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非線形積分可能系の確率モデル

A Stochastic Model of an Integrable Nonlinear System

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A stochastic model is introduced for a Lotka-Volterra sytem with a system of finite particles. For the case of infinite particles, the sytem of $2s+1$ species is described by an integrable dynamical system with $s+1$ conserved quantities. A natural extension of the $s+1$ conserved quantities is given for the stochastic sytem of finite particles.

The Toda lattice is a typical discrete system which has soliton solutions [1]. The cyclic Toda lattice is one of the typical nonlinear integrable system, which has m conserved quantities [2),3)] for $2m$ variables. The soliton solutions are known for systems of competing species[7)-14)]. A cyclic system of competing species

$$\frac{dP_i}{dt} = P_i(\sum_{j=1}^{s} P_{i-j} - \sum_{j=1}^{s} P_{i+j}) \tag{1}$$

for relative frequencies $P_i, i = 1, 2, \cdots, 2s+1$, has $s+1$ conserved quantities [7)-14)]. We have a symple combinatorial proof , to get the $s+1$ conserved quantities for this system. The Lax equation is obtained for a more general class [12)-14)] of equations. The system (1) is originally obtained from a deterministic approximation of a stochastic model[4)-11)]. Here we study the original stochastic system to see the behaviour of the conserved quantities of the system (1). Herė we discuss how the conserved quantities are extended to a stohchastic system for the integrable nonlinear system. To discuss the behaviour of the conserved quantities for the other stochastic systems of nonlinear integrable systems may be interesting problems. Our system could be one of the typical system with conserved quantities which can be naturally extended to stochastic system.

Define a_{ij} by the equation

$$\sum_{j=1}^{2s+1} a_{ij} P_j \equiv \sum_{j=1}^{s} P_{i-j} - \sum_{j=1}^{s} P_{i+j} \tag{2}$$

We say the type i dominates the type j if $a_{ij} = 1$. If $a_{ij} = -1$,we say the type i is dominated by the type j. Consider $2r+1$ species out of the $2s+1$ species. If each of the $2r+1$ species dominates the other r species and is dominated by the other remained r species, then we say the $2r+1$ species are in a regular tournament. Take $2r+1$ individuals(particles) at random from the system. Let I_r be the probability that the corresponding $2r+1$ species of the $2r+1$ particles are in a regular tournament, then the I_r, $r = 0, 1, 2, \cdots, s$, are conserved quantities, that is to say,

$$\frac{d}{dt} I_r = 0 \tag{3}$$

For example, for the case $2s+1 = 5$, we have the conserved quantities,

$I_1 = P_1 + P_2 + P_3 + P_4 + P_5,$
$I_2 = P_1P_2P_4 + P_2P_3P_5 + P_3P_4P_1 + P_4P_5P_2 + P_5P_1P_3,$
and
$I_3 = P_1P_2P_3P_4P_5.$

Consider the following stochastic sytem of i),ii),iii),which is a stochastic analogue of the above dynamical system.
i) There are three species $1, 2, ...2s+1$ whose numbers of particles are at time t are $n_1(t), n_2(t), ..., n_{2s+1}(t)$ respectively, where $n_1(t)+n_2(t)+...+n_{2s+1}(t) = n$ and n is a constant.
ii) A random collision takes place in a time interval Δt, that is , each colliding pair is equally likely chosen.
iii) A particle of species i and a particle of species j collide with each other and become two particles of species i,if $i - j \equiv 0, 1, 2, ..., s \pmod{2s + 1}$. If $i - j \equiv s + 1, s + 2, ..., 2s \pmod{2s + 1}$ they become two particles of species j.
For the case $s = 1$, From the above i),ii) and iii) we have a Markov chain for the probability $P(n_1, n_2, n_3; t)$ of each state (n_1, n_2, n_3) at time t whose transition probability is given by

$$\begin{aligned}
&P(n_1, n_2, n_3; t + \Delta t) \\
&= \frac{1}{n(n-1)} \quad \{(n_1(n_1 - 1) \\
&\quad +n_2(n_2 - 1) + n_3(n_3 - 1))P(n_1, n_2, n_3; t) \\
&\quad +2(n_1 + 1)(n_2 - 1)P(n_1 + 1, n_2 - 1, n_3; t) \\
&\quad +2(n_2 + 1)(n_3 - 1)P(n_1, n_2 + 1, n_3 - 1; t) \\
&\quad +2(n_3 + 1)(n_1 - 1)P(n_1 - 1, n_2, n_3 + 1; t)\}.
\end{aligned}$$

Consider the product $I_1(t)$ of the relative frequencies of three species at time t. For the stochastic process $I_1(t)$, we have the expectation conditioning the value I_t at time t ,

$$E(I_1(t + \Delta t) \mid I_1(t)) = (1 - 2\frac{{}_3C_2}{n(n-1)})I_1(t).$$

Let the frequencies of the three types be (n_1, n_2, n_3) at time t and consider the product of the numbers of three species. By i),ii) and iii) the frequencies at $t + \Delta t$ are

$$\begin{array}{lll} (n_1 - 1, n_2 + 1, n_3) & \text{with probability} & \dfrac{2n_1n_2}{n(n-1)} \\ (n_1, n_2 - 1, n_3 + 1) & \text{"} & \dfrac{2n_2n_3}{n(n-1)} \\ (n_1 + 1, n_2, n_3 - 1) & \text{"} & \dfrac{2n_3n_1}{n(n-1)} \\ (n_1, n_2, n_3) & \text{"} & \dfrac{n_1(n_1-1) + n_2(n_2-1) + n_3(n_3-1)}{n(n-1)}. \end{array}$$

So the expectation of the product is

$$\left(1 - 2\frac{{}_3C_2}{n(n-1)}\right) n_1n_2n_3.$$

For the general $2s + 1$ we have

$$E(I_r(t + \Delta t) \mid I_r(t)) = \left(1 - 2\frac{{}_{2r+1}C_2}{n(n-1)}\right) I_r(t) \tag{4}$$

for $r = 0, 1, 2, ..., s$. Put $\Delta t = dt$ and $2/n(n-1) = c_2dt$. We have,

$$E(I_r(t + dt) \mid I_r(t)) = (1 - {}_{2r+1}C_2c_2dt)I_r(t) \tag{5}$$

for $r = 0, 1, 2, ..., s$.

A continuous analogue of the discrete system is the stochastic differential equation,

$$dP_i(t) = c_1P_1(t)(\sum_{j=1}^{s} P_{i-j}(t) - \sum_{j=1}^{s} P_{i+j}(t))dt + \sqrt{c_2P_i(t)P_j(t)}db_{ij}(t) \tag{6}$$

for $i = 1, 2, \cdots, 2s + 1$. with $a_{ij} + a_{ji} = 0, b_{ij}(t) + b_{ji}(t) = 0$, where $b_{ij}(t)$ $(i > j)$ are mutually independent one-dimensional Wiener processes with the mean 0 and the variance t. For the case $c_1 = 0$, this system represents the

Wright-Fisher model in population genetics of the symplest case. In Wright-Fisher model it is supposed that each of the genes of the next generation is obtained by a random choice among the genes of the previous generation. The stochastic differential equation represents the fluctuation by the random sampling effect. For the case $c_2 = 0$, this equation coincides with Eq.(1). For $c_2 > 0$, we have

$$E(I_r(t+s) \mid I_r(t)) = I_r(t) \exp(-{}_{2r+1}C_2 c_2 s) \tag{7}$$

for $r = 0, 1, 2, ..., s$, which is equivalent to Eq.(5).

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