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1. Preliminaries

Here we give a brief résumé of [K1] to fix the notations. Let $M = \mathbb{C}^2 \cup v \mathbb{C}^2 \simeq S^4$; $w = v(z) = -\overline{z}/|z|^2$, and $E \simeq S^3$ be the equator. Let $S$ (resp. $S^+$ and $S^-$) be the spinor bundle (resp. of positive chirality and of negative chirality) on $M$. The inner product of two spinors $\phi, \varphi \in \Gamma(S^\pm)$ is defined by $<\phi(z), \varphi(z)> = \phi_1(z)\overline{\varphi}_1(z) + \phi_2(z)\overline{\varphi}_2(z)$. We denote by $\gamma_0$ Clifford multiplication of the radial vector field $n$ on $M$. $\gamma_0$ switches $S^+$ and $S^-$. Transition for spinors is given by $\hat{\varphi}(v(z)) = -(\gamma_0 \varphi)(z)$. Let $H$ (resp. $H^*$) be the space of square integrable spinors on $E$ of positive (resp. negative) chirality. From the definition $<\phi, \psi> = 0$ for all $\phi \in H$ and $\psi \in H^*$.

The Dirac operator is of the form $D = \begin{pmatrix} 0 & D^\dagger \\ D & 0 \end{pmatrix}$; $D : \Gamma(S^+) \rightarrow \Gamma(S^-)$. Let $\mathcal{P}$ be Hamiltonian on $E$. We have the radial decomposition of Dirac operator:

$$D = \gamma_0(n - \mathcal{P}), \quad D^\dagger = (n + \mathcal{P})\gamma_0.$$ 

The eigenvalues of $\mathcal{P}$ are $\pm(r + \frac{3}{2})$, $r = 0, 1, 2, \cdots$ with multiplicity $(r+1)(r+2)$. A complete orthonormal system of eigenspinors in $H$; \{ $\phi_{k,r-k}^q, \pi_{q}^{r-k,k}$ $\}_{r,q,k}$ was given explicit forms in [K1];

$$\mathcal{P}\phi_{k,r-k}^q = (r + \frac{3}{2})\phi_{k,r-k}^q, \quad \mathcal{P}\pi_{q}^{r-k,k} = -(r + \frac{3}{2})\pi_{q}^{r-k,k}.$$
\[ \phi_{k,r-k}^{q} = \left( \frac{q!k!(r-k)!}{(r+1-q)!} \right)^{-\frac{1}{2}} \left( q^{2-q+1}h_{k,r-k}^{q-1} \right) \]
\[ \pi_{q}^{r-k,k} = \left( \frac{q!k!(r-k)!}{(r+1-q)!} \right)^{-\frac{1}{2}} \left( 2^{-q} \hat{h}_{q}^{r-k+1,k} \right) \]

where
\[ 2^{-q}h_{k,r}^{q}(z_{1}, z_{2}) = \left( -\bar{z}_{2}\frac{\partial}{\partial z_{1}} + \bar{z}_{1}\frac{\partial}{\partial z_{2}} \right)^{q}(z_{1}^{k}z_{2}^{r-k}) \]
\[ 2^{-q}\hat{h}_{k}^{r-k,k}(z_{1}, z_{2}) = \left( \bar{z}_{2}\frac{\partial}{\partial z_{1}} - z_{1}\frac{\partial}{\partial z_{2}} \right)^{q}(\bar{z}_{1}^{k}z_{2}^{r-k}) \]

Let \( H_{+} \) (resp. \( H_{-} \)) be the subspace of \( H \) spanned by \( \phi_{k,r-k}^{q} \)'s (resp. \( \pi_{q}^{r-k,k} \)). We put \( H_{\pm} = \gamma_{0}H_{\pm} \).

b For a triplet \( \lambda = \{ \pm r; k, p \} \), \( 0 \leq r, 0 \leq k \leq r, 0 \leq p \leq r+1 \), we put \( -\lambda = \{ \mp r, r-k, r+1-p \} \). Lexicographic order for the triplets \( \lambda = \{ s, p, k \} \) is defined by \( \lambda \geq \lambda' \) if either (i) \( s \geq s' \), or (ii) \( s = s', k \geq k' \), or (iii) \( s = s', k = k' \) and \( p \geq p' \). Hence \( \lambda \geq \lambda' \) implies \( -\lambda \leq -\lambda' \). The smallest positive is \( o_{+} = (\frac{3}{2},0,0) \) while the largest negative is \( o_{-} = (-\frac{3}{2},0,1) \). Let \( \alpha(p) \) denote the triplet at the \( p \)-th place after \( o_{+} \) if \( p \) is non-negative (resp. at the \( p \)-th place before \( o_{-} \) if \( p \) is negative).

We denote by \( \mathcal{Z} \) (resp. \( \mathcal{Z}_{\geq 0} \) and \( \mathcal{Z}_{<0} \)) the set of all triplets \( \lambda \) (resp. \( \lambda \geq o_{+} \) and \( \lambda \leq o_{-} \)). We put also \( \mathcal{Z}_{\leq \alpha} = \{ \beta \in \mathcal{Z}; \beta \leq \alpha \} \) for \( \alpha \in \mathcal{Z} \).

A subset \( S \) of \( \mathcal{Z} \) is called Maya diagram if both \( S \cap \mathcal{Z}_{\geq 0} \) and \( S^{c} \cap \mathcal{Z}_{<0} \) are finite set. The integer \( \chi(S) = \#(\mathcal{Z}_{\geq 0} \cap S) - \#(\mathcal{Z}_{<0} \cap S^{c}) \) is called charge of \( S \). For each Maya diagram \( S \) with \( \chi(S) = p \) there corresponds a unique increasing function \( s: \mathcal{Z}_{\leq \alpha(p)} \rightarrow \mathcal{Z} \) such that (1) \( s(\nu) = \nu \) for sufficiently small \( \nu \) and (2) \( \text{Image}(s) = S \). The degree of a Maya diagram \( S \) is the number \( d(S) = \sum_{\nu}(s(\nu) - \nu) \).
2 Extensions and duality

a Let \( R = \{ z \in \mathbb{C}^2; |z| < 1 \} \) and \( \hat{R} = \{ w \in \hat{\mathbb{C}}^2; |w| < 1 \} \). Let
\[
\mathcal{N}(R) = \{ \phi \in \Gamma(R, S^+); \phi \text{ has } L^2\text{-boundary value on } |z| = 1, D\phi = 0 \},
\]
\[
\mathcal{N}^+(R) = \{ \psi \in \Gamma(R, S^-), \phi \text{ has } L^2\text{-boundary value on } |z| = 1, D^+\psi = 0 \}.
\]
\( \mathcal{N}(\hat{R}) \) and \( \mathcal{N}^+(\hat{R}) \) are defined similarly.

We have proved in [K1]:

Theorem 1.

(1) \( H_+ \cong \mathcal{N}(R) \), \( H_- \cong \mathcal{N}(\hat{R}) \),

(2) \( H_-^* \cong \mathcal{N}^+(R)_0 \), \( H_+^* \cong \mathcal{N}^+(\hat{R})_0 \),

where 0 indicates that the spinors in brace are 0 at 0 \( \in \mathbb{C}^2 \) or at \( \hat{0} \in \hat{\mathbb{C}}^2 \).

For instance, the isomorphism \( H_+^* \rightarrow \mathcal{N}^+(\hat{\mathbb{C}}^2)_0 \) is given as follows:

Let \( \psi = \gamma_0 \phi \in H_+^* \). We shall show that there is a \( \hat{\Psi} \in \mathcal{N}^+(\hat{R})_0 \) such that
\( \hat{\Psi}(w) = \hat{\psi}(w) \) for \( |w| = 1 \), where \( \hat{\psi}(v(z)) = -\overline{\gamma_0 \psi}(z) \). Let \( \phi = \sum_{\lambda>0} a_\lambda \phi_\lambda \in H_+ \) be the eigenfunction expansion.

Put \( \Phi(z) = \sum a_\lambda |z|^{-(\lambda+\frac{3}{2})}(\frac{2}{1+|z|^2})^{\frac{3}{2}} \phi_\lambda(\frac{z}{|z|}) \). The expression on \( \hat{\mathbb{C}}^2 \) becomes
\[
\hat{\Phi}(w) = \sum a_\lambda |w|^{(\lambda+\frac{3}{2})}(\frac{2}{1+|w|^2})^{\frac{3}{2}} \hat{\phi}_\lambda(\frac{w}{|w|}),
\]
\( \hat{\Phi}(v(z)) = -\overline{\gamma_0 \Phi(z)} \). \( \Phi \) is valued in \( \Delta^- \). We can verify that \( \hat{\Psi} = \overline{\gamma_0 \Phi} \in \mathcal{N}^+(\hat{R})_0 \) and \( \hat{\Psi}(w) = \hat{\psi}(w) \) for \( |w| = 1 \).

We define a pairing of \( H \) and \( H^* \) by
\[
(\psi|\phi) = \int_E <\phi, \gamma_0 \psi> \sigma(dz) \quad \text{for } \phi \in H \text{ and } \psi \in H^*.
\]

Theorem 1 and Stokes’ theorem yield that \( H_\pm \) and \( H_\mp^* \) are annihilated mutually by this pairing. On the other hand, \( H_\pm \) and \( H_\mp^* \) are respectively in duality. This is proved by Hahn-Banach’s extension theorem.

A coupling between \( \mathcal{N}(R) \) and \( \mathcal{N}^+(\hat{R})_0 \) is defined by
\[
-\int_E \Phi(z) \cdot \hat{\Psi}(v(z)) \sigma(dz) = \int_E <\Phi, \gamma_0 \Psi> \sigma(dz),
\]
for \( \Phi \in \mathcal{N}(R) \) and \( \hat{\Psi} \in \mathcal{N}^+(\hat{R})_0 \). Also the coupling of \( \Psi \in \mathcal{N}(\hat{R}) \) and \( \Phi \in \mathcal{N}^+(R)_0 \) is defined by the same integral.

The duality between \( H_\pm \) and \( H_\mp^* \) in the above and Theorem 1 prove the following:
Theorem 2.

(1) The dual of $\mathcal{N}(R)$ is isomorphic to $\mathcal{N}(\hat{R})_0$.
(2) The dual of $\mathcal{N}(\hat{R})$ is isomorphic to $\mathcal{N}^+(R)_0$.

3 Fockspace on $E$

Let

$$e_\lambda = \begin{cases} \phi_{r-k}^\rho \in H_+ & \text{if } \lambda \geq \rho \\ \pi_p^{-k-k} \in H_- & \text{if } \lambda \leq \rho \end{cases}.$$

We define the conjugation by $e^{*\lambda} = \gamma_0 e_{-\lambda}$. It follows that $e^{*\lambda} \in H^*_\pm$ if $\lambda \geq 0$ and $e^{*\lambda} \in H^*_+ \text{ if } \lambda < 0$. We have $(e^{*\lambda} | e_\mu) = \delta_{-\lambda, \mu}$. In particular $(e^{*0+} | e_{0-}) = 1$.

For a Maya-diagram $S$ we put $e_S = \wedge e_\lambda = e_{\max S} \wedge \cdots$, the wedge being taken on decreasing order. We denote in particular $|\alpha >= e_{\mathcal{Z}_{\alpha-}} = e_\alpha \wedge \cdots$.

The Fock space of charge $p$ and total Fock space are introduced as follows:

$$\mathcal{F}_p = \Pi_{\{S; \chi(S) = p\}} C e_S \quad \mathcal{F} = \bigoplus_p \mathcal{F}_p.$$ 

$\mathcal{F}_p$ is given a filtration by the degree of Maya-diagram introduced in section 1 and this filtration endows $\mathcal{F}_p$ with a complete vector space topology.

For a Maya-diagram $S$ we put $e^*_S = \wedge_{-\mu \in S} e^{*\mu} = \cdots \wedge e^{*\max S}$, the wedge being taken on decreasing order. We denote $< \alpha| = e_{\mathcal{Z}_{\alpha-}}^* = \cdots \wedge e^{*\alpha}$.

The dual Fock space is defined as a direct sum with discrete topology:

$$\mathcal{F}^* = \bigoplus S C e^*_S.$$ 

The coupling $(\ |)$ of $H_\pm$ and $H^*_\pm$ extends to give a coupling between $\mathcal{F}$ and $\mathcal{F}^*$. We have $(e^*_S | e_{S'}) = \delta_{S,S'}$. In particular we have $< \alpha| \beta >= \delta_{\alpha, \beta}$.

Differentiation $D_\alpha$ by $\alpha \in H$ is defined on $H$ by

$$D_\alpha \phi = (e^{*-\alpha} | \phi) = \int_E < \phi, \alpha > d\sigma$$ 

for $\phi \in H$.

It is extended to $\mathcal{F}$ by the rule

$$D_\alpha (\phi \wedge \psi) = D_\alpha \phi \wedge \psi + (-1)^{\deg \phi} \phi \wedge D_\alpha \psi$$.
for $\phi, \psi \in \mathcal{F}$. $D_\alpha$ acts on $\mathcal{F}$ from the left as an inner derivation.

We also define the differentiation on $H^*$ by

$$D_\alpha^* \phi^* = (\phi^* | e_\alpha), \quad \text{for } \phi^* \in H^*.$$  

It is extended to $\mathcal{F}^*$ by $D_\alpha^* (\phi^* \wedge \psi^*) = \phi^* \wedge D_\alpha^* \psi^* + (-1)^{\deg \psi^*} D_\alpha^* \phi^* \wedge \psi^*$ for $\phi^*, \psi^* \in \mathcal{F}^*$. $D_\alpha^*$ acts on $\mathcal{F}^*$ from the right.

b  We define the following actions $a_\nu, a_\nu^\dagger$ on $\mathcal{F}$ and $\mathcal{F}^*$:

$$a_\nu = D_\nu, \quad a_\nu^\dagger = e_\nu \wedge \quad \text{left action on } \mathcal{F},$$

$$a_\nu = e^{* - \nu}, \quad a_\nu^\dagger = D_\nu^* \quad \text{right action on } \mathcal{F}^*,$$

where exterior multiplications should be arranged in order. We have then the relations

$$\{a_\lambda, a_\nu\} = 0, \quad \{a_\lambda^\dagger, a_\nu^\dagger\} = 0$$

$$\{a_\lambda^\dagger, a_\nu\} = \{a_\lambda, a_\nu^\dagger\} = \delta_{\lambda, \nu}.$$  

Hence $\{a_\nu, a_\nu^\dagger\}$ generate a Clifford algebra $\mathcal{A}$, which is called fermion operator algebra. $\mathcal{A}$ acts on $\mathcal{F}$ and $\mathcal{F}^*$.

Proposition 3.

(1)  

$$a_\nu |\alpha> = 0 \quad \text{for } \nu > \alpha \quad a_\nu^\dagger |\alpha> = 0 \quad \text{for } \nu \leq \alpha$$

$$<\alpha|a_\nu = 0 \quad \text{for } \nu \leq \alpha \quad <\alpha|a_\nu^\dagger = 0 \quad \text{for } \nu > \alpha.$$  

(2)  

$$(e_S^* a_\alpha | e_{S'}) = (e_S^* | a_\alpha e_{S'})$$

$$(e_S^* | a_\alpha^\dagger e_{S'}) = (e_S^* a_\alpha^\dagger | e_{S'})$$

We shall introduce the following field operators of fermion:

$$\varphi_+(z) = \sum_{\nu \geq 0_+} \phi_{\nu}(z) a_\nu$$  

$$\varphi^+_\nu(z) = \sum_{\nu \geq 0_+} t_{\phi_{\nu}(z)} a_\nu^\dagger$$

$$\varphi_-(w) = \sum_{\nu \leq 0_-} \pi_{\nu}(w) a_\nu$$  

$$\varphi^-_\nu(w) = \sum_{\nu \leq 0_-} t_{\pi_{\nu}(w)} a_\nu^\dagger.$$
From the above proposition we have;

\[ \varphi_+(z)\left| o_- \right> = 0, \quad \varphi_+^\dagger(z)\left| o_- \right> = 0 \]

\[ \langle o_- | \varphi_-(z) = 0, \quad \langle o_- | \varphi_-^\dagger(z) = 0. \]

Proposition 4.

\[ < \varphi^\dagger(x)\varphi(y) >= < o_- | \varphi^\dagger(x) \cdot \varphi(y) | o_- > = \sum_{r} \sum_{q=0}^{r+1} \frac{r+1}{q!} h_{r+1-q,q}^{q}(A, B) \]

\[ < \varphi(x)\varphi^\dagger(y) >= < o_- | \varphi(x) \cdot \varphi^\dagger(y) | o_- > = \sum_{r} \sum_{q=0}^{r+1} \frac{r+2}{q!} h_{r-q,q}^{q}(C, D) \]

\[ < \varphi(x)\varphi(y) >= < \varphi^\dagger(x)\varphi^\dagger(y) >= 0 \]

where

\[ A = \bar{x}_1 y_1 + x_2 \bar{y}_2 \quad B = \bar{x}_1 y_2 - x_2 \bar{y}_1 \]

\[ C = x_1 \bar{y}_1 + x_2 \bar{y}_2 \quad D = x_1 y_2 - x_2 y_1. \]

References

[K] Kori, T., Dirac operators on $S^4$ and on $S^3$. Infinite dimensional Grassmanian on $S^3$.