数理解析研究所講究録
第 821 巻 1993 年 144-152

DIFFERENCE SCHEME OF SOLITON EQUATIONS*

Ryogo Hirota,[1] Satoshi Tsujimoto,[2] and Tatsuya Imai[2]
(広田良吾) (辻本 諭) (今井達也)

[1]Department of Information and Computer Science,
[2]Department of Electronics and Communication Engineering,
School of Science and Engineering,
Waseda University, Tokyo 169

INTRODUCTION

The study of discrete-time integrable systems is currently the focus of an intense activity[1, 2, 3, 4].

It is interesting to note that many of the well-known features of continuous-time integrable system such as Lax-pair, Bäcklund transformation, Painlevé property, etc., carry over to the case of discrete-time integrable system. As is noted in ref 4, intelligent space-time discretization of integrable systems is a notoriously difficult problem and versions of integrability are different depending on the way of discretization.

In a series of papers[5, 6, 7] one of the authors has proposed a method of constructing nonlinear partial difference equations that exhibit solitons. The method uses the bilinear formalism and follows 3 steps. First, a given nonlinear partial differential equation is transformed into the bilinear form by the dependent variable transformation. Secondly the bilinear differential equation is discretized. Thirdly the bilinear difference equation is transformed back into the nonlinear difference equation by the associated dependent variable transformation.

In this paper we explain why it is relatively easy in the bilinear formalism to go from a continuous system to a discrete one without destroying integrability. Then we construct discrete-time Toda equations and a discretized 2-wave interaction.

N-SOLITON SOLUTION OF THE BILINEAR EQUATIONS

Recent development of the bilinear formalism reveals that the bilinear equations have an extremely simple structure (the Plücker relation or the Jacobi formula) if the soliton solutions are expressed by the determinants.

*Submitted to proceedings of '*Future Directions of Nonlinear Dynamics in Physics and Biological Systems*',(*Copenhagen, July 24 - Aug. 1, 1992*)

Take the Kadomtsev-Petviashvili(KP) equation for example

$$(4u_t - 6uu_x - u_{xxx})_x - 3u_{yy} = 0, \tag{1}$$

which is transformed into the bilinear form

$$(D_x^4 - 4D_xD_t + 3D_y^2)\tau\cdot\tau = 0 \tag{2}$$

through the dependent variable transformation

$$u = 2(\log\tau)_{xx} \tag{3}$$

where integration constants are chosen to be zero. The wronskian form of the N-soliton solution to the KP equation is expressed by

$$\tau_{KP} = \det\left|\frac{\partial^{j-1}\phi_i}{\partial x^{j-1}}\right|_{1\leqq i,j\leqq N} \tag{4}$$

where all ϕ_i satisfy the following linear differential equations for $n = 1, 2, \cdots, \infty$

$$\frac{\partial\phi_i}{\partial x_n} = \frac{\partial^n\phi_i}{\partial x^n}, \tag{5}$$

where $x_1 = x, x_2 = y, x_3 = t, \cdots$.

Here we note that the Plücker relation are algebraic identities of determinants among which the simplest one is the following

$$\begin{vmatrix} a_0 & a_1 \\ b_0 & b_1 \end{vmatrix}\cdot\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \begin{vmatrix} a_0 & a_2 \\ b_0 & b_2 \end{vmatrix}\cdot\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \begin{vmatrix} a_0 & a_3 \\ b_0 & b_3 \end{vmatrix}\cdot\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = 0. \tag{6}$$

Recently Nakamura[8] has shown that the bilinear KP equation reduces to the Jacobi formula of determinants provided that the solutions are expressed by the gramian type determinant

$$\tau_{KP} = \det|M_{ij}|_{1\leqq i,j\leqq N}, \tag{7}$$

where

$$M_{ij} = c_{ij} + \int^x f_i g_j \mathrm{d}x, \tag{8}$$

f_i and g_j satisfy the linear differential equations

$$\frac{\partial f_i}{\partial x_n} = \frac{\partial^n f_i}{\partial x^n}, \quad \frac{\partial g_j}{\partial x_n} = (-1)^{n-1}\frac{\partial^n g_j}{\partial x^n}. \tag{9}$$

In ref 7, a discrete bilinear equation was obtained:

$$[Z_1\exp(D_1) + Z_2\exp(D_2) + Z_3\exp(D_3)]f\cdot f = 0 \tag{10}$$

where Z_i and D_i for $i = 1, 2, 3$ are arbitrary parameters and linear combinations of the bilinear operators D_t, D_x, D_y and D_n respectively. The equation generates various types of soliton equations including the KdV equation, the KP equation, the modified KdV equation, the sine-Gordon equation, etc. A Bäcklund Transformation and a Lax-pair for eq.(10) are also obtained.

As a special case of eq.(10) Miwa proposed the following bilinear difference equation:

$$\begin{aligned} &a(b-c)\tau(l-1,m,n)\tau(l,m-1,n-1) \\ &+b(c-a)\tau(l,m-1,n)\tau(l-1,m,n-1) \\ &+c(a-b)\tau(l,m,n-1)\tau(l-1,m-1,n)=0 \end{aligned} \tag{11}$$

where a,b,c are constants related to the intervals of the discrete space-time,[9] which is called the discrete KP(d-KP) equation. The N-soliton solution of the d-KP equation τ_{disc} is expressed by a Casorati determinant (a discrete analogue of a wronskian):

$$\tau_{disc}=det|\varphi_i(l,m,n,s+j-1)|_{1\leq i,j\leq N} \tag{12}$$

where $\varphi_i(i=1,2,\cdots,N)$ satisfies the linear difference equation

$$\begin{aligned} \Delta_{-l}\varphi_i(l,m,n,s)&=\Delta_{-m}\varphi_i(l,m,n,s)=\Delta_{-n}\varphi_i(l,m,n,s) \\ &=\varphi_i(l,m,n,s+1) \end{aligned} \tag{13}$$

where $\Delta_{-l},\Delta_{-m},\Delta_{-n}$ are the backward difference operators defined by

$$\Delta_{-l}\varphi_i\equiv[\varphi_i(l)-\varphi_i(l-1)]/a, \tag{14}$$

$$\Delta_{-m}\varphi_i\equiv[\varphi_i(m)-\varphi_i(m-1)]/b, \tag{15}$$

$$\Delta_{-n}\varphi_i\equiv[\varphi_i(n)-\varphi_i(n-1)]/c. \tag{16}$$

On the other hand the N-soliton solution τ_{cont} of the continuous KP equation is expressed by

$$\tau_{cont}=det\left|\frac{\partial^{j-1}}{\partial x_1^{j-1}}\varphi_i\right|_{1\leq i,j\leq N}. \tag{17}$$

We show in the following that

$$\tau_{disc}=\tau_{cont} \tag{18}$$

up to a trivial factor. A general solution φ_{cont} to linear differential equation(5) is expressed by

$$\varphi_{cont}=\sum_p C_p\exp[\sum_{k=1}^{\infty}p^k x_k]\cdot p^s \tag{19}$$

while a general solution φ_{disc} to the linear difference equation(13) is expressed by

$$\varphi_{disc}=\sum_p C_p(1-ap)^{-l}(1-bp)^{-m}(1-cp)^{-n}\cdots p^s. \tag{20}$$

According to Miwa, we introduce an infinite number of coordinates $x_k(k=1,2,\cdots)$ by

$$x_k=l\frac{a^k}{k}+m\frac{b^k}{k}+n\frac{c^k}{k}+\cdots. \tag{21}$$

Then we have

$$\begin{aligned}
&(1-ap)^{-l}(1-bp)^{-m}(1-cp)^{-n}\cdots \\
&\qquad = \exp[-l\log(1-ap) - m\log(1-bp) - n\log(1-cp)\cdots] \\
&\qquad = \exp[\sum_{k=1}^{\infty} p^k x_k], \qquad (22)
\end{aligned}$$

which gives

$$\varphi_{disc} = \varphi_{cont}. \qquad (23)$$

Using a relation

$$\frac{\partial}{\partial x_1}\varphi_{cont}(s) = \varphi_{cont}(s+1) \qquad (24)$$

we find

$$\begin{aligned}
\tau_{disc} &= det|\varphi_{disc,i}(l,m,n,s+j-1)|_{1\leqq i,j\leqq N} \\
&= det\left|\frac{\partial^{j-1}}{\partial x_1^{j-1}}\varphi_{cont,i}(x_1,x_2,\cdots,s)\right|_{1\leqq i,j\leqq N} \\
&= \tau_{cont}. \qquad (25)
\end{aligned}$$

Furthermore it can be shown that the bilinear d-KP eq. is reduced to the Plücker relation if τ_{disc} is expressed by the Casorati determinant and to the Jacobi formula of determinant provided that τ_{disc} is expressed by the discrete analogue of the gramian type determinant. The invariance of N-soliton solution of the bilinear equation under the transformation of the continuous coordinates $x_1, x_2, \cdots$, into the discrete ones $l, m, n, \cdots$, is a guiding principle of constructing discrete integrable systems.

DISCRETIZATION OF 2-D TODA MOLECULE EQUATION

We have the Toda lattice with free ends:

$$\frac{\partial^2}{\partial x \partial y}Q_n = V_{n+1} - 2V_n + V_{n-1}, \qquad (26)$$

$$V_n = \exp(Q_n), \qquad (27)$$

which has another expression:

$$\frac{\partial}{\partial x}V_n = V_n(J_n - J_{n+1}), \qquad (28)$$

$$\frac{\partial}{\partial y}J_n = V_{n-1} - V_n, \qquad (29)$$

for $n = 1, 2, 3, \cdots, N-1$ with the boundary condition

$$V_0 = V_N = 0, \qquad (30)$$

which is called the two-dimensional (2-D) Toda molecule equation. Let

$$V_n = \frac{\partial^2}{\partial x \partial y} \log(\tau_n). \tag{31}$$

Then eq.(26) is integrated with respect to x and y to give the bilinear form

$$\tau_n \frac{\partial^2 \tau_n}{\partial x \partial y} - \frac{\partial \tau_n}{\partial x}\frac{\partial \tau_n}{\partial y} = \tau_{n+1}\tau_{n-1} \tag{32}$$

where the integration constants are chosen to be zero. The boundary conditions eq.(30) are satisfied with $\tau_0 = 1$ and $\tau_N = \Phi(x)\chi(y), \Phi(x)$ and $\chi(y)$ being arbitrary functions of x and y respectively.

Let a solution $\tau_n(x, y)$ to the bilinear eq.(32) be expressed by a two-directional wronskian of order n,

$$\tau_n(x,y) = det \left| \left(\frac{\partial}{\partial x}\right)^{i-1} \left(\frac{\partial}{\partial y}\right)^{j-1} \psi(x,y) \right|_{1 \leq i,j \leq n} \tag{33}$$

where $\psi(x, y)$ is an arbitrary function of x and y. Then eq.(32) becomes the Jacobi formula of determinants.

We conjecture that a difference analogue of the bilinear eq.(32) is the Jacobi formula which is expressed by a Casorati determinant

$$\hat{\tau}_n(l, m) = det|\psi(l + i - 1, m + j - 1)|_{1 \leq i,j \leq n} \tag{34}$$

as follows

$$\hat{\tau}_n(l+1, m+1)\hat{\tau}_n(l, m) - \hat{\tau}_n(l+1, m)\hat{\tau}_n(l, m+1) = \hat{\tau}_{n+1}(l, m)\hat{\tau}_{n-1}(l+1, m+1). \tag{35}$$

The boundary conditions eq.(30) are satisfied with the following $\psi(l, m)$:

$$\psi(l, m) = \sum_{j=1}^{n+1} \hat{u}_j(l)\hat{v}_j(m). \tag{36}$$

Let $x = l\delta, y = m\epsilon$ where δ and ϵ are parameters specifying the intervals and $\hat{\tau}_n(l, m) = (\delta\epsilon)^{n(n-1)/2}\tau_n(x, y)$. Then eq.(35) is transformed into

$$\tau_n(x+\delta, y+\epsilon)\tau_n(x, y) - \tau_n(x+\delta, y)\tau_n(x, y+\epsilon) = \delta\epsilon\tau_{n+1}(x, y)\tau_{n-1}(x+\delta, y+\epsilon), \tag{37}$$

which is transformed back into a discretized 2-D Toda molecule equation:

$$\left\{ \begin{array}{rcl} I_{n+1}(x,y)V_n(x,y) & = & I_n(x, y+\epsilon)V_n(x+\delta, y) \\ I_n(x,y) + \delta\epsilon V_n(x,y) & = & I_n(x, y+\epsilon) + \delta\epsilon V_{n-1}(x+\delta, y) \end{array} \right. \tag{38}$$

through the dependent variable transformation

$$I_n(x,y) = \frac{\tau_{n-1}(x,y)\tau_n(x+\delta,y)}{\tau_{n-1}(x+\delta,y)\tau_n(x,y)}, \tag{39}$$

$$V_n(x,y) = \frac{\tau_{n+1}(x,y)\tau_{n-1}(x,y+\epsilon)}{\tau_n(x,y)\tau_n(x,y+\epsilon)}. \tag{40}$$

Let $I_n(x,y) = 1 - \delta J_n(x,y)$. Then eq.(38) is transformed into

$$\left\{\begin{array}{rcl} \delta^{-1}[V_n(x+\delta,y) - V_n(x,y)] &=& V_n(x+\delta,y)J_n(x,y+\epsilon) - V_n(x,y)J_{n+1}(x,y), \\ \epsilon^{-1}[J_n(x,y+\epsilon) - J_n(x,y)] &=& V_{n-1}(x+\delta,y) - V_n(x,y), \end{array}\right. \tag{41}$$

which clearly shows that it becomes the 2-D Toda molecule equation in the small limit of δ and ϵ.

THE BÄCKLUND TRANSFORMATION

We have the discrete Toda molecule equation in the bilinear form

$$\tau_n(x+\delta,y+\epsilon)\tau_n(x,y) - \tau_n(x+\delta,y)\tau_n(x,y+\epsilon) = \delta\epsilon\tau_{n+1}(x,y)\tau_{n-1}(x+\delta,y+\epsilon). \tag{42}$$

Following the standard procedure we obtain a Bäcklund transformation for eq.(42):

$$\left\{\begin{array}{rcl} \tau_n(x,y+\epsilon)\tau_n'(x,y) &-& \tau_n(x,y)\tau_n'(x,y+\epsilon) \\ &=& \epsilon\{\lambda\tau_{n+1}(x,y)\tau_{n-1}'(x,y+\epsilon) - \nu\tau_n(x,y+\epsilon)\tau_n'(x,y)\}, \\ \tau_n(x+\delta,y)\tau_{n-1}'(x,y) &-& \tau_n(x,y)\tau_{n-1}'(x+\delta,y) \\ &=& \delta\{-\lambda^{-1}\tau_{n-1}(x+\delta,y)\tau_n'(x,y) + \mu\tau_n(x,y)\tau_{n-1}'(x+\delta,y)\}. \end{array}\right. \tag{43}$$

Let

$$\tau_n'(x,y) = \Psi_{n+1}(x,y)\tau_n(x,y). \tag{44}$$

Then eq.(43) is expressed, using eqs.(39),(40) and (44), by

$$\left\{\begin{array}{rcl} \psi_n(x,y) &=& \psi_n(x,y+\epsilon) + \epsilon V_{n-1}(x,y)\psi_{n-1}(x,y+\epsilon), \\ \psi_n(x+\delta,y) &=& I_n(x,y)\psi_n(x,y) + \delta\psi_{n+1}(x,y), \end{array}\right. \tag{45}$$

for $n = 0,1,2,\cdots,N-1$ where we have chosen $\lambda = 1, \nu = \mu = 0$ for simplicity.

Let $\tilde{\psi}(x,y)$ be a N-dimensional vector:

$$\tilde{\psi}(x,y) = [\psi_1(x,y), \psi_2(x,y), \cdots, \psi_N(x,y)]^T, \tag{46}$$

where T denotes the transposed matrix, and $L(x,y)$ and $R(x,y)$ be $N \times N$ matrices:

$$L(x,y) = \begin{pmatrix} 1 & & & & 0 \\ \delta V_1(x,y) & 1 & & & \\ & \delta V_2(x,y) & 1 & & \\ & & \ddots & \ddots & \\ 0 & & & \delta V_{N-1}(x,y) & 1 \end{pmatrix},$$

$$R(x,y) = \begin{pmatrix} I_1(x,y) & \epsilon & & & 0 \\ & I_2(x,y) & \epsilon & & \\ & & I_3(x,y) & \ddots & \\ & & & \ddots & \epsilon \\ 0 & & & & I_N(x,y) \end{pmatrix}.$$

Then eq.(45) is expressed with the matrices:

$$\begin{cases} \tilde{\psi}(x,y) &= L(x,y)\tilde{\psi}(x+\delta,y), \\ \tilde{\psi}(x,y+\epsilon) &= R(x,y)\tilde{\psi}(x,y), \end{cases} \tag{47}$$

and the compatibility condition of eq.(47) becomes a matrix equation:

$$R(x,y)L(x,y) = L(x+\delta,y)R(x,y+\epsilon), \tag{48}$$

which gives the discrete 2-D Toda molecule eq.

$$\begin{cases} I_n(x,y+\epsilon) - I_n(x,y) &= \delta\epsilon[V_n(x,y) - V_{n-1}(x+\delta,y)], \\ I_n(x,y+\epsilon)V_n(x+\delta,y) &= I_{n+1}(x,y)V_n(x,y). \end{cases} \tag{49}$$

LR FACTORIZATION METHOD

It is known[10] that a discrete 2-D Toda molecule equation is reduced to a discrete 1-D Toda molecule equation by introducing a symmetric variable t with respect to x and y: $t = x+y, \delta = \epsilon$ and assuming that all dependent variables are functions of t only, namely $V_n(x,y) = V_n(t), I_n(x,y) = I_n(t), L(x,y) = L(t), R(x,y) = R(t)$, etc.

Accordingly we obtain a discrete 1-D Toda molecule equation:

$$\begin{cases} I_n(t+\delta) - I_n(t) &= \delta^2[V_n(t) - V_{n-1}(t+\delta)], \\ I_n(t+\delta)V_n(t+\delta) &= I_{n+1}(t)V_n(t), \end{cases} \tag{50}$$

which is expressed by a matrix equation:

$$R(t)L(t) = L(t+\delta)R(t+\delta). \tag{51}$$

Let us introduce a matrix A(t) by

$$A(t) = L(t)R(t). \tag{52}$$

Then eq.(51) implies

$$A(t+\delta) = R(t)L(t). \tag{53}$$

Eqs.(52) and (53) for $\delta = 1$ are nothing but the *LR* factorization method of calculating eigenvalues of a matrix A. Hence we obtain eigenvalues of a matrix A by investigating the time-development of the Toda molecule equation[12]. Details of the method will be published elsewhere.

Eqs.(52) and (53) show that

$$A(t+\delta) = L^{-1}(t)A(t)L(t). \tag{54}$$

Hence we have

$$Tr[A(t+\delta)]^m = Tr[A(t)]^m, \tag{55}$$

which implies that the trace of the m-th power of $A(t)$ is the m-th conserved quantity of the discrete 1-D Toda molecule equation. We note that the discrete 1-D Toda lattice equation with the periodic boundary conditions, $V_1(t) = V_{N+1}(t), I_1(t) = I_{N+1}(t)$ is expressed by the same matrix equation as eq.(51) by introducing the periodicity of $V_n(t)$ and $I_n(t)$ into the matrices $L(t)$ and $R(t)$.

2-WAVE INTERACTION

We consider an discrete analogue of a system of coupled nonlinear partial differential equations describing interaction of two waves with constant velocities c_1 and $c_2(\neq c_1)$

$$\left\{\begin{array}{rcl}\left(\frac{\partial}{\partial\tau}+c_1\frac{\partial}{\partial\xi}\right)u &=& uv,\\ \left(\frac{\partial}{\partial\tau}+c_2\frac{\partial}{\partial\xi}\right)v &=& -uv,\end{array}\right. \tag{56}$$

which is rewritten in the following form by the coordinates transformation

$$\left\{\begin{array}{rcl}\frac{\partial u}{\partial x} &=& uv,\\ \frac{\partial v}{\partial y} &=& -uv.\end{array}\right. \tag{57}$$

One of the authors has shown that eq.(57) is obtained by the Bäcklund transformation of the Toda molecule equation [11].

We show that the Bäcklund transformation of the discrete Toda molecule equation gives a discrete form of the 2-wave interaction. Let us introduce a couple of dependent variables $u_n(x,y)$ and $v_n(x,y)$ by

$$u_n \equiv \frac{\epsilon^{-1}[\tau_n(x,y+\epsilon)\tau'_n(x,y)-\tau_n(x,y)\tau'_n(x,y+\epsilon)]}{\tau_n(x,y+\epsilon)\tau'_n(x,y)} \tag{58}$$

$$= \lambda\frac{\tau_{n+1}(x,y)\tau'_{n-1}(x,y+\epsilon)}{\tau_n(x,y+\epsilon)\tau'_n(x,y)}-\nu \tag{59}$$

and

$$v_n \equiv -\frac{\delta^{-1}[\tau_n(x+\delta,y)\tau'_{n-1}(x,y)-\tau_n(x,y)\tau'_{n-1}(x+\delta,y)]}{\tau_n(x,y)\tau'_{n-1}(x+\delta,y)} \tag{60}$$

$$= \lambda^{-1}\frac{\tau_{n-1}(x+\delta,y)\tau'_n(x,y)}{\tau_n(x,y)\tau'_{n-1}(x+\delta,y)}-\mu, \tag{61}$$

where we have used eq.(43) in rewriting $u_n(x,y)$ and $v_n(x,y)$. Then we find that $u_n(x,y)$ and $v_n(x,y)$ satisfy a coupled difference equations.

$$\left\{\begin{array}{rcl}u_n(x+\delta,y) &-& u_n(x,y)\\ &=& \delta\{[\nu+u_n(x+\delta,y)\}v_n(x,y+\epsilon)-[\nu+u_n(x,y)]v_{n+1}(x,y)\},\\ v_n(x,y+\epsilon) &-& v_n(x,y)\\ &=& \epsilon\{[\mu+v_n(x,y+\epsilon)]u_{n-1}(x+\delta,y)-[\mu+v_n(x,y)]u_n(x,y)\},\end{array}\right. \tag{62}$$

for $n=1,2,\cdots,N$ which is a discrete analogue of 2N-wave interaction. As a very special case, let us take $N=1,\mu=\nu=0,v_2=u_0=0$. Then eq.(62) is reduced to

$$\left\{\begin{array}{rcl}u_1(x+\delta,y)-u_1(x,y) &=& \delta u_1(x+\delta,y)v_1(x,y+\epsilon),\\ v_1(x,y+\epsilon)-v_1(x,y) &=& -\epsilon v_1(x,y)u_1(x,y),\end{array}\right. \tag{63}$$

which is a discrete form of 2-wave interaction.

It is also possible, by using the bilinear formalism, to construct the following nonlinear difference equation.

$$u_n(t+\delta) - u_n(t) = \delta[u_{n-1}(t+\delta)u_n(t) - u_n(t+\delta)u_{n+1}(t)], \tag{64}$$

which is reduced in the small limit of δ to a system of Lotka-Volterra prey-predator equations

$$\frac{d}{dt}u_n = u_n(u_{n-1} - u_{n+1}), \qquad \text{for n} = -\infty, \cdots, -1, 0, 1, \cdots, \infty. \tag{65}$$

Details of derivation of eq.(64) will be published elsewhere.

References

1. B.Grammaticos, A.Ramani and V.Papageorgiou, Phys.Rev.Letters **67**(1991)1825.
2. A.Ramani, B.Grammaticos and J.Hietarinta, Phys.Rev.Letters **67**(1991)1829.
3. J.Gibbons and B.Kupershmidt, Phys.Letters A **165**(1992)105.
4. R.S.Ward, Phys.Letters A **165**(1992)325.
5. R.Hirota,J.Phys.Soc.Jpn. **43**(1977)1424.
6. R.Hirota,J.Phys.Soc.Jpn. **45**(1978)321.
7. R.Hirota, J.Phys.Soc.Jpn. **50**(1981)3785.
8. A.Nakamura, J.Phys.Soc.Jpn. **58**(1989)412.
9. T.Miwa, Proc.Jpn.Acad. **58**(1982)9.
10. R.Hirota, M.Ito and F.Kako, Progr.Theor.Phys.Suppl. **94**(1988)42.
11. R.Hirota, J.Phys.Soc.Jpn. **57**(1988)436.
12. W.W.Symes, Physica **4D**(1982)275.