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<th>Extension problems for spinors on $S^4$ (State of art and perspectives of studies on nonlinear integrable systems)</th>
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<td>KORI, Tosiaki</td>
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1. The space of spinors on $S^3$

Here we shall explain the complex analytic point of view of Dirac operator on $S^4$ and discuss the eigenvalues of Hamiltonian acting on spinors on the equator $\simeq S^3$. These were obtained in [K].

a. Let us consider two copies of complex planes $C^2_z$ and $\hat{C}^2_w$ and a smooth bijection $\nu : C^2_z \setminus \{0\} \to \hat{C}^2_w \setminus \{0\}$ given by $w = \nu(z) = -\frac{\overline{z}}{|z|^2}$. We patch $C^2_z$ and $C^2_w$ by $\nu$ to obtain a differentiable manifold $M = \bigcup_v \hat{C}^2$, which is homeomorphic to $S^4$.

We endow $M$ with a riemannian metric defined by

$$g = \begin{cases} 
(1 + |z|^2)^{-2} \sum_{i=1}^{2} dz_i \otimes d\overline{z}_i & \text{on } C^2_z \\
(1 + |w|^2)^{-2} \sum_{i=1}^{2} dw_i \otimes d\overline{w}_i & \text{on } \hat{C}^2_w.
\end{cases}$$

The Levi-Civita connection on $M$ is given by gauge potentials

$$\Gamma(z) = \frac{|z|^2}{1 + |z|^2} \sigma(z)^{-1} \cdot (d\sigma)_{z} \quad \text{for } z \in C^2_z$$

$$\hat{\Gamma}(w) = \frac{|w|^2}{1 + |w|^2} \sigma(w)^{-1} \cdot (d\sigma)_{w} \quad \text{for } w \in \hat{C}^2_w.$$
where \( \sigma(z) = |z|^2 (v_*)_z \), \( v_* \) being the differential of \( v \), and \( \sigma(z)^{-1}(d\sigma)_z \) is a one-form valued in \( G = \{ X \in gl(4, \mathbb{C}) : {}^tXK + KX = 0 \} \simeq o(4, \mathbb{C}) \), \( K = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix} \).

On \( M \) there are a unique spin-structure \( Spin(M) \) and the associated spinor bundle \( S = Spin(M) \times_{Spin(4)} \Delta \). \( \Delta \) is a basic representation space of \( Spin(4) \) which is the direct sum of two irreducible representations of \( \Delta^+ \) and \( \Delta^- \) each of dimension 2. Let \( S^+ \) and \( S^- \) be the corresponding bundles whose cross sections are spinors of positive (respectively negative) chirality. We shall choose a frame of \( S^\pm \) and denote the spinors in matrix form

\[
\left( \begin{array}{c} \phi \\ \psi \end{array} \right) \in \Gamma(S), \quad \phi = \left( \begin{array}{c} \phi_1 \\ \phi_2 \end{array} \right) \in \Gamma(S^+), \quad \psi = \left( \begin{array}{c} \psi_1 \\ \psi_2 \end{array} \right) \in \Gamma(S^-),
\]

where \( \Gamma \) signifies the sections of a bundle. The inner product of two spinors \( \phi, \varphi \in \Gamma(S^\pm) \) is defined by \( <\phi(z), \varphi(z)> = \phi_1(z)\overline{\varphi}_1(z) + \phi_2(z)\overline{\varphi}_2(z) \).

b) The Dirac operator acting on the spinors is defined as the composition \( D = \mu \cdot \nabla \) where \( \nabla \) is the covariant derivative induced by the Levi-Civita connection and \( \mu \) is Clifford multiplication. The Dirac operator switches \( S^+ \) and \( S^- \) and is of the form \( D = \begin{pmatrix} 0 & D^\dagger \\ D & 0 \end{pmatrix} \) where \( D : \Gamma(S^+) \rightarrow \Gamma(S^-) \).

We gave in [K] the following matrix representation of the Dirac operator.

\[
D = \begin{pmatrix} (1 + |z|^2)\frac{\partial}{\partial z_1} - \frac{3}{2}z_1 & -(1 + |z|^2)\frac{\partial}{\partial z_2} + \frac{3}{2}z_2 \\ (1 + |z|^2)\frac{\partial}{\partial z_2} - \frac{3}{2}z_2 & (1 + |z|^2)\frac{\partial}{\partial z_1} - \frac{3}{2}z_1 \end{pmatrix}
\]

\[
D^\dagger = \begin{pmatrix} (1 + |z|^2)\frac{\partial}{\partial \overline{z}_1} - \frac{3}{2}\overline{z}_1 & (1 + |z|^2)\frac{\partial}{\partial \overline{z}_2} - \frac{3}{2}\overline{z}_2 \\ -(1 + |z|^2)\frac{\partial}{\partial \overline{z}_2} + \frac{3}{2}\overline{z}_2 & -(1 + |z|^2)\frac{\partial}{\partial \overline{z}_1} - \frac{3}{2}\overline{z}_1 \end{pmatrix}
\]

We have a decomposition of \( D \) and \( D^\dagger \) to their longitudinal parts and radial parts:

\[
D = \gamma_0 (n - \mathcal{P}) \quad D^\dagger = (n + \mathcal{P})\gamma_0.
\]

Here \( \gamma_0 \) signifies Clifford multiplication of the radial vector \( n \). We shall explain \( \mathcal{P} \) soon after. First we introduce an orthonormal frame on \( M \), but here we shall write down it only on the local coordinate \( C^2 \subset M \), the formulas
on $\mathbb{C}^2 \subset M$ are easily obtained by the transition relation. This frame is important not only as it gives a neat expression of Dirac operators on $M$ and on the equator $\simeq S^3$ but also as is associated to the Lie group structure of $S^3 \simeq SU(2)$ (see c). Let

$$
\nu = \frac{1 + |z|^2}{|z|} \left( z_1 \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2} \right) \quad \epsilon = \frac{1 + |z|^2}{|z|} \left( -\overline{z}_2 \frac{\partial}{\partial z_1} + \overline{z}_1 \frac{\partial}{\partial z_2} \right)
$$

The radial vector field is given by

$$
\mathbf{n} = \frac{1}{2} (\nu + \overline{\nu}).
$$

Put

$$
\theta_0 = \frac{1}{2\sqrt{-1}} (\nu - \overline{\nu}) \quad \theta_1 = \frac{1}{2} (\epsilon + \overline{\epsilon}) \quad \theta_2 = \frac{1}{2\sqrt{-1}} (\epsilon - \overline{\epsilon}).
$$

Then $\sqrt{2}\mathbf{n}$, $\sqrt{2}\theta_0$, $\sqrt{2}\theta_1$, $\sqrt{2}\theta_2$ form an orthonormal frame on $M$ and $\theta_0$, $\theta_1$, $\theta_2$ are tangent to the constant altitude $\{|z| = \text{const}\}$.

$\mathcal{P} : S^+ \rightarrow S^+$ is given by $\mathcal{P} = - (\gamma_0 S^-) \sum_{i=0}^{2} \theta_i \nabla_{\theta_i}$ with $\gamma_0$ coming from Clifford multiplication of $\mathbf{n}$.

The matrix representation of $\mathcal{P}$ is written as

$$
\mathcal{P} = \left( \begin{array}{cc} -\sqrt{-1}\theta_0 & \overline{\epsilon} \\ -\epsilon & \sqrt{-1}\theta_0 \end{array} \right) + \frac{3}{2} |z| \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right).
$$

Let $E = \{|z| = 1\}$ be the equator of $M$; $E \simeq S^3$. $E$ is endowed with the riemannian metric $g|E$. Since Spin(3) has the spinor representation on $\Delta^\pm$ the restrictions on $E$ of bundle $S^\pm$ is the spinor bundle corresponding to the spin structure $Spin(E)$. $\gamma_0$ gives the isomorphism between $S^\pm$. The Dirac operator on $E$ acting on spinors of positive chirality is given by $-\gamma_0 \mathcal{P}|E$.

The restriction $\mathcal{P}$ on $E$ is called Hamiltonian on $E$.

Here we shall discuss a little about infinitesimal representations of $SU(2)$ given by the vector fields $\sqrt{-1}\theta_i$, $i = 0, 1, 2$. First we note the commutation relations same as those of $sl(2)$;

$$
[\sqrt{-1}\theta_0, \epsilon] = -2\epsilon, \quad [\sqrt{-1}\theta_0, \overline{\epsilon}] = 2\overline{\epsilon}, \quad [\epsilon, \overline{\epsilon}] = 4\sqrt{-1}\theta_0.
$$

We now follow the isomorphism $B \simeq S^3 \simeq SU(2)$ and look the point $z \in B$ as $\ddot{z} = \left( \begin{array}{c} z_1 \\ z_2 \end{array} \right) \in SU(2)$. We shall then define the right action on $E$.
of \( g \in SU(2) \) by \( z \cdot g = \) the first column of \( \ddot{z} \cdot g \). Put \( R_g f(z) = f(z \cdot g) \) for a continuous function \( f \) on \( E \). Then the differentials become \( dR(e_k) = -\theta_k, \quad k = 0, 1, 2 \), where

\[
\begin{bmatrix}
\sqrt{-1} & 0 \\
0 & -\sqrt{-1}
\end{bmatrix}, \quad
\begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix}, \quad
\begin{bmatrix}
0 & \sqrt{-1} \\
\sqrt{-1} & 0
\end{bmatrix}
\]

are the basis of \( su(2) \).

A polynomial that satisfies

\[
P(az_1, bz_2, b\overline{z}_1, a\overline{z}_2) = a^k b^l P(z_1, z_2, \overline{z}_1, \overline{z}_2)
\]

is called of class \((k, l)\). The set of polynomials of class \((k, l)\) is denoted by \( S_{k,l} \). Let \( \mathcal{H} \) be the set of harmonic polynomials on \( C^2 \) and put \( \mathcal{H}_{k,l} = \mathcal{H} \cap S_{k,l} \). We have \( \mathcal{H}_{k,l} = \mathcal{H}_{k,l} \oplus |z|^2 S_{k-1,l-1} \), hence \( \dim \mathcal{H}_{k,l} = k + l + 1 \). It follows also that, on \( E \), every polynomial is a sum of harmonic polynomials in \( \mathcal{H}_{k,l} \)'s. This ensures the fact that our family of eigenspinors of Hamiltonian on \( E \) obtained later is a complete system.

Put, for \( r \geq 0 \) and \( 0 \leq k \leq r \),

\[
h_{k,r-k}^q(z) = e^q(z_1^k z_2^{r-k}).
\]

For each pair \( r \) and \( k \leq r \) the set \( \{ h_{k,r-k}^q; q = 0, \cdots, r \} \) forms a basis of \( \mathcal{H}_{k,r-k} \).

Proposition.

(1) \( \sqrt{-1} \theta_0 h_{k,r-k}^q = (r - 2q)h_{k,r-k}^q \)

(2) \( \epsilon h_{k,r-k}^q = h_{k,r-k}^{q+1} \)

(3) \( \overline{\epsilon} h_{k,r-k}^q = -4q(r - q + 1)h_{k,r-k}^{q-1} \)

Hence the space of harmonic polynomials \( \mathcal{H} \) (restricted on \( B \)) is decomposed by the right action \( R \) of \( SU(2) \) into \( \mathcal{H} = \sum r \sum_{k=0}^r \mathcal{H}_{k,r-k} \). Each induced representation \( R_{k,r-k} = (R, \mathcal{H}_{k,r-k}) \) is an irreducible representation with the highest weight \( \frac{r}{2} \).

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Put, for \( r \leq 0, 0 \leq k \leq r \), and \( 0 \leq q \leq r + 1 \),

\[
\phi_{k,r-k}^q = \begin{pmatrix} q2^{-q+1}h_{k,r-k}^{q-1} \\ -2^{-q}h_{k,r-k}^q \\ -2^{-q}h_{k,r-k}^q \end{pmatrix}.
\]
Then we have from the matrix representation of the Hamiltonian and the Proposition in \(c\);
\[
\mathcal{P}\phi_{k,r-k}^{q} = (r + \frac{3}{2})\phi_{k,r-k}^{q}.
\]
Thus the positive eigenvalues and eigenfunctions of \(\mathcal{P}\) are obtained. In particular the multiplicity of the eigenvalue \(r\) is \((r + 1)(r + 2)\).

The investigation of negative eigenspinors is related to the left action of \(SU(2)\) on the harmonic polynomials. The left action of a \(g \in SU(2)\) on \(E\) is defined by \(g \cdot z = \text{the first column of } g \cdot \overline{z}\). Let \(L_{g} f(z) = f(g^{-1} \cdot z)\) for a continuous function on \(E\).

We introduce the following vector fields on \(M = \{0, \hat{0}\}\), that have the following local expressions on \(C^{2} - \{0\}\):
\[
\mu = \frac{1 + |z|^{2}}{|z|}(z_{2} \frac{\partial}{\partial z_{2}} + \overline{z}_{1} \frac{\partial}{\partial \overline{z}_{1}}), \quad \delta = \frac{1 + |z|^{2}}{|z|}(\overline{z}_{2} \frac{\partial}{\partial \overline{z}_{1}} - z_{1} \frac{\partial}{\partial z_{2}}).
\]
\[
\tau_{0} = \frac{1}{2\sqrt{-1}}(\mu - \overline{\mu}), \quad \tau_{1} = \frac{1}{2}(\delta + \overline{\delta}), \quad \tau_{2} = \frac{1}{2\sqrt{-1}}(\delta - \overline{\delta}).
\]
We have \(dL(e_{i}) = -\tau_{i} |E|\); \(i = 0, 1, 2\).

Let
\[
\hat{h}_{q}^{r-k,k}(z) = \delta^{q}(\overline{z}_{1}^{k}z_{2}^{r-k}).
\]
\(\{\hat{h}_{q}^{l,k}; q = 0, \cdots, r\}\) give a basis of \(\hat{\mathcal{H}}^{l,k}\): the space of harmonic polynomials that satisfy the condition \(P(az_{1}, az_{2}, b\overline{z}_{1}, b\overline{z}_{2}) = a^{l}b^{k}P(z_{1}, z_{2}, \overline{z}_{1}, \overline{z}_{2})\). Put, for \(r \geq 0, 0 \leq k \leq r\), and \(0 \leq q \leq r + 1\),
\[
\pi_{q}^{r-k,k} = \begin{pmatrix}
2^{-q}\hat{h}_{q}^{r-k+1,k} \\
2^{-q}\hat{h}_{q}^{r-k,k+1}
\end{pmatrix}.
\]
By an easy calculus we have
\[
\mathcal{P}\pi_{q}^{r-k,k} = -(r + \frac{3}{2})\pi_{q}^{r-k,k}.
\]
Thus we have
Theorem 1 [K]. The eigenvalues of $\mathcal{P}$ are $\pm \left( \frac{3}{2} + r \right)$; $r = 0, 1, 2, \cdots$ with multiplicity $(r + 1)(r + 2)$, in particular, there is no zero mode spinor of $\mathcal{P}$ and the spectrum are symmetric relative to 0.

Here we note corresponding subjects on the other coordinate neighborhood $\hat{C}_w^2$. The transition function to describe the bundle $Spin(M)$ is

\[ t'(\gamma_0) = -\overline{\gamma}_0 \]

and a spinor on $M$ is a pair of $\varphi(z) \in \Gamma(C_z^2 \times \Delta)$ and $\hat{\varphi}(w) \in \Gamma(\hat{C}^2 \times \Delta)$ that are patched by $\hat{\varphi}(u(z)) = -(\gamma_0 \varphi)(z))$. The matrix representations of the Dirac operator on $\hat{C} \subset M$ has the same form as those in (1-5) but the first and the second are changed since a section on $\hat{C}$ of the bundle $S^+$ (resp. $S^-$) is valued in $\Delta^-$ (resp. $\Delta^+$). This is "CPT"-theorem. The counterpart of $\mathcal{P}$ is defined as $\mathcal{P} = (\gamma_0 | S^+ \underline{\sum} \theta_1 \nabla_{\theta\underline{\sum}}$ acting on $\hat{\varphi} \in \Gamma(\hat{C}_w^2 \times \Delta^-)$. For a $\varphi \in \Gamma(C_z^2 \times \Delta^+)$, we have $D\varphi \wedge = D\varphi$ and $\mathcal{P}\varphi = \mathcal{P}\varphi \wedge$.

2 Extension of spinors from the equator

Let $H$ be the space of square integrable spinors of positive chirality on $E$. Let $H_{\pm}$ be the closed subspace of $H$ spanned by the eigenvectors $\phi_{\lambda}$ corresponding to the positive (resp. negative) eigenvalues $\lambda$ of $\mathcal{P}$.

Put $c(r, q, k) = (\frac{q!k!(r-k)!}{(r+1-q)!})^{-\frac{1}{2}}$. Then a complete orthonormal system of eigenspinors of $\mathcal{P}$ is given by

\[ \{ c(r, q, k)\phi_{k,r-k}^{q}, c(r, q, k)\pi_{q}^{r-k,k}, r \geq 0, 0 \leq k \leq r, 0 \leq q \leq r+1 \} . \]

Take an eigenspinor $\varphi_{\lambda}$ and extend it by $\Phi_{\lambda}(z) = r_{\lambda}(|z|)\varphi_{\lambda}(\frac{z}{|z|})$ to $C^2$, where $r_{\lambda}(t) = t^{\lambda-\frac{3}{2}}(\frac{1+t^2}{2})^{\frac{3}{2}}$. Then $\Phi_{\lambda}(z)$ is a zero-mode spinor of $D$ on $C^2$. This is proved by the following calculus:

\[ D\Phi_{\lambda}(z) = \gamma_0 (n - \mathcal{P})(\Phi_{\lambda}(z)) = \gamma_0 \left( (1 + |z|^2)r'_{\lambda}(|z|) - (\lambda - \frac{3}{2})\frac{1 + |z|^2}{|z|}r_{\lambda}(|z|) - 3|z|r_{\lambda}(|z|) \right) \varphi_{\lambda}(\frac{z}{|z|}) . \]

But $r_{\lambda}(t)$ satisfies the equation

\[ (1 + t^2)r'_{\lambda}(t) - (\lambda - \frac{3}{2})\frac{1 + t^2}{t}r_{\lambda}(t) - 3tr_{\lambda}(t) = 0 . \]

Therefore $D\Phi_{\lambda} = 0$.

Let $\mathcal{N}(U)$ (resp. $\mathcal{N}^\dagger(U)$) be the space of zero-mode spinors of Dirac operator $D$ (resp. $D^\dagger$) on $U$ that have $L^2$-boundary values.
\textbf{Theorem 2 [K].} Let $R = \{z \in C^2; |z| < 1\}$ and $\hat{R} = \{w \in \hat{C}^2; |w| < 1\}$.

1. $H_+$ is isomorphic to $\mathcal{N}(R)$,
2. $H_-$ is isomorphic to $\mathcal{N}(\hat{R})$,
3. Every spinor in $H$ is equal to the difference of the restrictions of zero mode spinors on $R$ and on $\hat{R}$.

\textbf{Proof:} Let $\varphi \in H_+$ and expand it in $\varphi = \sum_{\lambda > 0} a_{\lambda} \phi_{\lambda}$. The spinor on $R$; $\Phi(z) = \sum_{\lambda > 0} a_{\lambda} \Phi_{\lambda}(z)$ is well defined. In fact, consider the finite sum; $\Phi_m^n = \sum_{\lambda = m + 3/2}^{n + 3/2} a_{\lambda} \Phi_{\lambda}$. Then $\Phi_m^n \Phi_m^n(z)$ is subharmonic on $R$ and is dominated by some constant multiple of its $L^2$-norm on $E$, hence converges there to 0 compact uniformly as $m, n$ tend to infinity. If we note the fact that each component of $\Phi$ is harmonic we see that it has $L^2$-boundary value. Conversely let $\Phi \in \mathcal{N}(R)$ and let $\varphi$ be its restriction to $E$. We can show that the eigenfunction expansion of $\varphi$ by $\{\phi_{\lambda}\}$ can not contain the term with $\lambda < 0$ and $\varphi \in H_+$. As for (2) consider the function $r_{-\mu}(t) = t^{\mu - 3/2}(1 + t^2)^{3/2}$, $t \geq 0$, where $-\mu = -r - 3/2$, $r = 0, 1, \cdots$ and do the same argument as in (1).

Relations in e transform the result to that on $\hat{R}$.

b Let $H^*$ be the space of square integrable spinors of negative chirality on $E$. $\gamma_0$ switches $H$ and $H^*$; $(\gamma_0|S^+)H = H^*$, $(\gamma_0|S^-)H^* = H$. We shall define $H^*_+ = (\gamma_0|S^+)H_+$ and $H^*_- = (\gamma_0|S^+)H_-.$

Let $\psi^* \in H^*_-$ and suppose that $\psi = (\gamma_0|S^-)\psi^*$ is an eigenspinor belonging to a negative eigenvalue $\lambda = -(r + 3/2)$. Let $\Psi(z) = s_\lambda(|z|)\psi(z/|z|)$, where $s_\lambda(t) = t^{-(\lambda - 3/2)}(1 + t^2)^{3/2}$. Then as before we can verify that $\Psi(z)$ extend $\psi$ to $C^2$, $\Psi(0) = 0$ and $D^\dagger \psi^* = (n + \mathcal{P})\gamma_0 \psi^* = (n + \mathcal{P})\psi = 0$.

Thus in the same manner as in Theorem 2 we have the following;

\textbf{Theorem 3.}

1. $H^*_+$ is isomorphic to $\{\phi \in \mathcal{N}^\dagger(R); \phi(0) = 0\}$,
2. $H^*_-$ is isomorphic to $\{\psi \in \mathcal{N}^\dagger(\hat{R}); \psi(\hat{0}) = 0\}$,
3. Every spinor in $H^*$ is equal to the difference of the restrictions of zero mode spinors on $R$ and on $\hat{R}$.

c From the definition $<\phi, \psi> = 0$ for all $\phi \in H$ and $\psi \in H^*$.

Let $\phi$ and $\psi$ be spinors on $R = \{|z| \leq 1\}$, Stokes' theorem stats;

$$\int_R \frac{1}{(1 + |z|^2)^4} (<D\phi, \psi> + <\phi, D^\dagger \psi>) dV = \frac{1}{8} \int_E <\phi, \gamma_0 \psi> d\sigma.$$
Theorems 2, 3 and Stokes' theorem yield immediately that
\[ \int_E \langle \phi, \gamma_0 \psi \rangle d\sigma = 0 \quad \text{for } \phi \in H_+, \psi \in H_-^*. \]

Similarly
\[ \int_E \langle \phi, \gamma_0 \psi \rangle d\sigma = 0 \quad \text{for } \phi \in H_-, \psi \in H_+^*. \]

The coupling between \( H_\pm^* \) and \( H_\pm \) does not vanish and is important to construct the geometric model of conformal field theory on \( S^4 \) which will be treated in the next paper.

Actually eigenspinors \( \phi_\lambda; \lambda > 0 \) are extended to \( \mathcal{N}(C^2) \) and those for \( \lambda < 0 \) are extended to \( \mathcal{N}(\hat{C}^2) \). We list here a table of expansion formula for \( \phi_\lambda, \phi_\lambda^* = \gamma_0 \phi_\lambda \) for \( \lambda > 0 \) and \( \pi_\lambda, \pi_\lambda^* = \gamma_0 \phi_\lambda \) for \( \lambda < 0 \).

(1) \( \Phi_\lambda(z) = |z|^{\lambda - \frac{3}{2}} \left( \frac{1+|z|^2}{2} \right)^{\frac{3}{2}} \phi_\lambda \left( \frac{z}{|z|} \right) \in \mathcal{N}(C^2), \lambda > 0 \) and \( \Phi_\lambda(z) = \phi_\lambda(z) \) for \( |z| = 1 \).

(2) \( \Phi_\lambda^*(w) = |w|^{\lambda + \frac{3}{2}} \left( \frac{1+|w|^2}{2} \right)^{\frac{3}{2}} \phi_\lambda \left( \frac{w}{|w|} \right) \in \mathcal{N}(\hat{C}^2)_0, \lambda > 0 \) and \( \Phi_\lambda^*(-\overline{z}) = -\gamma_0 \phi_\lambda^*(z) \) for \( |z| = 1 \).

(3) \( \hat{\Pi}_\lambda(w) = |w|^{-\lambda - \frac{3}{2}} \left( \frac{1+|w|^2}{2} \right)^{\frac{3}{2}} \pi_\lambda \left( \frac{w}{|w|} \right) \in \mathcal{N}(\hat{C}^2), \lambda < 0 \) and \( \hat{\Pi}_\lambda(-\overline{z}) = -\gamma_0 \pi_\lambda(z) \) for \( |z| = 1 \).

(4) \( \Pi_\lambda^*(z) = |z|^{-\lambda + \frac{3}{2}} \left( \frac{1+|z|^2}{2} \right)^{\frac{3}{2}} \pi_\lambda \left( \frac{z}{|z|} \right) \in \mathcal{N}(C^2)_0, \lambda < 0 \) and \( \Pi_\lambda^*(z) = \pi_\lambda^*(z) \) for \( |z| = 1 \).

References

[K] Kori, T., Dirac operators on \( S^4 \) and on \( S^3 \). Infinite dimensional Grassmanian on \( S^3 \).

University of Waseda, Shinjuku-ku, Tokyo