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Extension problems for spinors on $S^4$

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$S^4$ 上のスピノールに対する延長問題

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1. The space of spinors on $S^3$

Here we shall explain the complex analytic point of view of Dirac operator on $S^4$ and discuss the eigenvalues of Hamiltonian acting on spinors on the equator $\simeq S^3$. These were obtained in [K].

a. Let us consider two copies of complex planes $C^2_z$ and $\hat{C}^2_w$ and a smooth bijection $\nu : C^2_z \setminus \{0\} \to \hat{C}^2_w \setminus \{0\}$ given by $w = \nu(z) = -\frac{\overline{z}}{|z|^2}$. We patch $C^2_z$ and $C^2_w$ by $\nu$ to obtain a differentiable manifold $M = C^2 \sqcup \hat{C}^2$, which is homeomorphic to $S^4$.

We endow $M$ with a riemannian metric defined by

$$g = \begin{cases} 
(1 + |z|^2)^{-2} \sum_{i=1}^{2} dz_i \otimes d\overline{z}_i & \text{on } C^2_z \\
(1 + |w|^2)^{-2} \sum_{i=1}^{2} dw_i \otimes d\overline{w}_i & \text{on } \hat{C}^2_w 
\end{cases}$$

The Levi-Civita connection on $M$ is given by gauge potentials

$$\Gamma(z) = \frac{|z|^2}{1 + |z|^2} \sigma(z)^{-1} \cdot (d\sigma)_z \quad \text{for } z \in C^2_z$$

$$\hat{\Gamma}(w) = \frac{|w|^2}{1 + |w|^2} \sigma(w)^{-1} \cdot (d\sigma)_w \quad \text{for } w \in \hat{C}^2_w$$
where \( \sigma(z) = |z|^2 (v_z)_z \), \( v_z \) being the differential of \( v \), and \( \sigma(z)^{-1} (d\sigma)_z \) is a one-form valued in \( G = \{ X \in gl(4, \mathbb{C}) : ^t X K + K X = 0 \} \simeq o(4, \mathbb{C}), K = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix} \).

On \( M \) there are a unique spin-structure \( Spin(M) \) and the associated spinor bundle \( S = Spin(M) \times \) \( Spin(4) \) \( \Delta \). \( \Delta \) is a basic representation space of \( Spin(4) \) which is the direct sum of two irreducible representations of \( \Delta^+ \) and \( \Delta^- \) each of dimension 2. Let \( S^+ \) and \( S^- \) be the corresponding bundles whose cross sections are spinors of positive (respectively negative) chirality. We shall choose a frame of \( S^\pm \) and denote the spinors in matrix form

\[
\begin{pmatrix} \phi \\ \psi \end{pmatrix} \in \Gamma(S), \quad \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \in \Gamma(S^+), \quad \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \in \Gamma(S^-),
\]

where \( \Gamma \) signifies the sections of a bundle. The inner product of two spinors \( \phi, \varphi \in \Gamma(S^\pm) \) is defined by \( <\phi(z), \varphi(z)> = \phi_1(z) \overline{\varphi}_1(z) + \phi_2(z) \overline{\varphi}_2(z) \).

b The Dirac operator acting on the spinors is defined as the composition \( D = \mu \cdot \nabla \) where \( \nabla \) is the covariant derivative induced by the Levi-Civita connection and \( \mu \) is Clifford multiplication. The Dirac operator switches \( S^+ \) and \( S^- \) and is of the form \( D = \begin{pmatrix} 0 & D^\dagger \\ D & 0 \end{pmatrix} \) where \( D : \Gamma(S^+) \rightarrow \Gamma(S^-) \).

We gave in [K] the following matrix representation of the Dirac operator.

\[
D = \begin{pmatrix} (1 + |z|^2) \frac{\partial}{\partial z_1} - \frac{3}{2} \overline{z}_1 & -(1 + |z|^2) \frac{\partial}{\partial \overline{z}_2} + \frac{3}{2} z_2 \\ (1 + |z|^2) \frac{\partial}{\partial \overline{z}_1} - \frac{3}{2} z_1 & (1 + |z|^2) \frac{\partial}{\partial z_2} - \frac{3}{2} \overline{z}_1 \end{pmatrix}
\]

\[
D^\dagger = \begin{pmatrix} (1 + |z|^2) \frac{\partial}{\partial \overline{z}_1} - \frac{3}{2} z_1 & (1 + |z|^2) \frac{\partial}{\partial z_2} - \frac{3}{2} z_2 \\ -(1 + |z|^2) \frac{\partial}{\partial z_2} + \frac{3}{2} \overline{z}_2 & (1 + |z|^2) \frac{\partial}{\partial \overline{z}_1} - \frac{3}{2} \overline{z}_1 \end{pmatrix}
\]

We have a decomposition of \( D \) and \( D^\dagger \) to their longitudinal parts and radial parts;

\[
D = \gamma_0 (n - P), \quad D^\dagger = (n + P) \gamma_0.
\]

Here \( \gamma_0 \) signifies Clifford multiplication of the radial vector \( n \). We shall explain \( P \) soon after. First we introduce an orthonormal frame on \( M \), but here we shall write down it only on the local coordinate \( C^2 \subset M \), the formulas
on $\hat{C}^2 \subset M$ are easily obtained by the transition relation. This frame is important not only as it gives a neat expression of Dirac operators on $M$ and on the equator $\simeq S^3$ but also as it is associated to the Lie group structure of $S^3 \simeq SU(2)$ (see c). Let
\[
\nu = \frac{1 + |z|^2}{|z|} \left( z_1 \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2} \right) \quad \epsilon = \frac{1 + |z|^2}{|z|} \left( -\overline{z}_2 \frac{\partial}{\partial z_1} + \overline{z}_1 \frac{\partial}{\partial z_2} \right)
\]
The radial vector field is given by
\[
\mathbf{n} = \frac{1}{2} (\nu + \overline{\nu}).
\]
Put
\[
\theta_0 = \frac{1}{2\sqrt{-1}} (\nu - \overline{\nu}) \quad \theta_1 = \frac{1}{2} (\epsilon + \overline{\epsilon}) \quad \theta_2 = \frac{1}{2\sqrt{-1}} (\epsilon - \overline{\epsilon}).
\]
Then $\sqrt{2}\mathbf{n}$, $\sqrt{2}\theta_0$, $\sqrt{2}\theta_1$, $\sqrt{2}\theta_2$ form an orthonormal frame on $M$ and $\theta_0$, $\theta_1$, $\theta_2$ are tangent to the constant altitude $\{|z| = \text{const}\}$.

$\mathcal{P} : S^+ \to S^+$ is given by $\mathcal{P} = - (\gamma_0 |S^-) \sum_{i=0}^{2} \theta_i \nabla_{\theta_i}$ with $\gamma_0$ coming from Clifford multiplication of $\mathbf{n}$.

The matrix representation of $\mathcal{P}$ is written as
\[
\mathcal{P} = \left( \begin{array}{cc}
-\sqrt{-1}\theta_0 & \overline{\epsilon} \\
-\epsilon & \sqrt{-1}\theta_0
\end{array} \right) + \frac{3}{2} |z| \left( \begin{array}{cc}
1 & 0 \\
0 & 1
\end{array} \right),
\]

Let $E = \{|z| = 1\}$ be the equator of $M$; $E \simeq S^3$. $E$ is endowed with the riemannian metric $g|E$. Since $\text{Spin}(3)$ has the spinor representation on $\Delta^\pm$ the restrictions on $E$ of bundle $S^\pm$ is the spinor bundle corresponding to the spin structure $\text{Spin}(E)$. $\gamma_0$ gives the isomorphism between $S^\pm$. The Dirac operator on $E$ acting on spinors of positive chirality is given by $-\gamma_0 \mathcal{P} |E$. The restriction $\mathcal{P}$ on $E$ is called Hamiltonian on $E$.

c Here we shall discuss a little about infinitesimal representations of $SU(2)$ given by the vector fields $\sqrt{-1}\theta_i$, $i = 0,1,2$. First we note the commutation relations same as those of $sl(2)$;
\[
[\sqrt{-1}\theta_0, \epsilon] = -2\epsilon, \quad [\sqrt{-1}\theta_0, \overline{\epsilon}] = 2\overline{\epsilon}, \quad [\epsilon, \overline{\epsilon}] = 4\sqrt{-1}\theta_0.
\]
We now follow the isomorphism $B \simeq S^3 \simeq SU(2)$ and look the point $z \in B$ as $\mathbf{z} = \left( \begin{array}{c}
z_1 \\
z_2 \\
\overline{z}_1
\end{array} \right) \in SU(2)$. We shall then define the right action on $E$
of $g \in SU(2)$ by $z \cdot g = \text{the first column of } \tilde{z} \cdot g$. Put $R_g f(z) = f(z \cdot g)$ for a continuous function $f$ on $E$. Then the differentials become $dR(e_k) = -\theta_k$, $k = 0, 1, 2$, where

$$e_0 = \begin{pmatrix} \sqrt{-1} & 0 \\
0 & -\sqrt{-1} \end{pmatrix}, \quad e_1 = \begin{pmatrix} 0 & 1 \\
-1 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & \sqrt{-1} \\
\sqrt{-1} & 0 \end{pmatrix}$$

are the basis of $su(2)$.

A polynomial $P(az_1, bz_2, b\overline{z}_1, a\overline{z}_2) = a^k b^l P(z_1, z_2, \overline{z}_1, \overline{z}_2)$ is called of class $(k, l)$. The set of polynomials of class $(k, l)$ is denoted by $S_{k,l}$. Let $\mathcal{H}$ be the set of harmonic polynomials on $C^2$ and put $\mathcal{H}_{k,l} = \mathcal{H} \cap S_{k,l}$. We have $S_{k,l} = \mathcal{H}_{k,l} \oplus |z|^2 S_{k-1,l-1}$, hence $\dim \mathcal{H}_{k,l} = k + l + 1$. It follows also that, on $E$, every polynomial is a sum of harmonic polynomials in $\mathcal{H}_{k,l}$'s. This ensures the fact that our family of eigenspinors of Hamiltonian on $E$ obtained later is a complete system.

Put, for $r \geq 0$ and $0 \leq k, q \leq r$,

$$h_{k,r-k}^q(z) = \epsilon^q(z_1^k z_2^{r-k}).$$

For each pair $r$ and $k \leq r$ the set $\{h_{k,r-k}^q; q = 0, \cdots, r\}$ forms a basis of $\mathcal{H}_{k,r-k}$.

**Proposition.**

1. $\sqrt{-1} \theta_0 h_{k,r-k}^q = (r - 2q)h_{k,r-k}^q$
2. $\epsilon h_{k,r-k}^q = h_{k,r-k}^{q+1}$
3. $\overline{\epsilon} h_{k,r-k}^q = -4q(r - q + 1)h_{k,r-k}^{q-1}$

Hence the space of harmonic polynomials $\mathcal{H}$ (restricted on $B$) is decomposed by the right action $R$ of $SU(2)$ into $\mathcal{H} = \sum_r \sum_{k=0}^r \mathcal{H}_{k,r-k}$. Each induced representation $R_{k,r-k} = (R, \mathcal{H}_{k,r-k})$ is an irreducible representation with the highest weight $k$. 

**d** Put, for $r \leq 0$, $0 \leq k \leq r$, and $0 \leq q \leq r + 1$,

$$\phi_{k,r-k}^q = \begin{pmatrix} q^{2q+1} h_{k,r-k}^{q-1} \\
-2^{q} h_{k,r-k}^q \end{pmatrix}.$$
Then we have from the matrix representation of the Hamiltonian and the Proposition in $c$;

$$\mathcal{P}\phi_{k,r-k}^{q} = (r + \frac{3}{2})\phi_{k,r-k}^{q}.$$ 

Thus the positive eigenvalues and eigenfunctions of $\mathcal{P}$ are obtained. In particular the multiplicity of the eigenvalue $r$ is $(r + 1)(r + 2)$.

The investigation of negative eigenspinors is related to the left action of $SU(2)$ on the harmonic polynomials. The left action of a $g \in SU(2)$ on $E$ is defined by $g \cdot z = \text{the first column of } g \cdot \bar{z}$. Let $L_g f(z) = f(g^{-1} \cdot z)$ for a continuous function on $E$.

We introduce the following vector fields on $M - \{0, \hat{0}\}$, that have the following local expressions on $C^2 - \{0\}$:

$$\mu = \frac{1 + |z|^2}{|z|} (z_2 \frac{\partial}{\partial z_2} + \overline{z}_1 \frac{\partial}{\partial \overline{z}_1}), \quad \delta = \frac{1 + |z|^2}{|z|} (\overline{z}_2 \frac{\partial}{\partial \overline{z}_2} - z_1 \frac{\partial}{\partial z_1}).$$

$$\tau_0 = \frac{1}{2\sqrt{-1}} (\mu - \overline{\mu}), \quad \tau_1 = \frac{1}{2} (\delta + \overline{\delta}), \quad \tau_2 = \frac{1}{2\sqrt{-1}} (\delta - \overline{\delta}).$$

We have $dL(e_i) = -\tau_i |E|; \quad i = 0, 1, 2$.

Let 

$$\hat{h}_{q}^{r-k,k}(z) = \delta^q (\overline{z}_1^k z_2^{r-k}).$$

$\{\hat{h}_{q}^{l,k}; q = 0, \cdots, r\}$ give a basis of $\hat{\mathcal{H}}^{l,k}$: the space of harmonic polynomials that satisfy the condition $P(az_1, az_2, b\overline{z}_1, b\overline{z}_2) = a^l b^k P(z_1, z_2, \overline{z}_1, \overline{z}_2)$. Put, for $r \geq 0, 0 \leq k \leq r$, and $0 \leq q \leq r+1$,

$$\pi_{q}^{r-k,k}(z; k) = \begin{pmatrix} 2^{-q} \hat{h}_{q}^{r-k+1,k} \\ 2^{-q} \hat{h}_{q}^{r-k,k+1} \end{pmatrix}.$$  

By an easy calculus we have

$$\mathcal{P}\pi_{q}^{r-k,k} = -(r + \frac{3}{2})\pi_{q}^{r-k,k}.$$ 

Thus we have
The eigenvalues of $\mathcal{P}$ are $\pm \left(\frac{3}{2} + r\right)$; $r = 0, 1, 2, \ldots$ with multiplicity $(r + 1)(r + 2)$, in particular, there is no zero mode spinor of $\mathcal{P}$ and the spectrum are symmetric relative to 0.

Here we note corresponding subjects on the other coordinate neighborhood $\hat{C}_w^2$. The transition function to describe the bundle $Spin(M)$ is $-t(\gamma_0) = -\overline{\gamma_0}$ and a spinor on $M$ is a pair of $\varphi(z) \in \Gamma(C^2_z \times \Delta)$ and $\hat{\varphi}(w) \in \Gamma(\hat{C}^2 \times \Delta)$ that are patched by $\hat{\varphi}(v(z)) = -(\gamma_0 \varphi)(z))$. The matrix representations of the Dirac operator on $\hat{C}^2 \subset M$ has the same form as those in (1-5) but the first and the second are changed since a section on $\hat{C}^2$ of the bundle $S^+$ (resp. $S^-$) is valued in $\triangle^-$ (resp. $\triangle^+$). This is "CPT"-theorem. The counterpart of $\mathcal{P}$ is defined as $\mathcal{P} = (\gamma_0|S^+\underline{)}\sum \theta_1 \nabla_{\theta\underline{:}}$ acting on $\hat{\varphi} \in \Gamma(\hat{C}^2_w \times \Delta^-)$. For $a \varphi \in \Gamma(C^2_z \times \Delta^+)$, we have $D\varphi \wedge = D\varphi$ and $\mathcal{P}\varphi = \mathcal{P}\varphi \wedge$.

2 Extension of spinors from the equator

Let $H$ be the space of square integrable spinors of positive chirality on $E$. Let $H_{\pm}$ be the closed subspace of $H$ spanned by the eigenvectors $\phi_\lambda$ corresponding to the positive (resp. negative) eigenvalues $\lambda$ of $\mathcal{P}$.

Put $c(r, q, k) = \left(\frac{q!k!(r-k)!}{(r+1-q)!}\right)^{-\frac{1}{2}}$. Then a complete orthonormal system of eigenspinors of $\mathcal{P}$ is given by

$$\{c(r, q, k)\phi_{k,r-k}^q, c(r, q, k)\pi_{r-k,k}^q; r \geq 0, 0 \leq k \leq r, 0 \leq q \leq r+1\}.$$

Take an eigenspinor $\varphi_\lambda$ and extend it by $\Phi_\lambda(z) = r_\lambda(|z|)\varphi_\lambda(\frac{z}{|z|})$ to $C^2$, where $r_\lambda(t) = t^{\lambda - \frac{3}{2}}(1 + t^2)^{\frac{3}{2}}$. Then $\Phi_\lambda(z)$ is a zero-mode spinor of $D$ on $C^2$. This is proved by the following calculus:

$$D\Phi_\lambda(z) = \gamma_0(n - \mathcal{P})(\Phi_\lambda(z))$$

$$= \gamma_0 \left( (1 + |z|^2)r_\lambda'(|z|) - (\lambda - \frac{3}{2}) \frac{1 + |z|^2}{|z|}r_\lambda(|z|) - 3|z|r_\lambda(|z|) \right) \varphi_\lambda(z)$$

But $r_\lambda(t)$ satisfies the equation

$$(1 + t^2)r_\lambda'(t) - (\lambda - \frac{3}{2}) \frac{1 + t^2}{t}r_\lambda(t) - 3tr_\lambda(t) = 0.$$

Therefore $D\Phi_\lambda = 0$.

Let $\mathcal{N}(U)$ (resp. $\mathcal{N}^\dagger(U)$) be the space of zero-mode spinors of Dirac operator $D$ (resp. $D^\dagger$) on $U$ that have $L^2$-boundary values.
Theorem 2 [K]. Let $R = \{ z \in C^2; \, |z| < 1 \}$ and $\hat{R} = \{ w \in \hat{C}^2; \, |w| < 1 \}$.

1. $H_+$ is isomorphic to $\mathcal{N}(R)$,
2. $H_-$ is isomorphic to $\mathcal{N}(\hat{R})$,
3. Every spinor in $H$ is equal to the difference of the restrictions of zero mode spinors on $R$ and on $\hat{R}$.

Proof: Let $\varphi \in H_+$ and expand it in $\varphi = \sum_{\lambda>0} a_\lambda \phi_\lambda$. The spinor on $R$; $\Phi(z) = \sum_{\lambda>0} a_\lambda \Phi_\lambda(z)$ is well defined. In fact, consider the finite sum; $\Phi_m = \sum_{\lambda=m+\frac{3}{2}}^{n} a_\lambda \Phi_\lambda$. Then $\langle \Phi_m^n, \Phi_m \rangle (z)$ is subharmonic on $R$ and is dominated by some constant multiple of its $L^2-$norm on $E$, hence converges there to 0 compact uniformly as $m, n$ tend to infinity. If we note the fact that each component of $\Phi$ is harmonic we see that it has $L^2-$boundary value. Conversely let $\Phi \in \mathcal{N}(R)$ and let $\varphi$ be its restriction to $E$. We can show that the eigenfunction expansion of $\varphi$ by $\{\phi_\lambda\}$ can not contain the term with $\lambda < 0$ and $\varphi \in H_+$. As for (2) consider the function $r_{-\mu}(t) = t^{\mu - \frac{3}{2}}(1 + t^2)^{\frac{3}{2}}$, $t \geq 0$, where $-\mu = -r - \frac{3}{2}$, $r = 0, 1, \cdots$ and do the same argument as in (1).

Relations in $e$ transform the result to that on $\hat{R}$.

b Let $H^*$ be the space of square integrable spinors of negative chirality on $E$. $\gamma_0$ switches $H$ and $H^*$: $(\gamma_0|S^+)H = H^*$, $(\gamma_0|S^-)H^* = H$. We shall define $H^*_+ = (\gamma_0|S^+)H_+$ and $H^*_- = (\gamma_0|S^+)H_-^*$.

Let $\psi^* \in H^*_-$ and suppose that $\psi = (\gamma_0|S^-)\psi^*$ is an eigenspinor belonging to a negative eigenvalue $\lambda = -(r + \frac{3}{2})$. Let $\Psi(z) = s_\lambda(|z|)\psi(\frac{z}{|z|})$, where $s_\lambda(t) = (\lambda - \frac{3}{2})(\frac{1 + t^2}{1 + t^2})^\frac{3}{2}$. Then as before we can verify that $\Psi(z)$ extend $\psi$ to $C^2$, $\Psi(0) = 0$ and $D^\uparrow \psi^* = (\mathbf{n} + \mathcal{P})\gamma_0 \psi^* = (\mathbf{n} + \mathcal{P})\psi = 0$.

Thus in the same manner as in Theorem 2 we have the following;

Theorem 3.

1. $H^*_-$ is isomorphic to $\{ \phi \in \mathcal{N}^\uparrow(R); \phi(0) = 0 \}$,
2. $H^*_+$ is isomorphic to $\{ \psi \in \mathcal{N}^\uparrow(\hat{R}); \psi(\hat{0}) = 0 \}$,
3. Every spinor in $H^*$ is equal to the difference of the restrictions of zero mode spinors on $R$ and on $\hat{R}$.

c From the definition $\langle \phi, \psi \rangle = 0$ for all $\phi \in H$ and $\psi \in H^*$.

Let $\phi$ and $\psi$ be spinors on $R = \{|z| \leq 1 \}$, Stokes' theorem stats;

$$\int_R \frac{1}{(1 + |z|^2)^4} (\langle D\phi, \psi \rangle + \langle \phi, D^\uparrow \psi \rangle) dV = \frac{1}{8} \int_E \langle \phi, \gamma_0 \psi \rangle d\sigma.$$
Theorems 2, 3 and Stokes’ theorem yield immediately that
\[ \int_E \langle \phi, \gamma_0 \psi \rangle d\sigma = 0 \quad \text{for} \ \phi \in H_+, \ \psi \in H_-^*. \]

Similarly
\[ \int_E \langle \phi, \gamma_0 \psi \rangle d\sigma = 0 \quad \text{for} \ \phi \in H_-, \ \psi \in H_+^*. \]

The coupling between \( H_{\pm}^* \) and \( H_{\mp} \) does not vanish and is important to construct the geometric model of conformal field theory on \( S^4 \) which will be treated in the next paper.

Actually eigenspinors \( \phi_{\lambda}; \lambda > 0 \) are extended to \( \mathcal{N}(C^2) \) and those for \( \lambda < 0 \) are extended to \( \mathcal{N}(\hat{C}^2) \). We list here a table of expansion formula for \( \phi_{\lambda}, \phi_{\lambda}^* = \gamma_0 \phi_{\lambda} \) for \( \lambda > 0 \) and \( \pi_{\lambda}, \pi_{\lambda}^* = \gamma_0 \pi_{\lambda} \) for \( \lambda < 0 \).

1. \( \Phi_{\lambda}(z) = |z|^{-\frac{3}{2}} \left( \frac{1+|z|^2}{2} \right)^{\frac{3}{2}} \phi_{\lambda}(\frac{z}{|z|}) \in \mathcal{N}(C^2), \ \lambda > 0 \) and \( \Phi_{\lambda}(z) = \phi_{\lambda}(z) \) for \( |z| = 1 \).

2. \( \overline{\Phi_{\lambda}^*}(w) = |w|^{\lambda+\frac{3}{2}} \left( \frac{2}{1+|w|^2} \right)^{\frac{3}{2}} \overline{\phi_{\lambda}^*}(\frac{w}{|w|}) \in \mathcal{N}_0^\dagger(\hat{C}^2), \ \lambda > 0 \) and \( \overline{\Phi_{\lambda}^*}(-\overline{z}) = -\gamma_0 \phi_{\lambda}^*(z) \) for \( |z| = 1 \).

3. \( \hat{\Pi}_{\lambda}(w) = |w|^{-\lambda-\frac{3}{2}} \left( \frac{1+|w|^2}{2} \right)^{\frac{3}{2}} \pi_{\lambda}(\frac{w}{|w|}) \in \mathcal{N}(\hat{C}^2), \ \lambda < 0 \) and \( \hat{\Pi}_{\lambda}(-\overline{z}) = -\gamma_0 \pi_{\lambda}(z) \) for \( |z| = 1 \).

4. \( \Pi_{\lambda}^*(z) = |z|^{-\lambda+\frac{3}{2}} \left( \frac{2}{1+|z|^2} \right)^{\frac{3}{2}} \pi_{\lambda}^*(\frac{z}{|z|}) \in \mathcal{N}_0^\dagger(C^2), \ \lambda < 0 \) and \( \Pi_{\lambda}^*(z) = \pi_{\lambda}^*(z) \) for \( |z| = 1 \).

References

[K] Kori, T., Dirac operators on \( S^4 \) and on \( S^3 \). Infinite dimensional Grassmanian on \( S^3 \).

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