we obtain (2.9).
It follows from (1.9) that for $s = \sigma + it$

$$\left\{\mathcal{H}^{m,n}_{p,q}\right\}'(s) = \mathcal{H}^{m,n}_{p,q}(s)\left[\sum_{j=1}^{m}\beta_j\psi(b_j+\beta_j s) - \sum_{i=1}^{n}\alpha_i\psi(1-a_i-\alpha_i s)\right.$$
$$\left. + \sum_{j=m+1}^{q}\beta_j\psi(1-b_j-\beta_j s) - \sum_{i=n+1}^{p}\alpha_i\psi(a_i+\alpha_i s)\right], \tag{2.13}$$

where $\psi(z) = \Gamma'(z)/\Gamma(z)$. In accordance with [6, 1.18(7)] for $c \in C$ we have, as $|t| \to \infty$,

$$\psi(c+\sigma \pm it) = \log(\pm it) \pm \frac{c+\sigma-1/2}{it} + O(t^{-2}). \tag{2.14}$$

Substituting this into (2.13) we arrive at (2.10).

Now we give an integral representation for the H-function (1.2) suitable for the space $\mathcal{L}_{\nu,2}$.

Theorem 1. *Let $\alpha < \gamma < \beta$. If either of the conditions*

(i) $a^* > 0$,

(ii) $a^* = 0, \Delta \neq 0$ and $\mathrm{Re}(\mu) + \Delta\gamma \leqq 0$,

(iii) $a^* = 0, \Delta = 0$ and $\mathrm{Re}(\mu) < 0$

holds, then for all $x \in \boldsymbol{R}_+$,

$$(2.15)\qquad H^{m,n}_{p,q}\left[x\left|\begin{matrix}(a_p,\alpha_p)\\ \\ (b_q,\beta_q)\end{matrix}\right.\right] = \frac{1}{2\pi i}\lim_{R\to\infty}\int_{\gamma-iR}^{\gamma+iR}\mathcal{H}^{m,n}_{p,q}\left[\left.\begin{matrix}(a_p,\alpha_p)\\ \\ (b_q,\beta_q)\end{matrix}\right| t\right] x^{-t}dt$$

except for $x = \delta$ *in the case* (iii) (*when* $H(x)$ *is not defined*).

Proof. When $a^* > 0$, then (2.15) follows from the definition of the function H (see [28, §8.3]). We prove (2.15) for $a^* = 0$ and either $\Delta < 0$, or $\Delta = 0$ and $x > \delta$. The proof for $a^* = 0$ and either $\Delta > 0$, or $\Delta = 0$ and $0 < x < \delta$ is exactly similar. In the case under consideration, it is followed from [28, §8.3] that

$$H(x) = \frac{1}{2\pi i}\int_L x^{-s}\mathcal{H}(s)ds$$

where L is a loop starting and ending at ∞ and encircling all of the poles of $\Gamma(1-a_i-\alpha_i s)$ ($i = 1, 2, ..., n$) once in the negative direction, but encircling none of the poles of $\Gamma(b_j + \beta_j s)$ ($j = 1, 2, ..., m$).

Let

$$\tau > \max\left[\left|\mathrm{Im}\left(\frac{a_1}{\alpha_1}\right)\right|, \cdots, \left|\mathrm{Im}\left(\frac{a_n}{\alpha_n}\right)\right|\right]$$

and choose $k, \gamma < k < \beta$. We choose L to be the loop consisting of the line $\mathrm{Im}(s) = -\tau$ from $\infty - i\tau$ to $k - i\tau$, the line $\mathrm{Re}(s) = k$ from $k - i\tau$ to $k + i\tau$ and the line $\mathrm{Im}(s) = \tau$ from $k + i\tau$ to $\infty + i\tau$. For $R > k - \gamma$, let L_R denote the portion of L on which $|s - \gamma| \leqq R$. Clearly that

$$(2.16)\qquad H(x) = \frac{1}{2\pi i}\lim_{R\to\infty}\int_{L_R} x^{-s}\mathcal{H}(s)ds.$$

For such R we denote by Λ the closed curve composed of the line $\mathrm{Re}(s) = \gamma$ from $\gamma - iR$ to $\gamma + iR$, the portion of the circle $|s - \gamma| = R$ clockwise from $\gamma + iR$ to the terminal point of L_R, L_R reversed, and the portion of the circle $|s - \gamma| = R$ clockwise from the initial point of L_R to $\gamma - iR$. Let Λ_1 and Λ_2 be the upper and lower circular parts of Λ, respectively. Since $\alpha < \gamma < \beta$, then according to Cauchy's theorem we have, for $x > 0$,

$$0 = \int_\Lambda x^{-s}\mathcal{H}(s)ds = \int_{\gamma-iR}^{\gamma+iR} x^{-s}\mathcal{H}(s)ds - \int_{L_R} x^{-s}\mathcal{H}(s)ds + \int_{\Lambda_1} x^{-s}\mathcal{H}(s)ds + \int_{\Lambda_2} x^{-s}\mathcal{H}(s)ds.$$

It is followed from (2.16) that to prove (2.15) it is sufficient to show that

$$(2.17)\qquad \lim_{R\to\infty}\int_{\Lambda_i} x^{-s}\mathcal{H}(s)ds = 0 \quad (i = 1, 2).$$

Now let us prove (2.17) for $i = 1$. Applying the relation $\Gamma(z)\Gamma(1-z) = \dfrac{\pi}{\sin \pi z}$, we represent $\mathcal{H}(s)$ as

$$\mathcal{H}(s) = \mathcal{H}_1(s)\mathcal{H}_2(s), \tag{2.18}$$

where

$$\mathcal{H}_1(s) = \frac{\prod_{j=1}^{q} \Gamma(b_j + \beta_j s)}{\prod_{i=1}^{p} \Gamma(a_i + \alpha_i s)}, \quad \mathcal{H}_2(s) = \pi^{n+m-q} \frac{\prod_{j=m+1}^{q} \sin \pi(b_j + \beta_j s)}{\prod_{i=1}^{n} \sin \pi(a_i + \alpha_i s)}. \tag{2.19}$$

We estimate first $\mathcal{H}_1(s)$. From Stirling's formula (2.11) we have that, if $\zeta = Re^{i\theta}$ $(-\pi/2 \leqq \theta \leqq \pi/2)$ and $c \in C$, then uniformly in θ, as $R \to \infty$,

$$|\Gamma(c+\zeta)| \sim \sqrt{2\pi} e^{-\theta \operatorname{Im}(c)} R^{R\cos\theta + \operatorname{Re}(c) - 1/2} e^{-R[\cos\theta + \theta\sin\theta + \operatorname{Re}(c)]}.$$

Hence, on Λ_1, putting $s = \gamma + \zeta, \zeta = Re^{i\theta}$, we have uniformly in θ, as $R \to \infty$,

$$|\mathcal{H}_1(s)| = |\mathcal{H}_1(\gamma+\zeta)| \sim (2\pi)^{(q-p)/2} \delta^{\gamma} \frac{\prod_{j=1}^{q} \beta_j^{\operatorname{Re}(b_j)-1/2}}{\prod_{i=1}^{p} \alpha_i^{\operatorname{Re}(a_i)-1/2}} \exp\left\{-\theta\left[\operatorname{Im}\left(\sum_{j=1}^{q} b_j - \sum_{i=1}^{p} a_i\right) + \gamma\Delta\right]\right\}$$

$$\cdot\ R^{\operatorname{Re}(\mu)+(\gamma+R\cos\theta)\Delta}\ \delta^{R\cos\theta}\ \exp\left\{-R\Delta(\cos\theta+\theta\sin\theta) - \operatorname{Re}\left(\sum_{j=1}^{q} b_j - \sum_{i=1}^{p} a_i\right) - \gamma\Delta\right\},$$

where δ, Δ and μ are given by (2.5)-(2.7). Since $0 \leqq \theta \leqq \pi/2$ on Λ_1, then

$$\exp\left\{-\theta\left[\operatorname{Im}\left(\sum_{j=1}^{q} b_j - \sum_{i=1}^{p} a_i\right) + \gamma\Delta\right]\right\} \leqq \exp\left\{\frac{\pi}{2}\left|\operatorname{Im}\left(\sum_{j=1}^{q} b_j - \sum_{i=1}^{p} a_i\right) + \gamma\Delta\right|\right\}.$$

Hence there is a constant A_1 such that

$$|\mathcal{H}_1(s)| \leqq A_1 R^{\operatorname{Re}(\mu)+(\gamma+R\cos\theta)\Delta}\ \delta^{R\cos\theta}\ e^{-R\Delta(\cos\theta+\theta\sin\theta)}, \tag{2.20}$$

if $s \in \Lambda_1$ and R is sufficiently large.

Now we estimate $\mathcal{H}_2(s)$. If $s \in \Lambda_1$, $s = \gamma + Re^{i\theta} = \gamma + \xi + i\eta$, then by invoking the estimate

$$\sinh y \leqq |\sin(x+iy)| \leqq \cosh y \tag{2.21}$$

we have

$$|\sin \pi(b_j + \beta_j s)| \leqq \cosh \pi(\operatorname{Im}(b_j) + \beta_j \eta) \leqq B_j e^{\pi\beta_j R\sin\theta}, \ B_j = e^{\pi|\operatorname{Im}(b_j)|}\ (j = m+1, \cdots, q).$$

Letting Λ_1 pass through the point $k+i\tau$, we have

$$|\sin\pi(a_i+\alpha_i s)| \geqq \sinh\pi(\mathrm{Im}(a_i)+\alpha_i\eta) > 0 \quad (i=1,2,\cdots,n)$$

by virtue of the left inequality of (2.21) and $\eta \geqq \tau > -\mathrm{Im}(a_i)/\alpha_i$ $(i=1,2,\cdots,n)$. Since

$$\frac{d}{d\eta}e^{-\pi\alpha_i\eta}\sinh\pi(\mathrm{Im}(a_i)+\alpha_i\eta) = \pi\alpha_i e^{-\pi(2\alpha_i\eta+\mathrm{Im}(a_i))} > 0,$$

then for $\eta \geqq \tau$ we have

$$e^{-\pi\eta\alpha_i}\sinh\pi(\mathrm{Im}(a_i)+\alpha_i\eta) \geqq C_i,\ \ C_i = e^{-\pi\tau\alpha_i}\sinh\pi(\mathrm{Im}(a_i)+\alpha_i\tau)\ (i=1,2,\cdots,n)$$

and therefore

$$|\sin\pi(a_i+\alpha_i s)| \geqq C_i e^{\pi\alpha_i R\sin\theta} \quad (i=1,2,\cdots,n).$$

Substituting these estimates into $\mathcal{H}_2(s)$ and taking into account the relations

$$a^*=0,\ \sum_{j=m+1}^{q}\beta_j-\sum_{i=1}^{n}\alpha_i=\sum_{j=1}^{m}\beta_j-\sum_{i=n+1}^{p}\alpha_j=\frac{\Delta}{2},$$

we obtain

$$(2.22)\qquad |\mathcal{H}_2(s)| \leqq A_2 e^{\pi R\Delta\sin\theta/2},\quad A_2=\pi^{m+n-q}\frac{\prod_{j=m+1}^{q}B_j}{\prod_{i=1}^{n}C_i}.$$

Since $\Delta\leqq 0$ and $0<\theta\leqq\pi/2$, then $R\Delta(\theta-\pi/2)\sin\theta\geqq 0$. Therefore it is followed from (2.18)-(2.22) that for $s\in\Lambda_1$ and for sufficiently large R, say $R>R_0$ with $A=A_1A_2$, we have

$$(2.23)\qquad \begin{aligned}|\mathcal{H}(s)| &\leqq AR^{\mathrm{Re}(\mu)+(\gamma+R\cos\theta)\Delta}\,\delta^{R\cos\theta}\,e^{-R\Delta[\cos\theta+(\theta-\pi/2)\sin\theta]}\\ &\leqq AR^{\mathrm{Re}(\mu)+(\gamma+R\cos\theta)\Delta}\,\delta^{R\cos\theta}\,e^{-R\Delta\cos\theta}.\end{aligned}$$

Let us consider the case $\Delta<0$. We assumed $\mathrm{Re}(\mu)+\gamma\Delta\leqq 0$ in the hypothesis (ii) of the theorem. Hence, if $x>0$ and $R>\max[R_0,K]$ with $K=e(x/\delta)^{1/\Delta}$, we have

$$\begin{aligned}\left|\int_{\Lambda_1}x^{-s}\mathcal{H}(s)ds\right| &\leqq Ax^{-\gamma}\,R^{\mathrm{Re}(\mu)+\gamma\Delta+1}\int_0^{\pi/2}R^{\Delta R\cos\theta}\,e^{-R\Delta\cos\theta}\left(\frac{x}{\delta}\right)^{-R\cos\theta}d\theta\\ &= Ax^{-\gamma}\,R^{\mathrm{Re}(\mu)+\gamma\Delta+1}\int_0^{\pi/2}e^{\Delta R\,(\log R-\log K)\cos\theta}d\theta\\ &= Ax^{-\gamma}\,R^{\mathrm{Re}(\mu)+\gamma\Delta+1}\int_0^{\pi/2}e^{\Delta R\,(\log R-\log K)\sin\theta}d\theta\\ &\leqq Ax^{-\gamma}\,R^{\mathrm{Re}(\mu)+\gamma\Delta+1}\int_0^{\pi/2}e^{2\Delta R\,(\log R-\log K)\theta/\pi}d\theta\\ &= A\pi x^{-\gamma}\,R^{\mathrm{Re}(\mu)+\gamma\Delta+1}\frac{1-e^{\Delta R(\log R-\log K)}}{2\Delta(\log R-\log K)}\to 0\end{aligned}$$

as $R \to \infty$, and thus (2.15) is proved when $\Delta < 0$.

When $\Delta = 0$ and $x > \delta$, we assumed $\mathrm{Re}(\mu) < 0$ in the hypothesis (iii) of the theorem. Therefore if $R > R_0$, then we have from (2.23) that

$$\begin{aligned}\left|\int_{\Lambda_1} x^{-s}\mathcal{H}(s)ds\right| &\leqq Ax^{-\gamma}\ R^{\mathrm{Re}(\mu)+1} \int_0^{\pi/2} (x/\delta)^{-R\cos\theta}d\theta \\ &= Ax^{-\gamma}\ R^{\mathrm{Re}(\mu)+1} \int_0^{\pi/2} e^{-R\cos\theta\log(x/\delta)}d\theta \\ &= Ax^{-\gamma}\ R^{\mathrm{Re}(\mu)+1} \int_0^{\pi/2} e^{-R\sin\theta\log(x/\delta)}d\theta \\ &\leqq Ax^{-\gamma}\ R^{\mathrm{Re}(\mu)+1} \int_0^{\pi/2} e^{-2R\theta\log(x/\delta)/\pi}d\theta \\ &= \pi Ax^{-\gamma}\ R^{\mathrm{Re}(\mu)}\ \frac{1-e^{-R\log(x/\delta)}}{2\log(x/\delta)} \to 0\end{aligned}$$

as $R \to \infty$. The proof of (2.17) for $i = 1$ is completed. The proof for the case $i = 2$ is similar. Thus the theorem is proved.

Theorem 2. *Suppose that $\alpha < \gamma < \beta$ and that either of the conditions $a^* > 0$ or $a^* = 0$ and $\Delta\gamma + \mathrm{Re}(\mu) < -1$ holds. Then for $x > 0$, except for $x = \delta$ when $a^* = 0$ and $\Delta = 0$, the relation*

$$H^{m,n}_{p,q}\left[x\,\middle|\,\begin{matrix}(a_p,\alpha_p)\\ \\(b_q,\beta_q)\end{matrix}\right] = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \mathcal{H}^{m,n}_{p,q}\left[\begin{matrix}(a_p,\alpha_p)\\ \\(b_q,\beta_q)\end{matrix}\,\middle|\,t\right] x^{-t}dt \tag{2.24}$$

holds and the estimate

$$\left|H^{m,n}_{p,q}\left[x\,\middle|\,\begin{matrix}(a_p,\alpha_p)\\ \\(b_q,\beta_q)\end{matrix}\right]\right| \leqq A_\gamma\ x^{-\gamma} \tag{2.25}$$

is valid, where A_γ is a positive constant depending only on γ.

Proof. If $a^* > 0$, or $a^* = 0$ and $\Delta\gamma + \mathrm{Re}(\mu) < -1$, then we obtain from (2.9) that $\mathcal{H}(\gamma + it) \in L_1(-\infty,\infty)$. So (2.24) follows from (2.15). The estimate (2.25) can be seen from the proof of Theorem 1.

Remark. If $\alpha_1 = \cdots = \alpha_p = \beta_1 = \cdots = \beta_q = 1$ (Meijer's G-function), then

$$a^* = 2c^*,\ \delta = 1,\ \Delta = q - p \tag{2.26}$$

and therefore Lemmas 3.1, 3.2 and 3.3 in [31] follow from Lemma 1, Theorems 1 and 2, respectively.

3. $\mathcal{L}_{\nu,2}$ theory of the transform $\boldsymbol{H}$

For two Banach spaces X and Y, we use the notation $[X, Y]$ to denote the collection of bounded linear operators from X to Y. To obtain $\mathcal{L}_{\nu,2}$ theory of the transform $\boldsymbol{H}$, we use the following statement on the integral transform $\boldsymbol{H}$ of the form

$$(\boldsymbol{H}f)(x) = hx^{1-(\lambda+1)/h} \frac{d}{dx} x^{(\lambda+1)/h} \int_0^\infty k(xt)f(t)dt, \tag{3.1}$$

where the kernel $k \in \mathcal{L}_{1-\nu,2}$, $\lambda \in C$ and $h \in \boldsymbol{R}\backslash\{0\}$.

Lemma 2. (a) *If* $k \in \mathcal{L}_{1-\nu,2}$, $\nu \neq 1 - (\mathrm{Re}(\lambda) + 1)/h$, *and*

$$(\mathfrak{M}k)(1 - \nu + it) = \frac{\omega(t)}{\lambda + 1 - (1 - \nu + it)h} \quad \text{almost everywhere,} \tag{3.2}$$

where $\omega \in L_\infty(-\infty, \infty)$, *then the transform* $\boldsymbol{H}$ *of the form* (3.1) *is in* $[\mathcal{L}_{\nu,2}, \mathcal{L}_{1-\nu,2}]$. *Moreover, if* $f \in \mathcal{L}_{\nu,2}$, *then*

$$(\mathfrak{M}\boldsymbol{H}f)(1 - \nu + it) = \omega(t)(\mathfrak{M}f)(\nu - it) \quad \text{almost everywhere.} \tag{3.3}$$

(b) *Conversely, given* $\omega \in L_\infty(-\infty, \infty)$, $\nu \in \boldsymbol{R}$ *and* $h \in \boldsymbol{R}_+$, *there is a transform* $\boldsymbol{H} \in [\mathcal{L}_{\nu,2}, \mathcal{L}_{1-\nu,2}]$ *so that* (3.3) *holds for* $f \in \mathcal{L}_{\nu,2}$. *Moreover, if* $\nu \neq 1 - (\mathrm{Re}(\lambda) + 1)/h$, *then* $\boldsymbol{H}f$ *is representable in the form* (3.1) *with the kernel* k *given by* (3.2).

(c) *Under the hypotheses of* (a) *or* (b) *with* $\omega \neq 0$ *almost everywhere,* $\boldsymbol{H}$ *is a one-to-one transform from* $\mathcal{L}_{\nu,2}$ *into* $\mathcal{L}_{1-\nu,2}$, *and if in addition* $1/\omega \in L_\infty(-\infty, \infty)$, *then* $\boldsymbol{H}$ *transforms* $\mathcal{L}_{\nu,2}$ *onto* $\mathcal{L}_{1-\nu,2}$. *For* $f \in \mathcal{L}_{\nu,2}$ *and* $g \in \mathcal{L}_{\nu,2}$, *the relation*

$$\int_0^\infty f(x)(\boldsymbol{H}g)(x)dx = \int_0^\infty g(x)(\boldsymbol{H}f)(x)dx \tag{3.4}$$

is valid.

The proof of this lemma is based on the relation (1.7) and the properties of Mellin transform (1.4), and is carried out similarly to the proof of Lemma 4.1 in [31] by taking

$$g_a(t) = \begin{cases} t^{(\lambda+1)/h} & \text{if } 0 < t < a, \\ 0 & \text{if } t > a, \end{cases}$$

and

$$h_a(t) = \begin{cases} 0 & \text{if } 0 < t < a, \\ t^{(\lambda+1)/h} & \text{if } t > a, \end{cases}$$

instead of (4,4a) and (4.6a) in [31] and $h(\nu - 1) + 1$ instead of μ.

We apply Lemma 2 to obtain $\mathcal{L}_{\nu,2}$ theory of the transform (1.1).

Definition. For the function $\mathcal{H}(s)$ given in (1.9) we call the *exceptional set* of $\mathcal{H}$ the set of real numbers ν such that $\alpha < 1-\nu < \beta$ and $\mathcal{H}(s)$ has a zero on the line $\mathrm{Re}(s) = 1-\nu$.

Theorem 3. *We suppose that*

(a) $\qquad \alpha < 1-\nu < \beta$

and that either of conditions

(b) $\qquad a^* > 0,$

(c) $\qquad a^* = 0,\ \Delta(1-\nu) + \mathrm{Re}(\mu) \leqq 0$

holds, then we have the following results:

(i) *There is a one-to-one transform* $\boldsymbol{H} \in [\mathfrak{L}_{\nu,2},\ \mathfrak{L}_{1-\nu,2}]$ *so that* (1.8) *holds for* $\mathrm{Re}(s) = 1-\nu$ *and* $f \in \mathfrak{L}_{\nu,2}$. *If* $a^* = 0$, $\Delta(1-\nu) + \mathrm{Re}(\mu) = 0$ *and* ν *is not in the exceptional set of* $\mathcal{H}$, *then the operator* $\boldsymbol{H}$ *transforms* $\mathfrak{L}_{\nu,2}$ *onto* $\mathfrak{L}_{1-\nu,2}$.

(ii) *If* $f \in \mathfrak{L}_{\nu,2}$ and $\mathrm{Re}(\lambda) > (1-\nu)h - 1$, *then* $\boldsymbol{H}f$ *is given by* (1.10), *namely*

$$(3.5) \qquad (\boldsymbol{H}f)(x) = hx^{1-(\lambda+1)/h}\frac{d}{dx}x^{(\lambda+1)/h} \cdot \int_0^\infty H^{m,n+1}_{p+1,q+1}\left[xt \,\middle|\, \begin{matrix}(-\lambda,h),(a_1,\alpha_1),\cdots,(a_p,\alpha_p)\\ (b_1,\beta_1),\cdots,(b_q,\beta_q),(-\lambda-1,h)\end{matrix}\right] f(t)dt.$$

If $f \in \mathfrak{L}_{\nu,2}$ and $\mathrm{Re}(\lambda) < (1-\nu)h - 1$, *then* $\boldsymbol{H}f$ *is given by*

$$(3.6) \qquad (\boldsymbol{H}f)(x) = -hx^{1-(\lambda+1)/h}\frac{d}{dx}x^{(\lambda+1)/h} \cdot \int_0^\infty H^{m+1,n}_{p+1,q+1}\left[xt \,\middle|\, \begin{matrix}(a_1,\alpha_1),\cdots,(a_p,\alpha_p),(-\lambda,h)\\ (-\lambda-1,h),(b_1,\beta_1),\cdots,(b_q,\beta_q)\end{matrix}\right] f(t)dt.$$

Further, if $f \in \mathfrak{L}_{\nu,2}$ and $g \in \mathfrak{L}_{\nu,2}$, *then the relation* (3.4) *holds for* $\boldsymbol{H}$.

(iii) *Moreover,* $\boldsymbol{H}$ *is independent of* ν *in the sense that if* ν_1 *and* ν_2 *satisfy* (a), *and* (b) *or* (c), *and if the transforms* $\boldsymbol{H}_1$ *and* $\boldsymbol{H}_2$ *are given by* (1.8), *then* $\boldsymbol{H}_1 f = \boldsymbol{H}_2 f$ *for* $f \in \mathfrak{L}_{\nu_1,2} \cap \mathfrak{L}_{\nu_2,2}$.

Proof. Let $\omega(t) = \mathcal{H}(1-\nu+it)$. By virtue of (1.9), (2.1), (2.2) and the condition (a) the function $\mathcal{H}(s)$ is analytic in the strip $\alpha < \mathrm{Re}(s) < \beta$. In accordance with (2.9) and the condition (b) or (c), $\omega(t) = O(1)$ as $|t| \to \infty$. Therefore $\omega \in L_\infty(-\infty,\infty)$, and hence we obtain from Lemma 2 (b) that there is a transform $\boldsymbol{H} \in [\mathfrak{L}_{\nu,2},\ \mathfrak{L}_{1-\nu,2}]$ such that

$$(\mathfrak{M}\boldsymbol{H}f)(1-\nu+it) = \mathcal{H}(1-\nu+it)(\mathfrak{M}f)(\nu-it)$$

for $f \in \mathfrak{L}_{\nu,2}$. This means that the equality (1.8) holds for $\mathrm{Re}(s) = 1-\nu$. Since $\mathcal{H}(s)$ is analytic in the strip $\alpha < \mathrm{Re}(s) < \beta$ and has isolated zeros, then $\omega(t) \neq 0$ almost everywhere.

Thus we obtain from Lemma 2 (c) that $\boldsymbol{H} \in [\mathcal{L}_{\nu,2}, \mathcal{L}_{1-\nu,2}]$ is a one-to-one transform. If $a^* = 0$, $\Delta(1-\nu)+\mathrm{Re}(\mu) = 0$ and ν is not in the exceptional set of $\mathcal{H}$, then $1/\omega \in L_\infty(-\infty,\infty)$, and again from Lemma 2 (c) we obtain that $\boldsymbol{H}$ transforms $\mathcal{L}_{\nu,2}$ onto $\mathcal{L}_{1-\nu,2}$. This comletes the proof of the first assertion (i) of the theorem.

Let us prove (3.5). Let $f \in \mathcal{L}_{\nu,2}$ and $\mathrm{Re}(\lambda) > (1-\nu)h - 1$. To show that in this case $\boldsymbol{H}f$ is given by (3.5), it is sufficient to calculate the kernel k in the transform (3.1) for such λ. From (3.2) we have the equality

$$(\mathfrak{M}k)(1-\nu+it) = \mathcal{H}(1-\nu+it)\,\frac{1}{\lambda+1-(1-\nu+it)h},$$

or, for $\mathrm{Re}(s) = 1-\nu$

$$(\mathfrak{M}k)(s) = \mathcal{H}(s)\,\frac{1}{\lambda+1-hs}.$$

Then from (1.5) we obtain the expression for the kernel k, namely

$$k(x) = \frac{1}{2\pi i}\lim_{R\to\infty}\int_{1-\nu-iR}^{1-\nu+iR} x^{-s}(\mathcal{H}f)(s)\,\frac{1}{\lambda+1-hs}ds, \tag{3.7}$$

where the limit is taken in the topology of $\mathcal{L}_{\nu,2}$.

According to (1.9) we have

$$\begin{aligned}\frac{\mathcal{H}(s)}{\lambda+1-hs} &= \mathcal{H}(s)\frac{\Gamma(1-(-\lambda)-hs)}{\Gamma(1-(-\lambda-1)-hs)} \\ &= \mathcal{H}^{m,n+1}_{p+1,q+1}\left[\begin{array}{c|c} (-\lambda,h),(a_1,\alpha_1),\cdots,(a_p,\alpha_p) & \\ & s \\ (b_1,\beta_1),\cdots,(b_q,\beta_q),(-\lambda-1,h) & \end{array}\right].\end{aligned} \tag{3.8}$$

We denote by $\alpha_1, \beta_1, a_1^*, \Delta_1, \mu_1$ that in (2.1), (2.2), (2.3), (2.7), (2.8) for this $\mathcal{H}^{m,n+1}_{p+1,q+1}$. Then $\alpha_1 = \alpha, \beta_1 = \min[\beta, (1+\mathrm{Re}(\lambda))/h], a_1^* = a^*, \Delta_1 = \Delta$, $\mu_1 = \mu - 1$. Thus, it follows that $\alpha_1 < 1-\nu < \beta_1$ from $\mathrm{Re}(\lambda) > (1-\nu)h-1$, and either of the following conditions hold:

$$a_1^* > 0,$$

$$a_1^* = 0,\ \Delta_1 \neq 0 \text{ and } \Delta_1(1-\nu)+\mathrm{Re}(\mu_1) = \Delta(1-\nu)+\mathrm{Re}(\mu)-1 \leqq -1,$$

or

$$a_1^* = 0,\ \Delta_1 = 0 \text{ and } \mathrm{Re}(\mu_1) = \mathrm{Re}(\mu)-1 \leqq -1.$$

Therefore, applying Theorem 2 for $x > 0$, except possibly for $x = \delta$ we obtain that the equality

$$\begin{aligned}&H^{m,n+1}_{p+1,q+1}\left[x\,\middle|\,\begin{array}{c} (-\lambda,h),(a_1,\alpha_1),\cdots,(a_p,\alpha_p) \\ (b_1,\beta_1),\cdots,(b_q,\beta_q),(-\lambda-1,h)\end{array}\right] \\ &= \frac{1}{2\pi i}\lim_{R\to\infty}\int_{1-\nu-iR}^{1-\nu+iR} x^{-s}(\mathcal{H}f)(s)\,\frac{1}{\lambda+1-hs}\,ds\end{aligned} \tag{3.9}$$

holds almost everywhere.

It follows from (3.7) and (3.9) that the kernel k is given by

$$k(x) = H^{m,n+1}_{p+1,q+1}\left[x \,\middle|\, \begin{matrix} (-\lambda, h), (a_1, \alpha_1), \cdots, (a_p, \alpha_p) \\ (b_1, \beta_1), \cdots, (b_q, \beta_q), (-\lambda - 1, h) \end{matrix}\right].$$

Hence (3.5) is proved.

The relation (3.6) can be proved similarly to (3.5), if we use the equality

$$\text{(3.10)} \qquad \frac{\mathcal{H}(s)}{\lambda + 1 - hs} = -\mathcal{H}(s)\,\frac{\Gamma(hs - \lambda - 1)}{\Gamma(hs - \lambda)}$$

$$= -\mathcal{H}^{m+1,n}_{p+1,q+1}\left[\begin{matrix} (a_1, \alpha_1), \cdots, (a_p, \alpha_p), (-\lambda, h) \\ (-\lambda - 1, h), (b_1, \beta_1), \cdots, (b_q, \beta_q) \end{matrix} \,\middle|\, s\right]$$

instead of (3.8).

If $f \in \mathcal{L}_{\nu,2}$ and $g \in \mathcal{L}_{\nu,2}$, then the relation (3.4) is valid according to Lemma 2 (c). The assertion (ii) is proved.

Lastly, let us prove (iii). If $f \in \mathcal{L}_{\nu_1,2} \cap \mathcal{L}_{\nu_2,2}$ and $\mathrm{Re}(\lambda) > \max[(1-\nu_1)h - 1, (1-\nu_2)h - 1]$, then both of transforms $\boldsymbol{H}_1 f$ and $\boldsymbol{H}_2 f$ (corresponding to ν_1 and ν_2, respectively) are given by (3.5). The right hand side of (3.5) is independent of ν and hence

$$(\boldsymbol{H}_1 f)(x) = (\boldsymbol{H}_2 f)(x)$$

$$= h x^{1-(\lambda+1)/h} \frac{d}{dx} x^{(\lambda+1)/h} \int_0^\infty H^{m,n+1}_{p+1,q+1}\left[xt \,\middle|\, \begin{matrix} (-\lambda, h), (a_1, \alpha_1), \cdots, (a_p, \alpha_p) \\ (b_1, \beta_1), \cdots, (b_q, \beta_q), (-\lambda - 1, h) \end{matrix}\right] f(t)dt,$$

which completes the proof of the theorem.

Corollary 1. *Let $\alpha < \beta$ and one of the following conditions holds*

(b) $\qquad a^* > 0,$

(e) $\qquad a^* = 0,\ \Delta > 0$ *and* $\alpha < -\dfrac{\mathrm{Re}(\mu)}{\Delta},$

(f) $\qquad a^* = 0,\ \Delta < 0$ *and* $\beta > -\dfrac{\mathrm{Re}(\mu)}{\Delta},$

(g) $\qquad a^* = 0,\ \Delta = 0$ *and* $\mathrm{Re}(\mu) \leqq 0.$

Then the transform $\boldsymbol{H}$ can be defined on $\mathcal{L}_{\nu,2}$ with $1 - \beta < \nu < 1 - \alpha$.

Proof. If $\alpha < 1-\nu < \beta$, or $1-\beta < \nu < 1-\alpha$, then by Theorem 3 either of the conditions $a^* > 0$ or $a^* = 0,\ \Delta(1 - \nu) + \mathrm{Re}(\mu) \leqq 0$ must be satisfied in order that the transform $\boldsymbol{H}$ can be defined on $\mathcal{L}_{\nu,2}$. Hence the corollary is clear in the former case and in the latter one with $\Delta = 0$. We consider the latter case when $\Delta \neq 0$. If $\Delta > 0$ then the inequality $\Delta(1-\nu) + \mathrm{Re}(\mu) \leqq 0$ is equivalent to $\nu \geqq 1 + \mathrm{Re}(\mu)/\Delta$, and this is compatible with the condition $\nu < 1-\alpha$ only if $\alpha < -\mathrm{Re}(\mu)/\Delta$. Similarly, if $\Delta < 0$, then $\Delta(1-\nu)+\mathrm{Re}(\mu) \leqq 0$ is equivalent to $\nu \leqq 1+\mathrm{Re}(\mu)/\Delta$, and this is compatible with $1-\beta < \nu$ only if $\beta > -\mathrm{Re}(\mu)/\Delta$. This completes the proof.

Theorem 4. *Let $\alpha < 1-\nu < \beta$ and either of the the following conditions holds*

(b) $a^* > 0$,

(d) $a^* = 0, \ \Delta(1-\nu) + \mathrm{Re}(\mu) < -1.$

Then for $x > 0$ $(\boldsymbol{H}f)(x)$ is given by (1.1) *for $f \in \mathfrak{L}_{\nu,2}$, namely,*

$$(\boldsymbol{H}f)(x) = \int_0^\infty H_{p,q}^{m,n}\left[xt \left| \begin{matrix} (a_1,\alpha_1),\cdots,(a_p,\alpha_p) \\ \\ (b_1,\beta_1),\cdots,(b_q,\beta_q) \end{matrix} \right.\right] f(t)dt. \tag{3.11}$$

Proof. It follows from [28, §8.3] that if $\mathrm{Re}(\lambda) > (1-\nu)h - 1$, then $H_{p+1,q+1}^{m,n+1}$ in (3.5) is continuously differentiable on $(0,\infty)$. Therefore we can differentiate under the integral sign in (3.5). Applying the relations [28, (8.3.1.16)] and [28, (8.3.1.6)], we arrive at (3.11) provided that the integral in (3.11) exists.

The existence of this integral is proved on the basis of (2.25) similarly to those in [31] for the G-transform. Indeed, we choose γ_1 and γ_2 so that $\alpha < \gamma_1 < 1-\nu < \gamma_2 < \beta$. According to (2.25) there are constants A_1 and A_2 such that for almost all $t > 0$, the inequalities

$$|H_{p+1,q+1}^{m,n+1}(t)| \leqq A_i t^{-\gamma_i}, \quad (i = 1,2).$$

hold. Therefore, using the Schwartz ineguality, we have

$$\int_0^\infty |H_{p+1,q+1}^{m,n+1}(xt)f(t)|dt$$

$$\leqq \left[A_1 x^{-\gamma_1}\left(\int_0^{1/x} t^{2(1-\nu-\gamma_1)-1}dt\right)^{1/2} + A_2 x^{-\gamma_2}\left(\int_{1/x}^\infty t^{2(1-\nu-\gamma_2)-1}dt\right)^{1/2}\right] ||f||_{\nu,2}$$

$$= Cx^{\nu-1} < \infty,$$

where

$$C = \left\{A_1[2(1-\nu-\gamma_1)]^{-1/2} + A_1[2(\gamma_2-\nu+1)]^{-1/2}\right\} ||f||_{\nu,2},$$

and the theorem is proved.

In conclusion of this paper we indicate the conditions for the transform $\boldsymbol{H}$ (3.11) to be defined on some $\mathfrak{L}_{\nu,2}$ space.

Corollary 2. *Let $\alpha < \beta$ and one of the following conditions holds*

(b) $a^* > 0,$

(h) $a^* = 0,\ \Delta > 0 \text{ and } \alpha < -\dfrac{\operatorname{Re}(\mu)+1}{\Delta},$

(i) $a^* = 0,\ \Delta < 0 \text{ and } \beta > -\dfrac{\operatorname{Re}(\mu)+1}{\Delta},$

(j) $a^* = 0,\ \Delta = 0 \text{ and } \operatorname{Re}(\mu) < -1.$

Then the transform $\boldsymbol{H}$ *can be defined* (3.11) *on* $\mathfrak{L}_{\nu,2}$ *with* $1-\beta < \nu < 1-\alpha$.

Proof. The proof of this statement is similar to those of Corollary 1.

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