数理解析研究所講究録
第821巻 1993年 56-69

Application of Generalized Fractional Calculus Operators in the Solution of Certain Dual Integral Equations

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Abstract

A formal solution of certain dual integral equations involving H-functions is derived by the application of the operators of fractional calculus due to Saigo [14], [15]. It has been shown that the given dual integral equations can be transformed, by the application of the operators, into two others with a common kernel and the problem then reduces to that of solving a single integral equation. Since the common kernel comes out to be a symmetrical Fourier kernel investigated by Fox [8], the formal solution readily follows.

1. Introduction and Preliminaries

Following Fox [7], we define the H-function in the notation of Saxena [20] in the form:

(1.1)
$$H_{P,Q}^{M,N}(x) \equiv H_{P,Q}^{M,N}\left[x \,\middle|\, \begin{matrix}[a_P, A_P]\\ \\ [b_Q, B_Q]\end{matrix}\right] = H_{P,Q}^{M,N}\left[x \,\middle|\, \begin{matrix}(a_1, A_1), \cdots, (a_P, A_P)\\ \\ (b_1, B_1), \cdots, (b_Q, B_Q)\end{matrix}\right]$$
$$= \frac{1}{2\pi\omega}\int_C \chi(s) x^{-s} ds,$$

where $\omega = \sqrt{-1}$ and

(1.2)
$$\chi(s) = \frac{\prod_{j=1}^{M}\Gamma(b_j + B_j s)\prod_{j=1}^{N}\Gamma(1 - a_j - A_j s)}{\prod_{j=M+1}^{Q}\Gamma(1 - b_j - B_j s)\prod_{j=N+1}^{P}\Gamma(a_j + A_j s)}.$$

Here an empty product is to be interpreted as unity and the following simplified assumptions are made:

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(i) P, Q, M, N are integers satisfying $0 \leqq M \leqq Q$, $1 \leqq N \leqq P$,

(ii) A_i's and B_i's are positive numbers for $i = 1, \cdots, P$ and $j = 1, \cdots, Q$,

(iii) a_j $(j = 1, \cdots, P)$ and b_j $(j = 1, \cdots, Q)$ are complex numbers,

(iv) The contour $\mathcal{C}$ is a straight line parallel to the imaginary axis in the s-plane with $s = \sigma + \tau\sqrt{-1}$ such that all the poles of $\Gamma(b_j + B_j s)$ for $j = 1, \cdots, M$ lie to the left and those of $\Gamma(1 - a_j - A_j s)$ for $j = 1, \cdots, N$ to the right of it.

A detailed account of the convergence conditions and analytical continuation of the H-functions is given by Braaksma [1]. Regarding applications of H-function in statistical distribution and integrals, series expansions of the H-function, the reader is referred to the monograph by Mathai and Saxena [11].

When $A_i = B_j = 1$ $(i = 1, \cdots, P;\ j = 1, \cdots, Q)$, the H-function reduces to Meijer's G-function. The result is

$$G_{P,Q}^{M,N}(x) \equiv G_{P,Q}^{M,N}\left(x \,\middle|\, \begin{matrix} a_1, \cdots, a_P \\ \\ b_1, \cdots, b_Q \end{matrix}\right) = \frac{1}{2\pi\omega}\int_{\mathcal{C}} \chi(s) x^{-s} ds, \tag{1.3}$$

where

$$\chi(s) = \frac{\prod_{j=1}^{M} \Gamma(b_j + s) \prod_{j=1}^{N} \Gamma(1 - a_j - s)}{\prod_{j=M+1}^{Q} \Gamma(1 - b_j - s) \prod_{j=N+1}^{P} \Gamma(a_j + s)}. \tag{1.4}$$

Here an empty product is to be interpreted as unity and a_j $(j = 1, \cdots, P)$, b_j $(j = 1, \cdots, Q)$ are complex numbers such that none of the poles of $\Gamma(b_j + s)$ $(j = 1, \cdots, M)$ coincide with any of the poles of $\Gamma(1 - a_j - s)$ $(j = 1, \cdots, N)$. The contour $\mathcal{C}$ separates these two sets of poles. General existence conditions are also available from Mathai and Saxena [10].

Fox [7] has shown that the function

$$H_{2m,2n}^{n,m}(x) \equiv H_{2m,2n}^{n,m}\left[x \,\middle|\, \begin{matrix} [1 - a_m, A_m], [a_m - A_m, A_m] \\ \\ [b_n, B_n], [1 - b_n - B_n, B_n] \end{matrix}\right] = \frac{1}{2\pi\omega}\int_{\mathcal{C}} \chi_{m,n}(s) x^{-s} ds, \tag{1.5}$$

where

$$\chi_{m,n}(s) = \prod_{j=1}^{n} \frac{\Gamma(b_j + B_j s)}{\Gamma(b_j + B_j - B_j s)} \prod_{j=1}^{m} \frac{\Gamma(a_j - A_j s)}{\Gamma(a_j - A_j + A_j s)} \tag{1.6}$$

behaves as a symmetrical Fourier kernel.

From (1.5), it follows that the Mellin transform of $H_{2m,2n}^{n,m}(x)$ is

$$\mathfrak{M}\left\{H_{2m,2n}^{n,m}(x)\right\}(s) = \chi_{m,n}(s), \tag{1.7}$$

where $\mathfrak{M}$ is the Mellin transform

$$\mathfrak{M}\{f(x)\}(s) = \int_0^\infty f(x)x^{s-1}dx.$$

Dual integral equations occur in many problems of Mathematical Physics especially those which are connected with mixed boundary conditions.

A well-known example of dual integral equations possessing ordinary Bessel functions $J_\nu(x)$ and $J_\mu(x)$, as their kernels, is

$$\text{(1.8)} \qquad \begin{cases} \int_0^\infty t^\rho J_\nu(tx)h(t)dt = \phi(x), & (0 < x < 1), \\ \int_0^\infty t^\sigma J_\mu(tx)h(t)dt = \psi(x), & (x > 1), \end{cases}$$

where $\phi(x)$ and $\psi(x)$ are given and $h(x)$ is to be determined.

Weber [25] solved the above equations for the case $\rho = \mu = \nu = 0$, $\sigma = 1$ in connection with the problem of finding the electrostatic field arising from a circular disk charged to a constant potential. Later on several workers developed various methods from time to time to solve the equations (1.8) notably by Busbridge [2], Erdélyi and Sneddon [6], Noble [12], Peters [13], Saxena and Kushwaha [22], Virchenko [24] etc.

A systematic analysis is developed by Fox [8] to derive the solution of dual integral equations of a general character than (1.8) associated with H-functions of order n by the application of Erdélyi-Kober operators [3] [9]. His results are further generalized by Saxena [20] [21] by considering the dual integral equations involving general H-functions which are more general character than the H-functions discussed by Fox [8].

The object of this paper is to develop a formal solution of certain dual integral equations associated with H-functions by the application of generalized fractional calculus operators introduced by Saigo [14] [15].

2. Generalized Fractional Calculus Operators

In order to provide an elegant generalization of Riemann-Liouville and Erdélyi-Kober operators of fractional calculus, Saigo [14] [15] introduced a generalization of operators of fractional calculus and derived in a series of papers [14] [15] [16] [17] [18] [19] their various properties and applications (cf. also [23]).

We give here a slight modification of such operators. Let α, β, η be complex numbers and $r > 0$. The Saigo operators are recognized as the case $r = 1$ of the following $I_{0,x;r}^{\alpha,\beta,\eta}$ and $J_{x,\infty;r}^{\alpha,\beta,\eta}$.

$$\text{(2.1)} \qquad I_{0,x;r}^{\alpha,\beta,\eta} f = \frac{rx^{-r(\alpha+\beta)}}{\Gamma(\alpha)} \int_0^x (x^r - t^r)^{\alpha-1} {}_2F_1\left(\alpha+\beta, -\eta; \alpha; 1 - \frac{t^r}{x^r}\right) t^{r-1} f(t)dt$$

for $\mathrm{Re}(\alpha) > 0$, and

$$\text{(2.2)} \qquad I_{0,x;r}^{\alpha,\beta,\eta} f = \frac{d^n}{d(x^r)^n} I_{0,x;r}^{\alpha+n,\beta-n,\eta-n} f$$

for $0<\mathrm{Re}(\alpha)+n\leqq 1$ $(n=1,2,3,\cdots)$, where ${}_2F_1(a,b;c;\cdot)$ is Gauss's hypergeometric function.

$$J^{\alpha,\beta,\eta}_{x,\infty;r}f=\frac{r}{\Gamma(\alpha)}\int_x^\infty (t^r-x^r)^{\alpha-1}t^{-r(\alpha+\beta)}{}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{x^r}{t^r}\right)t^{r-1}f(t)dt \tag{2.3}$$

for $\mathrm{Re}(\alpha)>0$, and

$$J^{\alpha,\beta,\eta}_{x,\infty;r}f=(-1)^n\frac{d^n}{d(x^r)^n}J^{\alpha+n,\beta-n,\eta}_{x,\infty;r}f \tag{2.4}$$

for $0<\mathrm{Re}(\alpha)+n\leqq 1$ $(n=1,2,3,\cdots)$.

The operators $I^{\alpha,\beta,\eta}_{0,x;r}$ and $J^{\alpha,\beta,\eta}_{x,\infty;r}$ involve as their special cases $\beta=-\alpha$ the fractional calculus operators of Riemann-Liouville and Weyl operators:

$$I^{\alpha,\beta,\eta}_{0,x;r}f\equiv R^{\alpha}_{0,x;r}f, \tag{2.5}$$

$$J^{\alpha,\beta,\eta}_{x,\infty;r}f\equiv W^{\alpha}_{x,\infty;r}f. \tag{2.6}$$

In a similar manner to the case $r=1$, we can obtain the following identities and inverses:

$$I^{0,0,\eta}_{0,x;r}f=f(x), \tag{2.7}$$

$$J^{0,0,\eta}_{x,\infty;r}f=f(x), \tag{2.8}$$

$$\left[I^{\alpha,\beta,\eta}_{0,x;r}\right]^{-1}=I^{-\alpha,-\beta,\alpha+\eta}_{0,x;r}, \tag{2.9}$$

$$\left[J^{\alpha,\beta,\eta}_{x,\infty;r}\right]^{-1}=J^{-\alpha,-\beta,\alpha+\eta}_{x,\infty;r}. \tag{2.10}$$

For the operators $I^{\alpha,\beta,\eta}_{0,x;r}$ and $J^{\alpha,\beta,\eta}_{x,\infty;r}$, there hold valid various interesting results discussed in the series [14] [15] [16] [17] [18] [19], in parallel.

In what follows, when $r=1$ we shall omit the index 1 in the operators.

3. Dual Integral Equations

The dual integral equations to be solved are

$$\begin{cases}\displaystyle\int_0^\infty H_1(xv)f(v)dv=\phi(x), & (0<x<1)\\[2ex] \displaystyle\int_0^\infty H_2(xv)f(v)dv=\psi(x), & (x>1),\end{cases} \tag{3.1}$$

where $\phi(x)$ and $\psi(x)$ are given and $f(x)$ is to be determined, and the functions $H_1(x)$ and $H_2(x)$ occuring in (3.1) possess the following definitions:

$$(3.2)\qquad H_1(x) \equiv H^{n,m+2k}_{2m+2k,2n+2k}(x)$$

$$= H^{n,m+2k}_{2m+2k,2n+2k}\left[x \left| \begin{matrix} [1-a_m, A_m], [1-\sigma_k, \tau_k], [1-\gamma_k-\delta_k-\eta_k-\sigma_k, \tau_k], [a_m - A_m, A_m] \\ [b_n, B_n], [1-b_n-B_n, B_n], [1-\delta_k-\sigma_k, \tau_k], [1-\eta_k-\sigma_k, \tau_k] \end{matrix} \right. \right]$$

$$= \frac{1}{2\pi\omega}\int_C \chi_{m,n,k}(s) x^{-s} ds$$

and

$$(3.3)\qquad H_2(x) \equiv H^{n+2l,m}_{2m+2l,2n+2l}(x)$$

$$= H^{n+2l,m}_{2m+2l,2n+2l}\left[x \left| \begin{matrix} [1-a_m, A_m], [a_m - A_m, A_m], [1-\lambda_l+\theta_l-\kappa_l, \xi_l], [1-\lambda_l+\zeta_l-\kappa_l, \xi_l] \\ [b_n, B_n], [1-\lambda_l-\kappa_l, \xi_l], [1+\theta_l+\zeta_l-\kappa_l, \xi_l], [1-b_n-B_n, B_n] \end{matrix} \right. \right]$$

$$= \frac{1}{2\pi\omega}\int_C \widetilde{\chi}_{m,n,l}(s) x^{-s} ds,$$

where

$$(3.4)\qquad \begin{aligned} \chi_{m,n,k}(s) &= \prod_{i=1}^{n} \frac{\Gamma(b_i + B_i s)}{\Gamma(b_i + B_i - B_i s)} \prod_{i=1}^{m} \frac{\Gamma(a_i - A_i s)}{\Gamma(a_i - A_i + A_i s)} \\ &\quad \cdot \prod_{i=1}^{k} \frac{\Gamma(\sigma_i - \tau_i s)\Gamma(\gamma_i + \delta_i + \eta_i + \sigma_i - \tau_i s)}{\Gamma(\delta_i + \sigma_i - \tau_i s)\Gamma(\eta_i + \sigma_i - \tau_i s)} \end{aligned}$$

and

$$(3.5)\qquad \begin{aligned} \widetilde{\chi}_{m,n,l}(s) &= \prod_{i=1}^{n} \frac{\Gamma(b_i + B_i s)}{\Gamma(b_i + B_i - B_i s)} \prod_{i=1}^{m} \frac{\Gamma(a_i - A_i s)}{\Gamma(a_i - A_i + A_i s)} \\ &\quad \cdot \prod_{i=1}^{l} \frac{\Gamma(1-\lambda_i-\kappa_i+\xi_i s)\Gamma(1+\theta_i+\zeta_i-\kappa_i+\xi_i s)}{\Gamma(1-\lambda_i+\theta_i-\kappa_i+\xi_i s)\Gamma(1-\lambda_i+\zeta_i-\kappa_i+\xi_i s)}. \end{aligned}$$

Here, we assume that the following conditions are satisfied:

(i) $m \leqq n-1$;

(ii) a_i $(i = 1,\cdots,m)$, b_i $(i = 1,\cdots,n)$, γ_j, δ_j, η_j, σ_j $(j = 1,\cdots,k)$; λ_j, θ_j, ζ_j, κ_j $(j = 1,\cdots,l)$ are all complex numbers and A_i $(i = 1,\cdots,m)$, B_i $(i = 1,\cdots,n)$, τ_j $(j = 1,\cdots,k)$, ξ_j $(j = 1,\cdots,l)$ are all positive numbers;

(iii) Let $s = \sigma + \tau\sqrt{-1}$, where σ and τ are real, then the contour C along which the integrals are taken is a straight line parallel to the imaginary axis in the complex s-plane whose equation is $\sigma = \sigma_0$, where σ_0 is a constant;

(iv) All the poles of functions $\chi_{m,n,k}(s)$ and $\tilde{\chi}_{m,n,l}(s)$ are simple. The common contour $\mathcal{C}$ is such that all the poles of $\Gamma(b_i + B_i s)$ for $i = 1, \cdots, n$, $\Gamma(1 - \lambda_j - \kappa_j + \xi_j s)$ and $\Gamma(1 + \theta_j + \zeta_j - \kappa_j + \xi_j s)$ for $j = 1, \cdots, l$ lie to the left and those of $\Gamma(a_i - A_i s)$ for $i = 1, \cdots, m$, $\Gamma(\sigma_j - \tau_j s)$ and $\Gamma(\gamma_j + \delta_j + \eta_j + \sigma_j - \tau_j s)$ for $j = 1, \cdots, k$ to the right of $\mathcal{C}$;

(v) $\epsilon = 2\left(\sum_{i=1}^{n} B_i - \sum_{i=1}^{m} A_i\right) > 0;$

(vi) $\sigma_0 < \frac{1}{2} - \frac{1}{\epsilon}\sum_{j=1}^{k} \mathrm{Re}\,(\gamma_j)$ for (3.2);

(vii) $\sigma_0 < \frac{1}{2} - \frac{1}{\epsilon}\sum_{j=1}^{l} \mathrm{Re}\,(\lambda_j)$ for (3.3).

4. The Reduction of (3.1) to Equations with a Common Kernel

In this section we will transform the dual integral equations (3.1) into others with the same kernel by the application of the Mellin transform and the generalized fractional calculus operators introduced in Section 2.

From (3.2) and (3.3), we know that

$$\mathfrak{M}\{H_1(x)\}(s) = \chi_{m,n,k}(s), \qquad \mathfrak{M}\{H_2(x)\}(s) = \tilde{\chi}_{m,n,l}(s). \tag{4.1}$$

On writing $\mathfrak{M}\{f(v)\}(s) = F(s)$ and applying the Parseval formula (see e.g. [5, Vol.1, p.308])

$$\mathfrak{M}\left\{\int_0^\infty \varphi_1(xv)\varphi_2(v)dv\right\}(s) = \Phi_1(s)\Phi_2(1-s) \tag{4.2}$$

for $\mathfrak{M}\{\varphi_j(x)\}(s) = \Phi_j(s)$ $(j = 1, 2)$ to (3.1) we find that

$$\begin{cases} \dfrac{1}{2\pi\omega}\displaystyle\int_C \chi_{m,n,k}(s)x^{-s}F(1-s)ds = \phi(x), & (0 < x < 1) \\ \dfrac{1}{2\pi\omega}\displaystyle\int_C \tilde{\chi}_{m,n,l}(s)x^{-s}F(1-s)ds = \psi(x), & (x > 1). \end{cases} \tag{4.3}$$

Now, we shall require the well known integral in [5, Vol.2, p.399]:

$$\int_0^1 x^{c-1}(1-x)^{d-1}{}_2F_1(a,b;c;x)dx = \frac{\Gamma(c)\Gamma(d)\Gamma(c+d-a-b)}{\Gamma(c+d-a)\Gamma(c+d-b)} \tag{4.4}$$

for $\mathrm{Re}(c) > 0$, $\mathrm{Re}(d) > 0$, $\mathrm{Re}(c+d-a-b) > 0$.

Replacing x by t in the first equation in (4.3), multiplying by

$$t^{c_k\delta_k+c_k\sigma_k-1}\left(x^{c_k}-t^{c_k}\right)^{\gamma_k-1}{}_2F_1\left(\gamma_k+\delta_k,-\eta_k;\gamma_k;1-\frac{t^{c_k}}{x^{c_k}}\right),$$

where $c_k = (\tau_k)^{-1}$ and integrating through the integral sign with respect to t from 0 to x $(0 < x < 1)$, we find that

$$\int_0^x \frac{1}{2\pi\omega}\int_C \chi_{m,n,k}(s)t^{-s}F(1-s)t^{c_k\delta_k+c_k\sigma_k-1}\left(x^{c_k}-t^{c_k}\right)^{\gamma_k-1}$$

$$\cdot\, {}_2F_1\left(\gamma_k+\delta_k,-\eta_k;\gamma_k;1-\frac{t^{c_k}}{x^{c_k}}\right)ds\,dt$$

$$=\int_0^x t^{c_k\delta_k+c_k\sigma_k-1}\left(x^{c_k}-t^{c_k}\right)^{\gamma_k-1}{}_2F_1\left(\gamma_k+\delta_k,-\eta_k;\gamma_k;1-\frac{t^{c_k}}{x^{c_k}}\right)\phi(t)dt$$

or in term of the fractional integral (2.1)

$$\frac{1}{2\pi\omega}\int_C \chi_{m,n,k}(s)F(1-s)\int_0^x t^{c_k\delta_k+c_k\sigma_k-s-1}\left(x^{c_k}-t^{c_k}\right)^{\gamma_k-1}$$

$$\cdot\, {}_2F_1\left(\gamma_k+\delta_k,-\eta_k;\gamma_k;1-\frac{t^{c_k}}{x^{c_k}}\right)dt\,ds$$

$$=\frac{\Gamma(\gamma_k)}{c_k}x^{c_k(\gamma_k+\delta_k)}I_{0,x;c_k}^{\gamma_k,\delta_k,\eta_k}x^{c_k(\delta_k+\sigma_k-1)}\phi(x).$$

Evaluating the inner integral on the left, say A, by means of (4.4), we find that

$$A=\frac{x^{c_k(\gamma_k+\delta_k+\sigma_k-1)-s}}{c_k}\frac{\Gamma(\gamma_k)\Gamma(\delta_k+\sigma_k-\tau_k s)\Gamma(\eta_k+\sigma_k-\tau_k s)}{\Gamma(\sigma_k-\tau_k s)\Gamma(\gamma_k+\delta_k+\eta_k+\sigma_k-\tau_k s)}, \tag{4.5}$$

where $c_k = (\tau_k)^{-1}$, $\mathrm{Re}(\gamma_k) > 0$, $\mathrm{Re}(\delta_k+\sigma_k-\tau_k s) > 0$, $\mathrm{Re}(\eta_k+\sigma_k-\tau_k s) > 0$, we can curtail the number of k in the kernel $\chi_{m,n,k}(s)$ such as

$$\frac{1}{2\pi\omega}\int_C \chi_{m,n,k-1}(s)x^{-s}F(1-s)ds = x^{-c_k(\sigma_k-1)}I_{0,x;c_k}^{\gamma_k,\delta_k,\eta_k}x^{c_k(\delta_k+\sigma_k-1)}\phi(x) \tag{4.6}$$

for $0 < x < 1$.

Let us introduce for convenience's sake the first operator of fractional integration $\mathfrak{J}$ which is a slight variant of the operator (2.1) in the form

$$\mathfrak{J}\left[\gamma,\delta,\eta,\sigma;c\right]f(x) = x^{c(1-\sigma)}I_{0,x;c}^{\gamma,\delta,\eta}x^{c(\delta+\sigma-1)}f(x) \tag{4.7}$$

for $\mathrm{Re}(\gamma) > 0$, which can be determined as far as the operator $I_{0,x;c}^{\gamma,\delta,\eta}$ exists on a certain class of functions. When $\mathrm{Re}(\gamma) \leq 0$, the operator $\mathfrak{J}$ can be also considered by noting the formula (2.2). In particular, by virtue of (2.7)

$$\mathfrak{J}\left[0,0,\eta,\sigma;c\right]f(x) = f(x). \tag{4.8}$$

Let us set

$$\mathfrak{Z}_j f(x) = \mathfrak{Z}\left[\gamma_j, \delta_j, \eta_j, \sigma_j; c_j\right] f(x) \quad (j = 1, 2, \cdots, k) \tag{4.9}$$

for the parameters appearing in (3.2). Then it can be easily seen that the R.H.S. of (4.6) is equal to $\mathfrak{Z}_j\phi(x)$ with $0 < x < 1$. On transforming the first equation of (4.3) step by step by the application of the operator $\mathfrak{Z}_j$ $(j = k, k-1, \cdots, 2, 1)$, successively, it is observed that

$$\frac{1}{2\pi\omega}\int_C \chi_{m,n}(s) x^{-s} F(1-s)ds = \mathfrak{Z}_1\mathfrak{Z}_2\cdots\mathfrak{Z}_k\phi(x) \quad (0 < x < 1) \tag{4.10}$$

Further, in the second equation of (4.3) replace x by t, multiply by

$$t^{-d_l(\theta_l-\kappa_l)-1}(t^{d_l} - x^{d_l})^{\lambda_l-1} {}_2F_1\left(\lambda_l + \theta_l, -\zeta_l; \lambda_l; 1 - \frac{x^{d_l}}{t^{d_l}}\right)$$

and then integrate through the integral sign with respect to t from x to ∞ with $x > 1$, we find that

$$\begin{aligned}\frac{1}{2\pi\omega}\int_C \tilde{\chi}_{m,n,l}(s)F(1-s)&\left[\int_x^\infty t^{-d_l(\theta_l-\kappa_l)-s-1}\left(t^{d_l} - x^{d_l}\right)^{\lambda_l-1}\right.\\ &\left.\cdot\, {}_2F_1\left(\lambda_l + \theta_l, -\zeta_l; \lambda_l; 1 - \frac{x^{d_l}}{t^{d_l}}\right) dt\right] ds\\ = \int_x^\infty t^{-d_l(\theta_l-\kappa_l)-s-1}&\left(t^{d_l} - x^{d_l}\right)^{\lambda_l-1} {}_2F_1\left(\lambda_l + \theta_l, -\zeta_l; \lambda_l; 1 - \frac{x^{d_l}}{t^{d_l}}\right)\psi(t)dt.\end{aligned}$$

Evaluating the inner integral on the left by means of the formula (4.4), we have

$$\begin{aligned}&\int_x^\infty t^{-d_l(\theta_l-\kappa_l)-s-1}(t^{d_l} - x^{d_l})^{\lambda_l-1} {}_2F_1\left(\lambda_l + \theta_l, -\zeta_l; \lambda_l; 1 - \frac{x^{d_l}}{t^{d_l}}\right) dt\\ &= \frac{x^{d_l(\lambda_l-\theta_l+\kappa_l-1)-s}}{d_l}\,\frac{\Gamma(\lambda_l)\Gamma(1-\lambda_l+\theta_l-\kappa_l+\xi_l s)\Gamma(1-\lambda_l+\zeta_l-\kappa_l+\xi_l s)}{\Gamma(1-\lambda_l-\kappa_l+\xi_l s)\Gamma(1+\theta_l+\zeta_l-\kappa_l+\xi_l s)},\end{aligned} \tag{4.11}$$

where $\xi_l = (d_l)^{-1}$, $\mathrm{Re}(\lambda_l) > 0$, $\mathrm{Re}(1 - \lambda_l + \theta_l - \kappa_l + \xi_l s) > 0$, $\mathrm{Re}(1 - \lambda_l + \zeta_l - \kappa_l + \xi_l s) > 0$. Thus we find that

$$\begin{aligned}&\frac{1}{2\pi\omega}\int_C \tilde{\chi}_{m,n,l-1}(s)F(1-s)x^{-s}ds\\ &= \frac{d_l x^{d_l(\theta_l-\lambda_l-\kappa_l+1)}}{\Gamma(\lambda_l)}\int_x^\infty t^{-d_l(\theta_l-\kappa_l)-1}\left(t^{d_l} - x^{d_l}\right)^{\lambda_l-1} {}_2F_1\left(\lambda_l + \theta_l, -\zeta_l; \lambda_l; 1 - \frac{x^{d_l}}{t^{d_l}}\right)\psi(t)dt,\end{aligned} \tag{4.12}$$

for $x > 1$.

Let us introduce the second operator of fractional integration $\mathfrak{K}_j$ which is also a slight variant of the operator (2.3) in the form

$$\mathfrak{K}_j f(x) \equiv \mathfrak{K}\left[\lambda_j, \theta_j, \zeta_j, \kappa_j; d_j\right] f(x) = x^{d_j(-\lambda_j+\theta_j-\kappa_j+1)} J^{\lambda_j,\theta_j,\zeta_j}_{x,\infty;d_j} x^{d_j(\lambda_j+\kappa_j-1)} f(x) \tag{4.13}$$

for $j = 1, 2, \cdots, l$. To this operator a similar comment is valid to that following the formula (4.7) and

$$\mathfrak{K}[0, 0, \zeta, \kappa; d]\, f(x) = f(x). \tag{4.14}$$

It is evident that the R.H.S. of (4.12) is $\mathfrak{K}_l\psi(x)$ with $x > 1$. The successive application of the operators $\mathfrak{K}_j$ for $j = l, l-1, l-2, \cdots, 2, 1$ to the second equation of (4.3) transforms it into the desired form

$$\frac{1}{2\pi\omega}\int_C \chi_{m,n}(s)x^{-s}F(1-s)ds = \mathfrak{K}_1\mathfrak{K}_2\cdots\mathfrak{K}_l\psi(x), \quad (x > 1) \tag{4.15}$$

If we write

$$g(x) = \begin{cases} \mathfrak{J}_1\mathfrak{J}_2\cdots\mathfrak{J}_k\phi(x), & (0 < x < 1) \\ \mathfrak{K}_1\mathfrak{K}_2\cdots\mathfrak{K}_l\psi(x), & (x > 1), \end{cases} \tag{4.16}$$

(4.10) and (4.15) can be put into a compact form

$$\frac{1}{2\pi\omega}\int_C \chi_{m,n}(s)x^{-s}F(1-s)ds = g(x), \tag{4.17}$$

or in view of (1.7)

$$\frac{1}{2\pi\omega}\int_C \mathfrak{M}\left\{H^{n,m}_{2m,2n}(v)\right\}(s)F(1-s)x^{-s}ds = g(x). \tag{4.18}$$

Applying the formula (4.2) to the left-hand side of (4.18), we see that it can be expressed by an integral involving the product of $H^{n,m}_{2m,2n}(x)$ and $f(v)$. The result is

$$\int_0^\infty H^{n,m}_{2m,2n}(xv)f(v)dv = g(x), \tag{4.19}$$

where the kernel $H^{n,m}_{2m,2n}(x)$ is given by (1.5).

Since $H^{n,m}_{2m,2n}(x)$ is a symmetrical Fourier kernel, we, therefore, obtain the formal solution as

$$\begin{aligned} f(x) &= \int_0^\infty g(v)H^{n,m}_{2m,2n}(xv)dv \\ &= \int_0^1 \mathfrak{J}_1\mathfrak{J}_2\cdots\mathfrak{J}_k\phi(v)H^{n,m}_{2m,2n}(xv)dv + \int_1^\infty \mathfrak{K}_1\mathfrak{K}_2\cdots\mathfrak{K}_l\psi(v)H^{n,m}_{2m,2n}(xv)dv, \end{aligned} \tag{4.20}$$

where $\mathfrak{J}$'s and $\mathfrak{K}$'s are defined by (4.9) and (4.13).

Note. Since our method is formal, it does not give any condition of the validity of the solution.

5. Special Cases

(i) For $\beta = 0$, we obtain the results due to Saxena [20].

(ii) If we set

$$m = 0,\ n = 1,\ k = l = 1,\ b_1 = b,\ B_1 = 1,$$

$$\gamma_1 = \delta_1 = 0,\ \eta_1 = 1,\ \sigma_1 = a,\ \lambda = -\tfrac{1}{2},\ \theta_1 = -1,\ \zeta_1 = 0,\ \kappa_1 = 0,\ \tau_1 = 1,\ \xi_1 = 1$$

and use the identities in [4, p.216, 217], then the equations (3.2) and (3.3) are given by

$$(5.1)\qquad \begin{cases} H_1(x) = G_{0,2}^{1,0}\left(x \middle| \begin{matrix} \cdots \\ b, -b \end{matrix}\right) = J_{2b}(2\sqrt{x}), \\ \\ H_2(x) = G_{1,3}^{2,0}\left(x \middle| \begin{matrix} \frac{1}{2} \\ 0, b, -b \end{matrix}\right) = -\sqrt{\pi} J_b(\sqrt{x}) Y_b(\sqrt{x}). \end{cases}$$

We see that the formal solution of the dual integral equations

$$(5.2)\qquad \begin{cases} \displaystyle\int_0^\infty J_{2b}\left(2\sqrt{xv}\right) f(v)dv = \phi(x), & (0 < x < 1) \\ \\ \displaystyle -\sqrt{\pi}\int_0^\infty J_b\left(\sqrt{xv}\right) Y_b\left(\sqrt{xv}\right) f(v)dv = \psi(x), & (x > 1) \end{cases}$$

is given by

$$(5.3)\qquad f(x) = \int_0^1 J_{2b}\left(2\sqrt{xv}\right)\phi(v)dv$$

$$+\frac{1}{\sqrt{\pi}}\int_1^\infty \sqrt{v}\frac{\partial}{\partial v}\left[\int_v^\infty (t-v)^{-1/2}\psi(t)dt\right] J_{2b}\left(2\sqrt{xv}\right) dv$$

in view of (4.8), (4.13), (2.3) and (2.4). If we assume

$$(5.4)\qquad \psi(x) \in C^0[1,\infty),\ \psi(x) = O\left(x^\alpha\right),\ (x \to \infty) \text{ with } \alpha < -3/4,$$

then the solution $f(t)$ can be written in the form

$$(5.5)\qquad f(x) = \int_0^1 J_{2b}\left(2\sqrt{xv}\right)\phi(v)dv - \frac{1}{\sqrt{\pi}}J_{2b}\left(2\sqrt{x}\right)\int_1^\infty (t-1)^{-1/2}\psi(t)dt$$

$$-\frac{1}{\sqrt{\pi}}\int_1^\infty \left\{\sqrt{x}J_{2b-1}\left(2\sqrt{xv}\right) + (1-2b)\frac{1}{2\sqrt{v}}J_{2b}\left(2\sqrt{xv}\right)\right\}\int_v^\infty (t-v)^{-1/2}\psi(t)dtdv,$$

by the integration by parts and by virtue of the formulas

$$\frac{d}{dx}\left\{x^\nu J_\nu(x)\right\} = x^\nu J_{\nu-1}(x)$$

and

$$J_\nu(x) = O\left(x^{-1/2}\right), \quad (x \to \infty).$$

(iii) Setting

$$m = 0,\ n = 1,\ k = l = 2,\ b_1 = b,\ B_1 = 1,$$

$$\gamma_1 = -\tfrac{1}{2} - c,\ \gamma_2 = c,\ \delta_1 = \tfrac{1}{2} + c,\ \delta_2 = -c,\ \eta_1 = 0,\ \eta_2 = 1,\ \sigma_1 = \tfrac{1}{2},\ \sigma_2 = 1,\ \tau_1 = \tau_2 = 1,$$

$$\lambda_1 = -\tfrac{1}{2},\ \lambda_2 = 0,\ \theta_1 = -1,\ \theta_2 = 0,\ \zeta_1 = 0,\ \zeta_2 = 1,\ \kappa_1 = \kappa_2 = 0,\ \xi_1 = \xi_2 = 1$$

and using the identities [4, p.218]:

$$G_{2,4}^{1,2}\left(x \,\middle|\, \begin{matrix} \tfrac{1}{2}, 0 \\ b, -b, c, -c \end{matrix}\right) = \sqrt{\pi} J_{b+c}\left(\sqrt{x}\right) J_{b-c}\left(\sqrt{x}\right) \tag{5.6}$$

and the second formula of (5.1), we find that the formal solution of the dual integral equations

$$\begin{cases} \sqrt{\pi} \int_0^\infty J_{b+c}\left(\sqrt{xv}\right) J_{b-c}\left(\sqrt{xv}\right) f(v)dv = \phi(x), & (0 < x < 1) \\ -\sqrt{\pi} \int_0^\infty J_b\left(\sqrt{xv}\right) Y_b\left(\sqrt{xv}\right) f(v)dv = \psi(x), & (x > 1) \end{cases} \tag{5.7}$$

is given by

$$f(x) = \int_0^1 J_{2b}\left(2\sqrt{xv}\right) \sqrt{v} R_{0,v}^{1/2} v^{-c} R_{0,v}^{-1} v\phi(v)dv + \frac{1}{\sqrt{\pi}} \int_1^\infty J_{2b}\left(2\sqrt{xv}\right) \sqrt{v} W_{v,\infty}^{-1/2} \psi(v)dv. \tag{5.8}$$

(iv) Next if we set

$$m = 0,\ n = 1,\ k = 1,\ l = 1,\ b_1 = b,\ B_1 = 1,$$

$$\gamma_1 = -\tfrac{1}{2},\ \delta_1 = -1,\ \eta_1 = 0,\ \sigma_1 = a + 2,\ \tau_1 = 1,$$

$$\lambda_1 = -\tfrac{1}{2},\ \theta_1 = -1,\ \zeta_1 = 0,\ \kappa_1 = 0,\ \xi_1 = 1$$

and use the second identity of (5.1) and

$$G_{1,3}^{1,1}\left(x \,\middle|\, \begin{matrix} \tfrac{1}{2}, 0 \\ b, -b, a \end{matrix}\right) = x^b \frac{\Gamma(-2b)\Gamma(-a-b)}{\Gamma(\frac{1}{2} - b)} {}_1F_2\left(\begin{matrix} \tfrac{1}{2} - b; \\ -2b, -a-b; \end{matrix} \ -x\right), \tag{5.9}$$

we now find that the formal solution of

$$(5.10)\qquad \begin{cases} \dfrac{\Gamma(-2b)\Gamma(-a-b)}{\Gamma(\frac{1}{2}-b)}x^b\displaystyle\int_0^\infty v^b\,{}_1F_2\left(\begin{matrix}\frac{1}{2}-b; \\ -2b,-a-b;\end{matrix}\; -xv\right)f(v)dv = \phi(x), \\ \hfill (0<x<1) \\ -\sqrt{\pi}\displaystyle\int_0^\infty J_b\left(\sqrt{xv}\right)Y_b\left(\sqrt{xv}\right)f(v)dv = \psi(x), \qquad (x>1) \end{cases}$$

is given by

$$(5.11)\qquad \begin{aligned} f(x) = {} & \int_0^1 v^{-a-1}I_{0,v}^{-1/2,-1,0}v^a\phi(v)J_{2b}\left(2\sqrt{xv}\right)dv \\ & + \int_1^\infty \sqrt{v}J_{x,\infty}^{-1/2,-1,0}v^{-3/2}\psi(v)J_{2b}\left(2\sqrt{xv}\right)dv. \end{aligned}$$

(v) Finally if we set

$m=0,\ n=1,\ k=2,\ l=2,\ b_1=b,\ B_1=1,$

$\gamma_1=-\frac{1}{2}-c,\ \gamma_2=c,\ \delta_1=\frac{1}{2}+c,\ \delta_2=-c,\ \eta_1=0,\ \eta_2=1,\ \sigma_1=\frac{1}{2},\ \sigma_2=1,\ \tau_1=\tau_2=1,$

$\lambda_1=-\frac{1}{2}-c,\ \lambda_2=c,\ \theta_1=\frac{1}{2}+c,\ \theta_2=-c,\ \zeta_1=0,\ \zeta_2=1,\ \kappa_1=\kappa_2=0,\ \xi_1=\xi_2=1$

and use the identity (5.6), we find that the formal solution of

$$(5.12)\qquad \begin{cases} \sqrt{\pi}\displaystyle\int_0^\infty J_{b+c}\left(\sqrt{xv}\right)J_{b-c}\left(\sqrt{xv}\right)f(v)dv = \phi(x), \qquad (0<x<1) \\ \displaystyle\int_0^\infty G_{2,4}^{3,0}\left(xv\,\middle|\,\begin{matrix}2+2c,1-2c \\ b,\frac{3}{2}+c,1-c,-b\end{matrix}\right)f(v)dv = \psi(x), \quad (x>1) \end{cases}$$

is given by

$$(5.13)\qquad \begin{aligned} f(x) = {} & \int_0^1 \sqrt{v}I_{0,v}^{-1/2-c,1/2+c,0}v^{1+c}I_{0,v}^{c,-c,0}v^{-c}\phi(v)J_{2b}\left(2\sqrt{xv}\right)dv \\ & + \int_1^\infty v^{2+2c}J_{v,\infty}^{-1/2-c,1/2+c,0}v^{-3c-5/2}J_{v,\infty}^{c,-c,1}v^{c-1}\psi(v)J_{2b}\left(2\sqrt{xv}\right)dv. \end{aligned}$$

6. Concluding Remarks

It is interesting to observe that the method of Saigo operators of fractional integration described in this article can be applied fairly easily in deriving the solution of integral equations involving H-functions and their various generalizations. This will form the subject matter of a future communication.

Acknowledgement. The third author expresses his gratitude to the Council of Scientific and Industrial Research (India) for awarding a Senior Research Fellowship to enable him to do the present investigations.

References

[1] B.L.J. Braaksma, Asymptotic expansions and analytic continuation for a class of Barnes integrals, *Comp. Math.*, **15**(1973), 239-341.

[2] I.W. Busbridge, Dual integral equation, *Proc. London Math. Soc.*, **44**(1935), 115-129.

[3] A. Erdélyi, On some functional transforms, *Univ. e Politec. Torino Rend. Sem. Mat.*, **10**(1950-51), 217-234.

[4] A. Erdélyi et al., *Higher Transcendental Functions*, Vol.1, McGraw-Hill, New York, 1953.

[5] A. Erdélyi et al., *Tables of Integral Transforms*, Vols.1, 2, McGraw-Hill, New York, 1954.

[6] A. Erdélyi and I.N. Sneddon, Fractional integration and dual integral equations, *Canad. J. Math.*, **14**(1962), 685-693.

[7] C. Fox, The G and H-functions as symmetrical Fourier kernels, *Trans. Amer. Math. Soc.*, **98**(1961), 395-429.

[8] C. Fox, A formal solution of certain dual integral equations, *Trans. Amer. Math. Soc.*, **119**(1965), 389-398.

[9] H. Kober, On fractional integrals and derivatives, *Quart. J. Math. Oxford*, **11**(1940), 193-211.

[10] A.M. Mathai and R.K. Saxena, *Generalized Hypergeometric Functions with Applications in Statistics and Phisical Sciences*, Lecture Notes in Math., Vol. 348, Springer-Verlag, Berlin-Heidelberg-New York, 1973.

[11] A.M. Mathai and R.K. Saxena, *The H-functions with Applications in Statistics and Other Disciplines*, Halsted Press, (John Wiley & Sons, Wiley Eastern Ltd.), New York-London-Sydney-Toronto, 1978.

[12] B. Noble, The solution of Bessel function dual integral equations by a multiplying factor method, *Proc. Cambridge Philos. Soc.*, **59**(1963), 351-362.

[13] A.S. Peters, Certain dual integral equations and Sonine's integrals, *IMM-NYU* 285, Institute of Math. Sci., New York Univ., New York, 1961.

[14] M. Saigo, A remark on integral operators involving the Gauss hypergeometric functions, *Rep. College General Ed. Kyushu Univ.*, **11**(1978), 135-143.

[15] M. Saigo, A certain boundary value problem for the Euler-Darboux equation, *Math. Japon.*, **24**(1979), 377-385.

[16] M. Saigo, A certain boundary value problem for the Euler-Darboux equation, II, *Math. Japon.*, **25**(1980), 211-220.

[17] M. Saigo, A certain boundary value problem for the Euler-Darboux equation, III, *Math. Japon.*, **26**(1981), 103-119.

[18] M. Saigo, A generalization of fractional calculus, *Fractional Calculus*, Research Notes in Math., **138**, Pitman, 1984.

[19] M. Saigo and R.K. Raina, Fractional calculus operators associated with a general class of polynomials, *Fukuoka Univ. Sci. Rep.*, **18**(1988), 15-22.

[20] R.K. Saxena, A formal solution of certain dual integral equations involving H-functions, *Proc. Cambridge Philos. Soc.*, **63**(1967a), 171-178.

[21] R.K. Saxena, On the formal solution of certain dual integral equations, *Proc. Amer. Math. Soc.*, **18**(1967b), 1-8.

[22] R.K. Saxena and R.S. Kushwaha, Certain dual integral equations associated with a kernel of Fox, *Proc. Nat. Acad. Sci. India*, **42(B)**(1972), 39-54.

[23] H.M. Srivastava and M. Saigo, Multiplication of fractional calculus operators and boundary value problems involving the Euler-Darboux equation, *J. Math. Anal. Appl.*, **121**(1987), 325-369.

[24] N.A. Virchenko, *Dual / Triple/ Integral Equations* (in Russian), Vyshcha Shkola, Kiev, 1989.

[25] H. Weber, Über die Besselschen Funktionen und ihre Anwendung an die theorie der Elektrischen strome, *J. f. Math.*, **17**(1873), 75.